On density-critical matroids

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Abstract

For a matroid M having m rank-one flats, the density $d(M)$ is $\frac{m}{r(M)}$ unless $m = 0$, in which case $d(M) = 0$. A matroid is density-critical if all of its proper minors of non-zero rank have lower density. By a 1965 theorem of Edmonds, a matroid that is minor-minimal among simple matroids that cannot be covered by k independent sets is density-critical. It is straightforward to show that $U_{1,k+1}$ is the only minor-minimal loopless matroid with no covering by k independent sets. We prove that there are exactly ten minor-minimal simple obstructions to a matroid being able to be covered by two independent sets. These ten matroids are precisely the density-critical matroids M such that $d(M) > 2$ but $d(N) \leqslant 2$ for all proper minors N of M. All density-critical matroids of density less than 2 are series-parallel networks. For $k \geqslant 2$, although finding all density-critical matroids of density at most k does not seem straightforward, we do solve this problem for $k = \frac{9}{4}$.

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1 Introduction

Our notation and terminology follow Oxley [7]. For a positive integer k, let $\mathcal{M}_k$ be the class of matroids M for which $E(M)$ is the union of k independent sets. We say such a matroid can be *covered* by k independent sets. Edmonds [3] gave the following characterization of the members of $\mathcal{M}_k$.

Theorem 1. *A matroid M has k independent sets whose union is $E(M)$ if and only if, for every subset A of $E(M)$,*

$$k\,r(A) \geqslant |A|.$$

Clearly, $\mathcal{M}_k$ is closed under deletion. However, $\mathcal{M}_k$ is not closed under contraction. For example, the 6-element rank-3 uniform matroid $U_{3,6}$ can be covered by two independent sets, yet contracting a point of this matroid gives $U_{2,5}$, which cannot. For all k, the loop is the unique minor-minimal matroid not in $\mathcal{M}_k$. On that account, we limit the types of obstructions we consider. We first examine the minor-minimal loopless matroids that are not in $\mathcal{M}_k$. We find the following result.

Proposition 2. *The unique minor-minimal loopless matroid that cannot be covered by k independent sets is $U_{1,k+1}$.*

Restricting attention to minor-minimal simple matroids not in $\mathcal{M}_k$, we find much more structure. We have the following collection of ten matroids for the case when k is two. In this result, $P(M_1, M_2)$ denotes the parallel connection of matroids M_1 and M_2, this matroid being unique when both M_1 and M_2 have transitive automorphism groups. Geometric representations of the nine of these ten matroids of rank at most four are shown in Figure 1. A diagram representing the tenth matroid, $P(M(K_4), M(K_4))$ is also given where we note that this matroid has rank five.

