Structural results for conditionally intersecting families and some applications

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Abstract

Let $k \geqslant d \geqslant 3$ be fixed. Let $\mathcal{F}$ be a k-uniform family on $[n]$. Then $\mathcal{F}$ is (d,s)-conditionally intersecting if it does not contain d sets with union of size at most s and empty intersection. Answering a question of Frankl, we present some structural results for families that are (d,s)-conditionally intersecting with $s \geqslant 2k+d-3$, and families that are $(k,2k)$-conditionally intersecting. As applications of our structural results we present some new proofs to the upper bounds for the size of the following k-uniform families on $[n]$:

(a) $(d, 2k+d-3)$-conditionally intersecting families with $n \geqslant 3k^5$;

(b) $(k,2k)$-conditionally intersecting families with $n \geqslant k^2/(k-1)$;

(c) Nonintersecting $(3,2k)$-conditionally intersecting families with $n \geqslant 3k\binom{2k}{k}$.

Our results for (c) confirms a conjecture of Mammoliti and Britz for the case $d=3$.

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1 Introduction

Let V be a set, and let S,T be two subsets of V. Then we use $S-T$ to denote the set $S \setminus T$, and use $\binom{V}{k}$ to denote the collection of all k-subsets of V. Let $[n]$ denote the set $\{1,\ldots,n\}$. A *d-cluster* of k-sets is a collection of d different k-subsets $A_1,\ldots,A_d$ of $[n]$ such that

$$|A_1 \cup \cdots \cup A_d| \leqslant 2k, \quad \text{and} \quad |A_1 \cap \cdots \cap A_d| = 0.$$

Let $\mathcal{F}$ be a k-uniform family on $[n]$. Then $\mathcal{F}$ is (d,s)-*conditionally intersecting* if it does not contain d sets with union of size at most s and empty intersection. In particular, a family $\mathcal{F}$ is $(d, 2k)$-conditionally intersecting if it does not contain d-clusters. We use $h(n, k, d, s)$ to denote the maximum size of a (d, s)-conditionally intersecting family $\mathcal{F}$.

Note that a k-uniform family is $(2, 2k)$-conditionally intersecting if and only if it is intersecting. The celebrated Erdős-Ko-Rado theorem [4] states that $h(n, k, 2, 2k) \leqslant \binom{n-1}{k-1}$ for all $n \geqslant 2k$, and when $n > 2k$ equality holds only if $\mathcal{F}$ is a *star*, i.e. a collection of k-sets that contain a fixed vertex. In [5], Frankl showed that the same conclusion holds for $n \geqslant dk/(d-1)$ when the intersecting condition is replaced by the *d-wise intersecting* condition, i.e. every d sets of $\mathcal{F}$ have nonempty intersection.

Theorem 1 (Frankl [5])**.** *Let $k \geqslant d \geqslant 3$ be fixed and $n \geqslant dk/(d-1)$. If $\mathcal{F} \subset \binom{[n]}{k}$ is a d-wise intersecting family, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$, with equality only if $\mathcal{F}$ is a star.*

Later, Frankl and Füredi [7] extended Theorem 1 and proved that $h(n, k, 3, 2k) \leqslant \binom{n-1}{k-1}$ for all $n \geqslant k^2 + 3k$, and they conjectured that the same inequality holds for all $n \geqslant 3k/2$. In [11], Mubayi settled their conjecture and posed the following more general conjecture.

Conjecture 2 (Mubayi [11])**.** Let $k \geqslant d \geqslant 3$ and $n \geqslant dk/(d-1)$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a $(d, 2k)$-conditional intersecting family. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$, with equality only if $\mathcal{F}$ is a star.

Conjecture 2 has been intensively studied in the past decade. Mubayi [12] proved this conjecture for the case $d = 4$ with n sufficiently large. Later, Mubayi and Ramadurai [13], and independently, Füredi and Özkahya [8] settled this conjecture for all $d \geqslant 3$ with n sufficiently large. In [2], Chen, Liu and Wang confirmed this conjecture for the case $d = k$, and they also showed that $h(n, k, d, (d+1)k/2) \leqslant \binom{n-1}{k-1}$ for all $n \geqslant dk/(d-1)$. Very recently, Conjecture 2 was completely solved by Currier [3].

In this paper, we consider the structure of conditionally intersecting families, which is motivated by a structural theorem for $(3, 6)$-conditionally intersecting family proved by Frankl [6].

Definition 3. Let $\mathcal{H} \subset 2^{[n]}$, and let $H \in \mathcal{H}$. A subset $G \subset H$ is called unique if there is no other set in $\mathcal{H}$ containing G.

The following result of Bollobás [1] gives an upper bound for the size of a family in which every set has a unique subset.

Theorem 4 (Bollobás [1])**.** *Suppose that for every member H of the family $\mathcal{H} \subset 2^{[n]}$ the set $G(H) \subset H$ is a unique subset. Then*

$$\sum_{H \in \mathcal{H}} \frac{1}{\binom{n-|H-G(H)|}{|G(H)|}} \leqslant 1.$$

Frankl [6] proved the following structural result for $(3, 6)$-conditionally intersecting families.

Theorem 5 (Frankl [6])**.** *Suppose that $\mathcal{F} \subset \binom{[n]}{3}$ is a $(3, 6)$-conditionally intersecting family. Then $\mathcal{F}$ can be partitioned into two families $\mathcal{H}$ and $\mathcal{B}$, and the ground set $[n]$ can be partitioned into two disjoint subsets Y and Z such that the following statements hold.*

(a) $\mathcal{H} \subset \binom{Y}{3}$ *and every set* $H \in \mathcal{H}$ *contains a unique* 2*-subset.*

(b) $\mathcal{B} \subset \binom{Z}{3}$ *and* $\mathcal{B}$ *is the vertex disjoint union of* $|Z|/4$ *copies of complete* 3*-graphs on* 4 *vertices.*

First, let us show how to use Theorem 5 to get an upper bound for $|\mathcal{F}|$. Let $\mathcal{F} \subset \binom{[n]}{3}$ be a $(3,6)$-conditionally intersecting family, and let $Y, Z, \mathcal{B}$ and $\mathcal{H}$ be given by Theorem 5. Since every set in $\mathcal{H}$ contains a unique 2-subset, it follows from Theorem 4 that $|\mathcal{H}| \leqslant \binom{|Y|-1}{2}$. On the other hand, it is easy to see that $|\mathcal{B}| = |Z|$. Therefore,

$$|\mathcal{F}| = |\mathcal{H}| + |\mathcal{F}| \leqslant \binom{|Y|-1}{2} + |Z| \leqslant \binom{n-1}{2},$$

and equality holds only if $Z = \emptyset$.