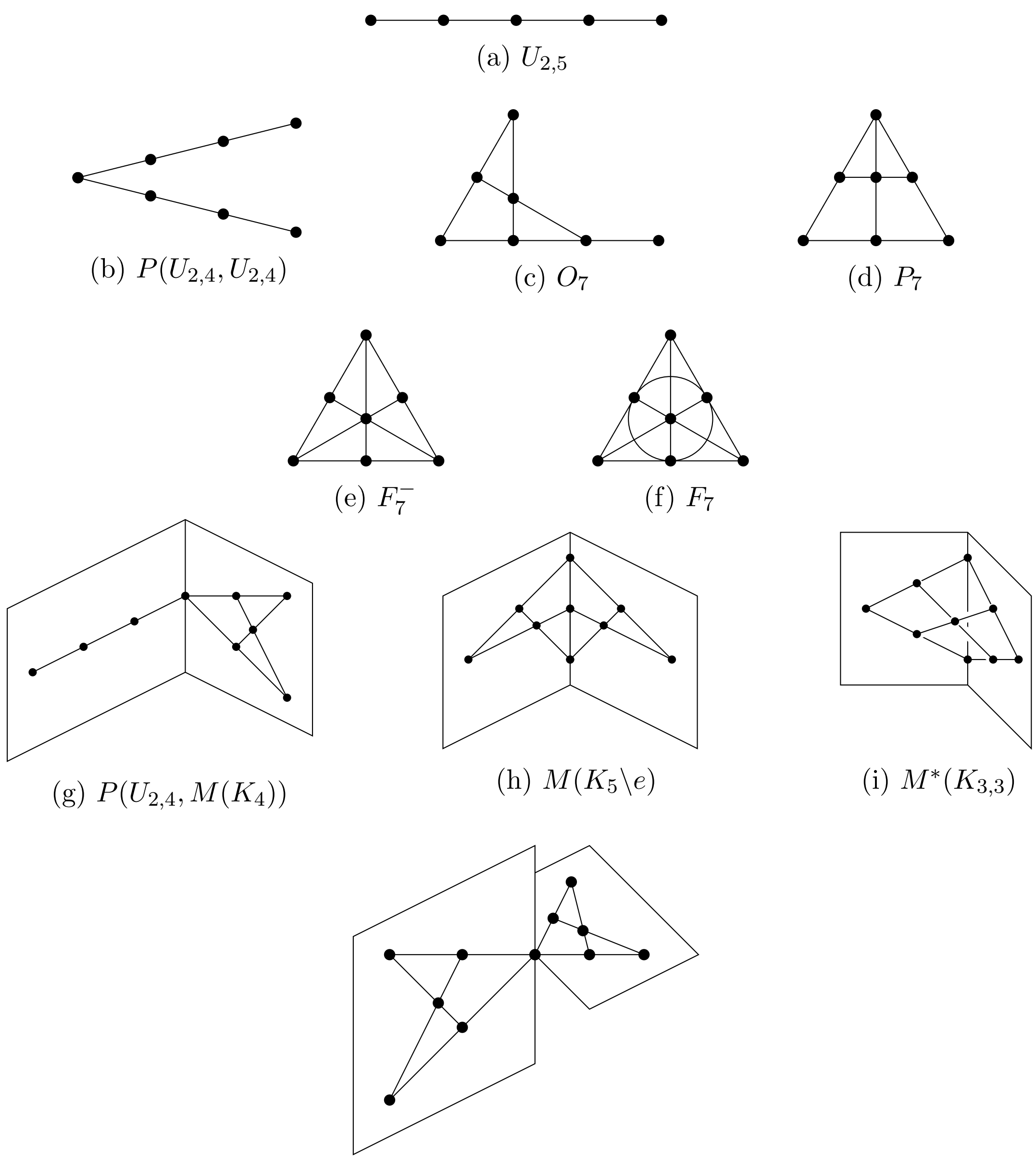
Theorem 3. *The minor-minimal simple matroids that cannot be covered by two independent sets are $U_{2,5}$, $P(U_{2,4}, U_{2,4})$, O_7, P_7, F_7^-, F_7, $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, $M^*(K_{3,3})$, and $P(M(K_4), M(K_4))$.*

The following consequence of Theorem 1 will be helpful.

Lemma 4. *Let M be a minor-minimal matroid that cannot be covered by k independent sets. Then*

$$k\,r(M) = |E(M)| - 1.$$

Moreover, M has no coloops.



(a) $U_{2,5}$

(b) $P(U_{2,4}, U_{2,4})$

(c) O_7

(d) P_7

(e) F_7^-

(f) F_7

(g) $P(U_{2,4}, M(K_4))$

(h) $M(K_5 \backslash e)$

(i) $M^*(K_{3,3})$

(j) $P(M(K_4), M(K_4))$

Figure 1: The minor-minimal simple matroids not in $\mathcal{M}_2$.

For a matroid M, we write $\varepsilon(M)$ for $|E(\mathrm{si}(M))|$, the number of rank-one flats of M. The *density* $d(M)$ of M is $\frac{\varepsilon(M)}{r(M)}$ unless $r(M) = 0$. In the exceptional case, $\varepsilon(M) = 0$ and we define $d(M) = 0$. We say that M is *density-critical* when $d(N) < d(M)$ for all proper minors N of M. Note that all density-critical matroids are simple. By Lemma 4 and Theorem 1, M is a minor-minimal simple matroid that cannot be covered by k independent sets if and only if $d(M) > k$ but $d(N) \leqslant k$ for all proper minors N of M. Such matroids are strictly k-density-critical where, for $t \geqslant 0$, we say a matroid is *strictly t-density-critical* when its density is strictly greater than t while all its proper minors have density at most t. Thus Theorem 3 explicitly determines all ten strictly 2-density-critical matroids.

We propose the following.

Conjecture 5. For all positive integers k, there are finitely many minor-minimal simple matroids that cannot be covered by k independent sets.

More generally, we make the following conjectures. For $t > 0$, we say a matroid is *t-density-critical* when its density is at least t while all of its proper minors have density strictly less than t.

Conjecture 6. For all $t \geqslant 0$, there are finitely many strictly t-density-critical matroids.

Conjecture 7. For all $t > 0$, there are finitely many t-density-critical matroids.

We also propose the following weakening of the last conjecture.

Conjecture 8. For all $t \geqslant 0$, there are finitely many density-critical matroids with density exactly t.

We note that these conjectures hold over any class of matroids that is well-quasi-ordered with respect to minors. In particular, by a result announced by Geelen, Gerards, and Whittle (see, for example, [4]), these conjectures hold within the class of matroids representable over a fixed finite field.

Because the two excluded minors for series-parallel networks, $U_{2,4}$ and $M(K_4)$, have density exactly two, for $k < 2$, all density-critical matroids of density at most k are series-parallel networks. For $k > 2$, finding all density-critical matroids of density at most k does not seem straightforward. However, we were able to solve this problem when $k = \frac{9}{4}$. For all $n \geqslant 2$, we denote by P_n any matroid that can be constructed from n copies of $M(K_3)$ via a sequence of $n-1$ parallel connections. In particular, $P_2 \cong M(K_4\backslash e)$. There are two choices for P_3 depending on which element of $M(K_4\backslash e)$ is used as the basepoint of the parallel connection with the third copy of $M(K_3)$. We denote by M_{18} the 18-element matroid that is obtained by attaching, via parallel connection, a copy of $M(K_4)$ at each element of an $M(K_3)$.

Theorem 9. *The following is a list of all pairs (M, d) where M is a density-critical matroid of density d and $d \leqslant \frac{9}{4}$: $(U_{1,1}, 1)$, $(U_{2,3}, \frac{3}{2})$, $\left(M\left(P_n\right), \frac{2n+1}{n+1}\right)$ for all $n \geqslant 2$, $(U_{2,4}, 2)$, $(M(K_4), 2)$, $(P(M(K_4), M(K_4)), \frac{11}{5})$, $(P(U_{2,4}, M(K_4)), \frac{9}{4})$, $(M(K_5\backslash e), \frac{9}{4})$, $(M^*(K_{3,3}), \frac{9}{4})$, $(M_{18}, \frac{9}{4})$.*

2 Preliminaries

This section proves some preliminary results beginning with two that were stated in the introduction.

Proof of Proposition 2. Clearly, $U_{1,k+1}$ is a minor-minimal loopless matroid that cannot be covered by k independent sets. Conversely, suppose that M is a minor-minimal loopless matroid that cannot be covered by k independent sets. Certainly, M contains some element e. Let $P \cup \{e\}$ be the parallel class of M that contains e where $P = \{e_1, e_2, \dots, e_\ell\}$ and $e \notin P$. Now $M/e \setminus P$ is loopless, so, by minimality, $M/e \setminus P$ can be covered by k independent sets $\{A_1, A_2, \dots, A_k\}$. Note that each $A_i \cup \{e\}$ is independent in M, so if $|P| = \ell \leqslant k-1$, then $\{A_1 \cup \{e_1\}, A_2 \cup \{e_2\}, \dots, A_\ell \cup \{e_\ell\}, A_{\ell+1} \cup \{e\}, \dots, A_k \cup \{e\}\}$ is a set of k independent sets that covers M. Thus $|P| \geqslant k$, and so $M \cong U_{1,k+1}$. □