In [6], Frankl also asked for a structural result for a $(3,2k)$-conditionally intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ which can imply the $\binom{n-1}{k-1}$ bound for $|\mathcal{F}|$. Here we consider a more general question, namely the structures of $(d, 2k+d-3)$-conditionally intersecting families for all $k \geqslant d \geqslant 3$, and we obtain the following result.

Let $\mathcal{L}_k$ denote the collection of all k-graphs on at most $2k$ vertices.

Theorem 6. *Let* $k \geqslant d \geqslant 3$ *be fixed. Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *is a* $(d, 2k+d-3)$*-conditionally intersecting family. Then* $\mathcal{F}$ *can be partitioned into three families* $\mathcal{H}$, $\mathcal{B}$ *and* $\mathcal{S}$, *and the ground set* $[n]$ *can be partitioned into two subsets* Y *and* Z *such that the following statements hold.*

(a) $\mathcal{H} \subset \binom{Y}{k}$ *and every set* $H \in \mathcal{H}$ *contains a unique* $(k-1)$*-subset.*

(b) Z *has a partition* $V_1 \cup \cdots \cup V_t$ *with each* V_i *of size at most* $2k$ *such that* $\mathcal{B} \subset \bigcup_{i=1}^{t} \binom{V_i}{k}$, *i.e., the family* $\mathcal{B}$ *is the vertex disjoint union of copies of* k*-graphs in* $\mathcal{L}_k$

(c) $\mathcal{S} \subset \binom{[n]}{k} - \binom{Y}{k}$, *and for every set* $S \in \mathcal{S}$ *and every* $V_i \subset Z$ *the size of* $S \cap V_i$ *is either* 0 *or at least* d.

Note that the constraint on $|S \cap V_i|$ in (c) for $S \in \mathcal{S}$ and $V_i \subset Z$ implies that the family $\mathcal{S}$ is actually very sparse. Therefore, the term $|\mathcal{S}|$ contributes very little to $|\mathcal{F}|$.

Our next result gives a structure for $(k, 2k)$-intersecting families for all $k \geqslant 3$.

Theorem 7. *Let* $k \geqslant 3$ *be fixed. Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *is a* $(k, 2k)$*-conditionally intersecting family. Then* $\mathcal{F}$ *can be partitioned into two families* $\mathcal{H}$ *and* $\mathcal{B}$, *and the ground set* $[n]$ *can be partitioned into two subsets* Y *and* Z *such that the following statements hold.*

(a) $\mathcal{H} \subset \binom{Y}{k}$ *and every set* $H \in \mathcal{H}$ *contains a unique* $(k-1)$*-subset.*

(b) $\mathcal{B} \subset \binom{Z}{k}$ *and* $\mathcal{B}$ *is the vertex disjoint union of* $\frac{|Z|}{k+1}$ *copies of complete* k*-graphs on* $(k+1)$ *vertices.*

Applying the structural results above we are able to give some new proofs to the following theorems.

Theorem 8. *Let $k \geqslant d \geqslant 3$ be fixed and $n \geqslant 3k^5$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a $(d, 2k + d - 3)$-conditionally intersecting family. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.*

Note that Theorem 8 is true for every $n \geqslant 3k/2$ according to the result in [11], but in our proof we need the assumption that $n \geqslant 3k^5$ to keep the calculations simple.

Theorem 9. *Let $k \geqslant 3$ be fixed and $n \geqslant k^2/(k-1)$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a $(k, 2k)$-conditionally intersecting family. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.*

Theorem 10. *Let $k \geqslant 3$ be fixed and $n \geqslant 3k\binom{2k}{k}$. Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family that is $(3, 2k)$-conditionally intersecting but not intersecting. Then $|\mathcal{F}| \leqslant \binom{n-k-1}{k-1} + 1$.*

Theorem 10 shows that Mammoliti and Britz's conjecture (Conjecture 4.1 in [10]) is true for the case $d = 3$. Note that in [9] the author considered Mammoliti and Britz's conjecture for all $d \geqslant 3$, and showed that their conjecture is true for $d = 3$, but false for all $d \geqslant 4$. However, the method we used here is completely different from the method used in [9].

The remaining part of this paper is organized as follows. In Section 2, we prove Theorems 6 and 7. In Section 3, we prove Theorems 8, 9, and 10.

2 Structural Results

Let $\mathcal{F}$ be a k-uniform family on $[n]$ and $B \in \mathcal{F}$. We say B is *bad* if it does not contain any unique $(k-1)$-subset. Suppose that $B = \{b_1, \ldots, b_k\}$ is a bad set in $\mathcal{F}$, then there exist k distinct sets $C_1, \ldots, C_k$ in $\mathcal{F}$ such that $B \cap C_i = B - \{b_i\}$ for all $i \in [k]$. Let $V_B = B \cup C_1 \cdots \cup C_k$ and $H_B = \{B, C_1, \ldots, C_k\}$. First let us prove Theorem 7.

Proof of Theorem 7. Suppose that $\mathcal{F}$ is a $(k, 2k)$-conditionally intersecting family, and suppose that $B = \{b_1, \ldots, b_k\}$ is a bad set in $\mathcal{F}$. Let $C_1, \cdots, C_k, V_B, H_B$ be defined as above. Since $|V_B| \leqslant 2k$, by assumption we have $C_1 \cap \cdots \cap C_k \neq \emptyset$. It follows that $|V_B| = k + 1$ and, hence, the family H_B is a complete k-graph on V_B. Let b_{k+1} denote the vertex in $V_B - B$, and let $F \in \mathcal{F} - H_B$. Then we claim that $F \cap V_B = \emptyset$. Indeed, suppose that $F \cap V_B \neq \emptyset$. We may assume that $F \cap V_B = \{b_1, \ldots, b_\ell\}$ for some $\ell \in [k-1]$. Now, rename the edges in H_B as $B_i = V_B - b_i$ for all $i \in [k+1]$. Since $|F \cup B_1 \cup \cdots \cup B_{k-1}| \leqslant 2k$ and $F \cap B_1 \cap \cdots \cap B_{k-1} = \emptyset$, the k sets $F, B_1, \ldots, B_{k-1}$ form a k-cluster in $\mathcal{F}$, a contradiction. Therefore, $F \cap V_B = \emptyset$. To finish the proof we just let $\mathcal{B}$ be the collection of all bad sets in $\mathcal{F}$, and let $\mathcal{H} = \mathcal{F} - \mathcal{B}$. □

Before proving Theorem 6 let us present a useful lemma. Let $s = 2k + d - 3$.

Lemma 11. *Suppose that $\mathcal{F}$ is a (d, s)-conditionally intersecting family and B is a bad set in $\mathcal{F}$. Then for every $F \in \mathcal{F}$ either $|F \cap V_B| = 0$ or $|F \cap V_B| \geqslant d$.*

Proof. Let B is a bad set in $\mathcal{F}$ and let V_B be the set as we defined before. Suppose that $F \in \mathcal{F}$ has nonempty intersection with V_B. It suffices to show that $|F \cap V_B| \geqslant d$. For contradiction, suppose that $|F \cap B| = x$, $|F \cap (V_B - B)| = y$ and $x + y \leqslant d - 1$. Suppose that $F \cap B = \{b_{m_1}, \dots, b_{m_x}\}$ and $F \cap (V_B - B) = \{c_{n_1}, \dots, c_{n_y}\}$.

If $x = d - 1$, then $y = 0$ and, hence, the d sets $F, C_{m_1}, \dots, C_{m_{d-1}}$ satisfy $|F \cup C_{m_1} \cup \dots \cup C_{m_{d-1}}| \leqslant 2k$ and $F \cap C_{m_1} \cap \dots \cap C_{m_{d-1}} = \emptyset$, a contradiction. If $x = d - 2$, then the d sets $F, B, C_{m_1}, \dots, C_{m_{d-2}}$ satisfy $|F \cup B \cup C_{m_1} \cup \dots \cup C_{m_{d-2}}| \leqslant 2k$ and $F \cap B \cap C_{m_1} \cap \dots \cap C_{m_{d-2}} = \emptyset$, a contradiction. Therefore, we may assume that $x \leqslant d - 3$. Let $p = d-(x+2)$. Choose p sets $C_{q_1}, \dots, C_{q_p}$ from $\{C_1, \dots, C_k\} - \{C_{m_1}, \dots, C_{m_x}\}$. Then the d sets $F, B, C_{m_1}, \dots, C_{m_x}, C_{q_1}, \dots, C_{q_p}$ satisfy $|F \cup B \cup C_{m_1} \cup \dots \cup C_{m_x} \cup C_{q_1} \cup \dots \cup C_{q_p}| \leqslant 2k+p$ and $F \cap B \cap C_{m_1} \cap \dots \cap C_{m_x} \cap C_{q_1} \cap \dots \cap C_{q_p} = \emptyset$. By assumption we have $2k + p \geqslant s$ and, hence, $x = 0$ and $y \geqslant 1$.

Let $p' = d-(y+2)$, and choose p' sets $C_{q_1}, \dots, C_{q_{p'}}$ from $\{C_1, \dots, C_k\} - \{C_{n_1}, \dots, C_{n_y}\}$. Then the d sets $F, B, C_{n_1}, \dots, C_{n_y}, C_{q_1}, \dots, C_{q_{p'}}$ satisfy $|F \cup B \cup C_{n_1} \cup \dots \cup C_{n_y} \cup C_{q_1} \cup \dots \cup C_{q_{p'}}| \leqslant 2k + p' \leqslant s$ and $F \cap B \cap C_{n_1} \cap \dots \cap C_{n_y} \cap C_{q_1} \cap \dots \cap C_{q_{p'}} = \emptyset$, a contradiction. Therefore, we have $|F \cap V_b| \geqslant d$. □

Now we are ready to prove Theorem 6.

Proof of Theorem 6. Let $\mathcal{F}$ be a (d, s)-conditionally intersecting family. Choose a collection of bad sets $\{B_1, \dots, B_t\}$ for some t from $\mathcal{F}$ such that the sets $V_{B_1}, \dots, V_{B_t}$ are pairwise disjoint, and any other bad set in $\mathcal{F}$ has nonempty intersection with V_{B_i} for some $i \in [t]$. Note that this can be done by greedy choosing each B_i from $\mathcal{F}$ such that B_i is disjoint from $\bigcup_{j<i} V_{B_j}$, and by Lemma 11 the set V_{B_i} is also disjoint from $\bigcup_{j<i} V_{B_j}$.

Now let $V_i = V_{B_i}$ and $H_i = H_{B_i}$ for $i \in [t]$. Let $Z = \bigcup_{i\in[t]} V_i$ and $Y = [n] - Z$. Let $\mathcal{B} = \bigcup_{i\in[t]} H_i$, $\mathcal{H} = \mathcal{F} \cap \binom{Y}{k}$ and $\mathcal{S} = \mathcal{F} - \mathcal{B} - \mathcal{H}$. Suppose that $S \in \mathcal{S}$. Then by Lemma 11, either $|S \cap V_i| = 0$ or $|S \cap V_i| \geqslant d$ for every $i \in [t]$, and this completes the proof of Theorem 6. □

3 Applications

In this section we show some applications of Theorems 6 and 7 by giving new proofs to Theorems 8, 9, and 10. First let us prove Theorem 9.

Proof of Theorem 9. Suppose that $\mathcal{F}$ is a $(k, 2k)$-conditionally intersecting family on $[n]$. Let $Y, Z, \mathcal{B}$ and $\mathcal{H}$ be given by Theorem 7. By Theorem 4, $\mathcal{H} \leqslant \binom{|Y|-1}{k-1}$. On the other hand, it is easy to see that $|\mathcal{B}| = (k+1) \times |Z|/(k+1) = |Z|$. Therefore, $|\mathcal{F}| = |\mathcal{H}| + |\mathcal{B}| \leqslant \binom{|Y|-1}{k-1} + |Z| \leqslant \binom{n-1}{k-1}$, and equality holds only if $Z = \emptyset$. □

Now we apply Theorem 6 to prove Theorem 8.

Proof of Theorem 8. Let $\mathcal{F}$ be a $(d, 2k+d-3)$-conditionally intersecting family on $n \geqslant 3k^5$ vertices. Let $Y, Z, \mathcal{B}, \mathcal{H}$ and $\mathcal{S}$ be given by Theorem 6. Let $v_i = |V_i|$ for $i \in [t]$. Let $Y_0 = Y$ and $Y_i = Y_{i-1} \cup V_i$ for $i \in [t]$ and let $y_i = |Y_i|$ for $0 \leqslant i \leqslant t$. Define $\mathcal{H}_i = \mathcal{F} \cap \binom{Y_i}{k}$ and let

$h_i = |\mathcal{H}_i|$. By Lemma 11, every set $H \in \mathcal{H}_i$ is either disjoint from V_i or has an intersection of size at least d with V_i. Therefore, $|\mathcal{H}_i| \leqslant |\mathcal{H}_{i-1}| + \sum_{\ell=d}^{k} \binom{v_i}{\ell}\binom{y_{i-1}}{k-\ell}$. Inductively, we obtain

$$|\mathcal{F}| \leqslant |\mathcal{H}| + \sum_{i=0}^{t-1}\sum_{\ell=d}^{k} \binom{v_{i+1}}{\ell}\binom{y_i}{k-\ell} \leqslant \binom{y_0-1}{k-1} + \sum_{i=0}^{t-1}\sum_{\ell=d}^{k}\binom{2k}{\ell}\binom{n-k-1}{k-\ell}.$$

Since $\binom{2k}{\ell}\binom{n-k-1}{k-\ell} \geqslant \binom{2k}{\ell+1}\binom{n-k-1}{k-\ell-1}$, we obtain

$$\begin{aligned}|\mathcal{F}| &\leqslant \binom{y_0-1}{k-1} + \sum_{i=0}^{t-1}(k-d)\binom{2k}{d}\binom{n-k-1}{k-d} \\ &\leqslant \binom{y_0-1}{k-1} + (k-d)\binom{2k}{d}\binom{n-k-1}{k-d}\frac{n-y_0}{k+1} \\ &\leqslant \binom{y_0-1}{k-1} + \binom{2k}{3}\binom{n-k-1}{k-3}(n-y_0).\end{aligned}$$

Now let $\delta = \left(2\binom{2k}{3}\right)^{-1}$. If $n - y_0 \leqslant \delta n$, then

$$|\mathcal{F}| < \binom{n-1}{k-1} - k\binom{n-k-1}{k-2} + \frac{n}{2}\binom{n-k-1}{k-3} < \binom{n-1}{k-1},$$

and we are done. Therefore, we may assume that $y_0 \leqslant (1-\delta)\, n$. Then

$$|\mathcal{F}| \leqslant \left(1 - \frac{1}{4\binom{2k}{3}}\right)\binom{n-1}{k-1} + \binom{n-k-1}{k-3}\frac{n}{2} \leqslant \binom{n-1}{k-1},$$

and this completes the proof of Theorem 8. □

The remaining part of this section is devoted to prove Theorem 10. We will use the following lemma in our proof.

The *shadow* $\partial\mathcal{H}$ of a family $\mathcal{H} \subset \binom{[n]}{k}$ is defined as follows:

$$\partial\mathcal{H} = \left\{ G \in \binom{[n]}{k-1} : \exists H \in \mathcal{H} \text{ such that } G \subset H \right\}.$$

Lemma 12. *Suppose that $\mathcal{H} \subset \binom{[n]}{k}$, and every set $H \in \mathcal{H}$ has a unique $(k-1)$-subset $G(H) \subset H$. Then*

$$|\mathcal{H}| \leqslant \frac{n-k+1}{n}|\partial\mathcal{H}|.$$

Proof. Consider a weight function $\omega(G,H)$ for all pairs $G \subset H \in \mathcal{F}$ with $|G| = k-1$. For every $G \in \partial\mathcal{H}$ and every $H \in \mathcal{H}$ assign weight 1 to (G,H) if $G = G(H)$ and $(n-k+1)^{-1}$ if $G \neq G(H)$. Then an easy double counting gives

$$\left(1 + \frac{k-1}{n-k+1}\right)|\mathcal{H}| = \sum_{(G,H)} \omega(G,H) \leqslant |\partial\mathcal{H}|,$$

which implies $|\mathcal{H}| \leqslant (n-k+1)|\partial\mathcal{H}|/n$. □

Definition 13. Let $\mathcal{F} \subset \binom{[n]}{k}$ and $S \subset [n]$. Then $\mathcal{F}$ is a full star on S if it is the collection of all k-subsets of S that contain a fixed vertex v, and $\mathcal{F}$ is a star if it is a subfamily of some full star on S. In either case, we call v the core of $\mathcal{F}$.

Now we prove Theorem 10.

Proof of Theorem 10. Let $n \geqslant 3k\binom{2k}{k}$ and let $\mathcal{F}$ be a family on $[n]$ such that $\mathcal{F}$ is $(3,2k)$-conditionally intersecting but not intersecting. Suppose that $B \in \mathcal{F}$ is a bad set. Let V_B, H_B be as defined at the beginning of this section and let $\mathcal{F}' = \mathcal{F} \cap \binom{[n]-V_B}{k}$. Since $\mathcal{F}'$ is also $(3,2k)$-intersecting, by result in [11], $|\mathcal{F}'| \leqslant \binom{n-|V_B|-1}{k-1})| \leqslant \binom{n-k-2}{k-1}$. Then by Lemma 11,

$$\begin{aligned}
|\mathcal{F}| &\leqslant |\mathcal{F}'| + \sum_{i=3}^{k} \binom{2k}{i}\binom{n-k-1}{k-i} \\
&\leqslant \binom{n-k-2}{k-1} + k\binom{2k}{3}\binom{n-k-1}{k-3} \\
&= \binom{n-k-1}{k-1} - \left(\binom{n-k-2}{k-2} - k\binom{2k}{3}\binom{n-k-1}{k-3}\right) < \binom{n-k-1}{k-1} + 1,
\end{aligned}$$

and we are done. So we may assume that every $F \in \mathcal{F}$ has a unique $(k-1)$-subset $G(F)$.

Since $\mathcal{F}$ is not intersecting, there exist two disjoint sets A, B in $\mathcal{F}$. Assume that $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. Let $I = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ and let $U = [n]-I$. For every set $C \subset U$ of size at most $k-1$ define the family $\mathcal{F}(C)$ on I as follows:

$$\mathcal{F}(C) = \{F - C : F \in \mathcal{F} \text{ and } F \cap U = C\}.$$

For every $i \in \{0, 1, \ldots, k\}$ let

$$\mathcal{F}_i = \{F \in \mathcal{F} : |F \cap I| = i\}.$$

First notice that $\mathcal{F}_k = \{A, B\}$, since any extra edge in $\mathcal{F}_k$ together with A, B would form a 3-cluster in $\mathcal{F}$. Next, we will prove

$$\sum_{i=0}^{\ell} |\mathcal{F}_i| \leqslant \sum_{i=1}^{\ell} \binom{n-2k}{k-i}\binom{k-1}{i-1}. \tag{1}$$

for all $\ell \in [k]$. Suppose that (1) is true, then by letting $\ell = k$ we obtain

$$|\mathcal{F}| = \sum_{i=0}^{k} |\mathcal{F}_i| \leqslant \sum_{i=1}^{k-1} \binom{n-2k}{k-i}\binom{k-1}{i-1} + 2 = \binom{n-k-1}{k-1} + 1,$$

and this will complete the proof of Theorem 10. One could compare (1) with a similar inequality in [11], which is

$$|\mathcal{F}| \leqslant \sum_{\ell=1}^{k} \binom{n-tk}{k-\ell}\binom{tk-1}{\ell-1} = \binom{n-1}{k-1}, \tag{2}$$

where t is the maximum number of pairwise disjoint sets in $\mathcal{F}$. For the case $t=2$, the summand in (2) is $\binom{n-2k}{k-\ell}\binom{2k-1}{\ell-1}$, but the summand in (1) is $\binom{n-2k}{k-\ell}\binom{k-1}{\ell-1}$, which is smaller when $\ell \geqslant 2$.

Claim 14. *Let $F \in \mathcal{F}_1$. Then the set $F \cap U$ is a unique $(k-1)$-subset of F in $\mathcal{F}$.*

Proof of Claim 14. Without loss of generality, we may assume that $F = \{a_1, f_1, \ldots, f_{k-1}\}$, where $f_1, \ldots, f_{k-1}$ are contained in U. Suppose that there is another edge $F' \in \mathcal{F}$ containing $\{f_1, \ldots, f_{k-1}\}$. Then the three sets A, F, F' form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, $F \cap U = \{f_1, \ldots, f_{k-1}\}$ is a unique $(k-1)$-subset of F in $\mathcal{F}$. □

Now we prove (1) for $\ell = 1$. Let us consider the family $\mathcal{F}_0 \cup \mathcal{F}_1$. Define

$$\mathcal{M} = \left\{ G \in \binom{U}{k-1} : \exists F \in \mathcal{F}_0 \cup \mathcal{F}_1 \text{ such that } G \subset F \right\}.$$

By assumption, every set $F \in \mathcal{F}_0 \cup \mathcal{F}_1$ has a unique $(k-1)$-subset $G(F)$, and by Claim 14, we may assume that $G(F) \subset U$. Let $\mathcal{G} = \{G(F) : F \in \mathcal{F}_1\}$. For every set $F_1 \in \mathcal{F}_1$, the set $G(F_1)$ cannot be contained in $\partial \mathcal{F}_0$, since otherwise one could easily find a 3-cluster. Therefore, $\mathcal{G}$ and $\partial \mathcal{F}_0$ are disjoint. Since $|\mathcal{G}| = |\mathcal{F}_1|$, by Lemma 12, we have

$$\frac{|U|}{|U|-k+1}|\mathcal{F}_0| + |\mathcal{F}_1| \leqslant |\mathcal{M}| \leqslant \binom{n-2k}{k-1},$$

and hence $|\mathcal{F}_0| + |\mathcal{F}_1| \leqslant \binom{n-2k}{k-1}$.

To prove (1) for $\ell \geqslant 2$, we need to give an upper bound for $|\mathcal{F}_i|$ for every $2 \leqslant i \leqslant k-1$. Since $|\mathcal{F}_i| = \sum_{C \in \binom{U}{k-i}} |\mathcal{F}(C)|$, it suffices to give an upper bound for $|\mathcal{F}(C)|$ for every $C \in \binom{U}{k-i}$. Unfortunately, the inequality $|\mathcal{F}(C)| \leqslant \binom{k-1}{i-1}$ is not true in general. So, in our proof, we will build a relationship between $\mathcal{F}_i$ and $\bigcup_{j<i} \mathcal{F}_j$ and then use this relation to prove (1).

The basic idea in our proof is showing that if $|\mathcal{F}(C)|$ is bigger than its expected value $\binom{k-1}{k-|C|-1}$, then there must be many sets D containing C such that the size of $\mathcal{F}(D)$ is smaller than its expected value $\binom{k-1}{k-|D|-1}$.

Let $C \subset U$ be a set of size at most $k-2$. We say C is *perfect* if the family $\mathcal{F}(C)$ is a full star on either A or B. Let $D \subset U$ be a set of size $k-1$. We say D is *perfect* if there exists a set F in $\mathcal{F}$ that contains D.

For every $i \in [k-1]$ let $\mathcal{P}_i$ be the collection of all perfect sets in $\binom{U}{k-i}$, and let $\mathcal{N}_i$ be the collection of non-perfect sets in $\binom{U}{k-i}$. Let $p_i = |\mathcal{P}_i|$ and $n_i = |\mathcal{N}_i|$ for $i \in [k-1]$ and notice that $p_i + n_i = \binom{|U|}{k-i}$.

For every $i \in \{2, \ldots, k-1\}$ let $\mathcal{P}'_i$ denote the collection of all sets $C \in \binom{U}{k-i}$ such that C is contained in a perfect set in $\binom{U}{k-i+1}$, and let $\mathcal{N}'_i$ denote the collection all of sets $D \in \binom{U}{k-i}$ such that D is not contained in any perfect set in $\binom{U}{k-i+1}$. Let $p'_i = |\mathcal{P}'_i|$ and $n'_i = |\mathcal{N}'_i|$ for $i \in \{2, \ldots, k-1\}$. Let $\mathcal{G}_i = \mathcal{N}_i \cap \mathcal{P}'_i$ and $\mathcal{B}_i = \mathcal{N}_i \cap \mathcal{N}'_i$, and let $g_i = |\mathcal{G}_i|$

and $b_i = |\mathcal{B}_i|$ for $i \in \{2, \dots, k-1\}$. Let $\mathcal{G}_1 = \mathcal{N}_1$, and let $g_1 = n_1$, $b_1 = 0$. Note that by definition, $b_i + g_i = n_i$ and $n'_i \geqslant b_i$ for $i \in [k-1]$.

By the definition of perfect sets, $|\mathcal{F}(C)| = \binom{k-1}{i-1}$ for all $C \in \mathcal{P}_i$. Later we will show that $|\mathcal{F}(C)| < \binom{k-1}{i-1}$ for all $C \in \mathcal{G}_i$. For every $C \in \mathcal{B}_i$ it could be true that $|\mathcal{F}(C)| > \binom{k-1}{i-1}$. However, for every $C \in \mathcal{B}_i$ there are either many sets in $\mathcal{G}_{i-1}$ containing C, which means that there are many sets $D \in \binom{U}{k-i+1}$ with $|\mathcal{F}(D)|$ smaller than its expected value, or there are many sets in $\mathcal{B}_{i-1}$, in which case we turn to consider sets in $\binom{U}{k-i+2}$ and repeat this argument until we end up with many sets P in $\binom{U}{k-1}$ with $|\mathcal{F}(P)|$ smaller than its expected value.

The next claim gives a relation between n_i and b_{i+1}.

Claim 15. *For every $i \in [k-2]$ we have*

$$n_i \geqslant \frac{n-3k}{k} b_{i+1}.$$

Proof of Claim 15. Let $C \in \mathcal{N}'_{i+1}$, and let $u \in U - C$. By definition $C \cup \{u\}$ is a non-perfect set in $\binom{U}{k-i}$. Therefore, we have $(k-i)n_i \geqslant n'_{i+1}(n-3k+i+1) \geqslant b_{i+1}(n-3k)$. It follows that $n_i \geqslant (n-3k)b_{i+1}/k$. □

Claim 16. *The following statement holds for all $\ell \geqslant (k+1)/2$. Suppose that $C \subset U$ is a perfect set of size ℓ, and $\mathcal{F}(C)$ is a full star on A (or on B) with core v. Then for every $(\ell-1)$-subset C' of C the family $\mathcal{F}(C')$ is a star on A (or on B) with core v.*

Proof of Claim 16. Let $C \subset U$ such that $\mathcal{F}(C)$ is a full star on A with core $v \in A$. Without loss of generality we may assume that $v = a_1$. Let $E' \in \mathcal{F}(C')$. If $E' \subset B$, then choose a set E from $\mathcal{F}(C)$, and the three sets $E \cup C, E' \cup C', B$ form a 3-cluster in $\mathcal{F}$, a contradiction. If $E' \cap A \neq \emptyset$ and $E' \cap B \neq \emptyset$, then let $x = |E' \cap A|$ and $y = |E' \cap B|$. Since $x + y = k - \ell + 1$, we have $x \leqslant k - \ell$ and $y \leqslant k - \ell$. If $a_1 \notin E' \cap A$, then by the assumption that $\ell \geqslant (k+1)/2$ and $\mathcal{F}(C)$ is a full star, there exists a set $E \in \mathcal{F}(C)$ such that $(E' \cap A) \cap E = \emptyset$. So the three sets $E' \cup C', E \cup C, A$ form a 3-cluster in $\mathcal{F}$, a contradiction. If $a_1 \in E' \cap A$, then by assumption there exists a set $E \in \mathcal{F}(C)$ such that $E' \cap A \subset E$. However, the three sets $E \cup C, E' \cup C', B$ form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, every set in $\mathcal{F}(C')$ is completely contained in A.

Next, we show that every set $E' \in \mathcal{F}(C')$ contains a_1. Suppose there exists a set $E' \in \mathcal{F}(C')$ such that $a_1 \notin E'$. By assumption we have $k - \ell + 1 + k - \ell \leqslant k$, so there exists a set $E \in \mathcal{F}(C)$ such that $E \cap E' = \emptyset$. However, the three sets $E' \cup C', E \cup C, A$ form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, the family $\mathcal{F}(C')$ is a star on A with core a_1. □

For every $i \in [k-1]$ let $w_i = \binom{k-1}{i-1}\binom{n-2k}{k-i}$ and $k_i = \binom{2k}{i} - \binom{k-1}{i-1} + 1$. Our next claim gives an upper bound for $|\mathcal{F}_i|$ for $2 \leqslant i \leqslant (k+1)/2$.

Claim 17. *For every i satisfying $2 \leqslant i \leqslant (k+1)/2$ we have*

$$|\mathcal{F}_i| \leqslant w_i + k_i b_i - n_i.$$

Proof of Claim 17. Let us give an upper bound for $|\mathcal{F}(C)|$ for every $C \in \binom{U}{k-i}$. First notice that by definition $|\mathcal{F}(C)| = \binom{k-1}{i-1}$ for all $C \in \mathcal{P}_i$. By Claim 16, $|\mathcal{F}(C)| \leqslant \binom{k-1}{i-1} - 1$ for all $C \in \mathcal{G}_i$. On the other hand, it is trivially true that $|\mathcal{F}(C)| \leqslant \binom{2k}{i}$ for all $C \in \mathcal{B}_i$. Therefore,

$$\begin{aligned}|\mathcal{F}_i| &= \sum_{C\in\mathcal{P}_i} |\mathcal{F}(C)| + \sum_{C\in\mathcal{G}_i} |\mathcal{F}(C)| + \sum_{C\in\mathcal{B}_i} |\mathcal{F}(C)| \\ &\leqslant \binom{k-1}{i-1} p_i + \left(\binom{k-1}{i-1} - 1\right) g_i + \binom{2k}{i} b_i \\ &= \binom{k-1}{i-1}\binom{n-2k}{k-i} + \left(\binom{2k}{i} - \binom{k-1}{i-1} + 1\right) b_i - n_i = w_i + k_i b_i - n_i.\end{aligned}$$

Here we used that fact that $b_i + g_i = n_i$ and $n_i + p_i = \binom{n-2k}{k-i}$. □

Recall that Claim 15 says that $n_i \geqslant (n-3k)b_{i+1}/k$. Since $n \geqslant 3k\binom{2k}{k}$ and $k_{i+1} < \binom{2k}{k}$, we have $n_i/2 \geqslant k_{i+1}b_{i+1}$. Combining this inequality with Claim 17 we obtain the following claim.

Claim 18. *For every ℓ satisfying $1 \leqslant \ell \leqslant (k+1)/2$ we have*

$$\sum_{i=0}^{\ell} |\mathcal{F}_i| \leqslant \sum_{i=1}^{\ell} w_i - \sum_{i=1}^{\ell} \frac{n_i}{2}.$$

Proof of Claim 18. The case $\ell = 1$ follows from the inequality that

$$|\mathcal{F}_0| + |\mathcal{F}_1| \leqslant |\mathcal{M}| = \binom{n-2k}{k-1} - n_1.$$

For $\ell \geqslant 2$ by Claim 17 we obtain

$$\sum_{i=0}^{\ell} |\mathcal{F}_i| \leqslant \sum_{i=1}^{\ell} (w_i + k_i b_i - n_i) = \sum_{i=1}^{\ell} w_i - \sum_{i=1}^{\ell-1} (n_i - k_{i+1} b_{i+1}) - n_\ell \leqslant \sum_{i=1}^{\ell} w_i - \sum_{i=1}^{\ell} \frac{n_i}{2}. \quad \square$$

The next step is to extend Claim 18 to all $\ell > (k+1)/2$.

Claim 19. *Let $C \subset U$ be a set of size $\ell \geqslant 2$. Suppose that $\mathcal{F}(C)$ is a full-star on A (or on B) with core v and there exists a perfect set $P \in \binom{U}{k-1}$ containing C. Then, for every $(\ell-1)$-subset $C' \subset C$ the family $\mathcal{F}(C')$ is a star on A (or on B) with core v.*

Proof of Claim 19. Let $C \subset U$ be a set of size ℓ such that $\mathcal{F}(C)$ is a full-star on A with core v. Without loss of generality we may assume that $v = a_1$. Let $P \in \binom{U}{k-1}$ be a perfect set containing C. By the definition of perfect set there exists a set $F \in \mathcal{F}$ containing P. Suppose that $F = P \cup \{u\}$, and we want to show that $u = a_1$. Suppose that $u \notin A$. Then for every $E \in \mathcal{F}(C)$ the three sets $A, F, E \cup C$ form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, $u \in A$.

Now suppose for the contrary that $u \neq a_1$. Then by assumption there exists a set $E \in \mathcal{F}(C)$ not containing u and, hence, the three sets $A, F, E \cup C$ form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, $u = a_1$.

Let $C' \subset C$ be a set of size $\ell - 1$ and $E' \in \mathcal{F}(C')$. If $E' \subset B$, then for every $E \in \mathcal{F}(C)$ the three sets $E \cup C, E' \cup C', B$ form a 3-cluster in $\mathcal{F}$, a contradiction. If $E' \cap A \neq \emptyset$ and $E' \cap B \neq \emptyset$, then let $x = |E' \cap A|$ and $y = |E' \cap B|$. Since $x + y = k - \ell + 1$, we have $x \leqslant k - \ell$ and $y \leqslant k - \ell$. If $x \leqslant k - \ell - 1$, then by assumption there exists a set $E \in \mathcal{F}(C)$ containing $E' \cap A$. However, the three sets $E \cup C, E' \cup C', B$ form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, we may assume that $x = k - \ell$. If $a_1 \in E' \cap A$, then there exists a set $E \in \mathcal{F}(C)$ such that $E' \cap A = E$. However, the three sets $E \cup C, E' \cup C', B$ form a 3-cluster in $\mathcal{F}$, a contradiction. If $a_1 \notin E' \cap A$, then the three sets $A, F, E' \cup C'$ form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, every set in $\mathcal{F}(C')$ is completely contained in A.

Suppose that there is a set $E' \in \mathcal{F}(C')$ not containing a_1, then the three sets $A, F, E' \cup C'$ would form a 3-cluster in $\mathcal{F}$, a contradiction. Therefore, every set in $\mathcal{F}(C')$ contains a_1, and this complete the proof of Claim 19. □

Let $c = \lfloor (k+1)/2 \rfloor$ and let $m = \lfloor k/2 \rfloor$, and notice that $m + c = k$. The next claim shows that (1) holds for $\ell = c + 1$.

Claim 20. *We have*

$$\sum_{i=0}^{c+1} |\mathcal{F}_i| \leqslant \sum_{i=1}^{c+1} w_i - \sum_{i=1}^{c+1} \frac{n_i}{4}.$$

Proof of Claim 20. Similar to the proof of Claim 17, for every $C \in \mathcal{P}_{c+1}$ we have $|\mathcal{F}(C)| = \binom{k-1}{c}$, and for every $C \in \mathcal{B}_{c+1}$ we have $|\mathcal{F}(C)| \leqslant \binom{2k}{c+1}$.

For every perfect set $D \in \binom{U}{m}$ we say that D is a *good container* if D itself is contained in a perfect $(k-1)$-set, otherwise we say that D is a *bad container*. Let $\mathcal{S}$ be the collection of all sets in $\mathcal{G}_{c+1}$ that are contained in a good container. Let $\mathcal{T}$ be the collection of all sets in $\mathcal{G}_{c+1}$ that are not contained in any good container. Let $s = |\mathcal{S}|$ and $t = |\mathcal{T}|$. Since every bad container in $\binom{U}{m}$ has m subsets of size $m - 1$, the number of bad containers in $\binom{U}{m}$ is at least t/m.

Let $D \in \binom{U}{m}$ be a bad container. Then for every $E \in \binom{U-D}{k-m-1}$ the set $D \cup E$ is non-perfect in $\binom{U}{k-1}$. Therefore, $n_1 \geqslant \binom{n-2k-m}{c-1} t / \left(m\binom{k-1}{m}\right)$. By definition, every set $C \in \mathcal{G}_{c+1}$ is contained in a perfect set $D \in \binom{U}{m}$. If $C \in \mathcal{S}$, then by Claim 19, $|\mathcal{F}(C)| \leqslant \binom{k-1}{c} - 1$. If $C \in \mathcal{T}$, then it is trivially true that $|\mathcal{F}(C)| \leqslant \binom{k}{c+1}$. Therefore,

$$\begin{aligned}
|\mathcal{F}_{c+1}| &= \sum_{C \in \mathcal{P}_{c+1}} |\mathcal{F}(C)| + \sum_{C \in \mathcal{B}_{c+1}} |\mathcal{F}(C)| + \sum_{C \in \mathcal{S}} |\mathcal{F}(C)| + \sum_{C \in \mathcal{T}} |\mathcal{F}(C)| \\
&\leqslant \binom{k-1}{c} p_{c+1} + \binom{2k}{c+1} b_{c+1} + \left(\binom{k-1}{c} - 1\right) s + \binom{2k}{c+1} t \\
&= w_{c+1} + k_{c+1} b_{c+1} + k_{c+1} t - n_{c+1}.
\end{aligned}$$

Here we used the fact that $s + t = g_{c+1}$, $g_{c+1} + b_{c+1} = n_{c+1}$ and $n_{c+1} + p_{c+1} = \binom{n-2k}{k-c-1}$. Combining the inequality above with Claim 17, we obtain

$$\sum_{i=0}^{c+1} |\mathcal{F}_i| \leqslant \sum_{i=1}^{c+1} (w_i + k_i b_i - n_i) + k_{c+1} t.$$

Since $n_1/4 \geqslant k_{c+1}t$ and $n_i/2 \geqslant k_{i+1}b_{i+1}$,

$$\sum_{i=0}^{c+1} |\mathcal{F}_i| \leqslant \sum_{i=1}^{c+1} w_i - \sum_{i=1}^{c+1} \frac{n_i}{4}. \qquad \square$$

Claim 21. *Every set $C \subset U$ of size at most $k - c$ is contained in a perfect $(k-1)$-set.*

Proof of Claim 21. Let $C \subset U$ be a set of size $\ell \leqslant k - c$. Suppose that C is not contained in any perfect $(k-1)$-set. Then for every $S \in \binom{U-C}{k-\ell-1}$ the set $C \cup S$ is non-perfect and of size $k-1$. Therefore, we have $n_1 \geqslant \binom{n-2k-\ell}{k-\ell-1}/\binom{k-1}{\ell} \geqslant \binom{n-2k-\ell}{c-1}/\binom{k-1}{\ell}$. On the other hand, we have $\sum_{i=c+2}^{k-1} |\mathcal{F}_i| \leqslant \sum_{i=c+2}^{k-1} \binom{2k}{i}\binom{n-2k}{k-i}$. Since $n \geqslant 3k\binom{2k}{k}$, $n_1/4 > \sum_{i=c+2}^{k-1} |\mathcal{F}_i|$. Therefore, by Claim 20,

$$\sum_{i=0}^{k-1} |\mathcal{F}_i| = \sum_{i=1}^{c+1} |\mathcal{F}_i| + \sum_{i=c+2}^{k-1} |\mathcal{F}_i| \leqslant \sum_{i=1}^{c+1} w_i - \sum_{i=1}^{c+1} \frac{n_i}{4} + \sum_{i=c+2}^{k-1} \binom{2k}{i}\binom{n-2k}{k-c-2} < \sum_{i=1}^{k-1} w_i,$$

and we are done. So we may assume that C is contained in a perfect $(k-1)$-set. $\square$

Claim 22. *The inequality $|\mathcal{F}_i| \leqslant w_i + t_i b_i - n_i$ holds for all $i \geqslant c + 1$.*

Proof. By Claim 21, every set $C \subset U$ of size at most $k - c$ is contained in a perfect $(k-1)$-set. Therefore, by Claim 19,

$$\begin{aligned} |\mathcal{F}_i| &= \sum_{C \in \mathcal{P}_i} |\mathcal{F}(C)| + \sum_{C \in \mathcal{G}_i} |\mathcal{F}(C)| + \sum_{C \in \mathcal{B}_i} |\mathcal{F}(C)| \\ &\leqslant \binom{k-1}{i-1} p_i + \left(\binom{k-1}{i-1} - 1\right) g_i + \binom{2k}{i} b_i. \end{aligned} \qquad \square$$

By Claims 15, 17, and 22,

$$\begin{aligned} \sum_{i=0}^{k-1} |\mathcal{F}_i| &\leqslant \sum_{i=1}^{k-1} (w_i + t_i b_i - n_i) = \sum_{i=1}^{k-1} w_i - \sum_{i=1}^{k-2} (n_i - t_{i+1} b_{i+1}) - n_{k-1} \\ &\leqslant \sum_{i=1}^{k-1} w_i - \sum_{i=1}^{k-1} \frac{n_i}{2}, \end{aligned}$$

which proves (1), and equality holds if and only if C is perfect for every $C \in \binom{U}{i}$ and for every $i \in [k-1]$, which implies that $\mathcal{F}$ is the disjoint union of a k-set and a full star. $\square$

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References

[1] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hung.*, 16:447–452, 1965.

[2] W. Y. C. Chen, J. Liu, and L. X. W. Wang. Families of sets with intersecting clusters. *SIAM J. Discrete Math.*, 23(3):1249–1260, 2009.

[3] G. Currier. On the d-cluster generalization of Erdős-Ko-Rado. *arXiv preprint* arXiv:1812.11153, 2018.

[4] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Q. J. Math., Oxf. II. Ser.*, 12:313–320, 1961.

[5] P. Frankl. On Sperner families satisfying an additional condition. *J. Comb. Theory, Ser. A*, 20:1–11, 1976.

[6] P. Frankl. A structural result for 3-graphs. *Discrete Math.*, 340(5):1039–1041, 2017.

[7] P. Frankl and Z. Füredi. A new generalization of the Erdős-Ko-Rado theorem. *Combinatorica*, 3:341–349, 1983.

[8] Z. Füredi and L. Özkahya. Unavoidable subhypergraphs: **a**-clusters. *J. Comb. Theory, Ser. A*, 118(8):2246–2256, 2011.

[9] X. Liu. d-cluster-free sets with a given matching number. *European J. Combin.*, 82:103000, 19, 2019.

[10] A. Mammoliti and T. Britz. On Mubayi's conjecture and conditionally intersecting sets. *SIAM J. Discrete Math.*, 32(3):2361–2380, 2018.

[11] D. Mubayi. Erdős–Ko–Rado for three sets. *J. Comb. Theory, Ser. A*, 113(3):547–550, 2006.

[12] D. Mubayi. An intersection theorem for four sets. *Adv. Math.*, 215(2):601–615, 2007.

[13] D. Mubayi and R. Ramadurai. Set systems with union and intersection constraints. *J. Comb. Theory, Ser. B*, 99(3):639–642, 2009.