Since $U_{1,k+1}$ is a $(k+1)$-element cocircuit, the matroids having no $U_{1,k+1}$-minor are precisely the matroids for which every cocircuit has at most k elements.

Proof of Lemma 4. Take x in $E(M)$. Then $M \setminus x$ can be covered by k independent sets. Thus, by Theorem 1,

$$|E(M)| > kr(M) \geqslant kr(M \setminus x) \geqslant |E(M \setminus x)| = |E(M)| - 1.$$

We deduce that $kr(M) = |E(M)| - 1$ and $r(M) = r(M \setminus x)$ so M has no coloops. □

Lemma 10. *Let M be a density-critical matroid of rank at least two. For each subset S of $E(M)$,*

$$|E(M)| - \varepsilon(M/S) > d(M)r(S).$$

In particular, every element of M is in a triangle and is in at least two triangles when $d(M) \geqslant 2$.

Proof. Since M is density-critical and therefore simple,

$$\frac{\varepsilon(M/S)}{r(M/S)} < \frac{\varepsilon(M)}{r(M)} = \frac{|E(M)|}{r(M)}.$$

Hence $r(M)\varepsilon(M/S) < |E(M)|(r(M) - r(S))$, so

$$r(M)d(M)r(S) = |E(M)|r(S) < r(M)\left(|E(M)| - \varepsilon(M/S)\right).$$

Thus $d(M)r(S) < |E(M)| - \varepsilon(M/S)$. In particular, $d(M) < |E(M)| - \varepsilon(M/e)$ for all e in $E(M)$. Hence every such element e is in at least one triangle, and e is in at least two triangles when $d(M) \geqslant 2$. □

The next result will be useful in the proof of Theorem 3.

Lemma 11. *Let F be a $2k$-element set $\{b_1, a_1, b_2, a_2, \dots, b_k, a_k\}$ in a 3-connected matroid M. Suppose $\{b_1, b_2, \dots, b_k\}$ is independent and $\{b_i, a_i, b_{i+1}\}$ is a circuit for all i, where $b_{k+1} = b_1$. Then $M|F$ is a wheel of rank at least three or a whirl of rank at least two.*

Proof. Since M is 3-connected with at least four elements, it is simple. Now $M|F$ has $\{a_i, b_{i+1}, a_{i+1}\}$ as a triad, where $a_{k+1} = a_1$. By a result of Seymour [8] (see also [7, Lemma 8.8.5(ii)]), $M|F$ is a wheel or a whirl of rank k. □

3 The matroids that cannot be covered by two independent sets

In this section, we prove Theorem 3, first restating it for convenience.

Theorem 12. *The minor-minimal simple matroids that cannot be covered by two independent sets are* $U_{2,5}$*,* $P(U_{2,4}, U_{2,4})$*,* O_7*,* P_7*,* F_7^-*,* F_7*,* $P(U_{2,4}, M(K_4))$*,* $M(K_5 \setminus e)$*,* $M^*(K_{3,3})$*, and* $P(M(K_4), M(K_4))$*.*

Proof. It is straightforward to check that each of the matroids listed is a minor-minimal simple matroid that cannot be covered by two independent sets. Now let M be such a matroid. The next two assertions are immediate consequences of Lemmas 4, 10, and Theorem 1. However, we include proofs independent of Edmonds's result for completeness.

12.1. *Every element of* M *is contained in at least two triangles.*

Let e be an element of M and let $M' = \mathrm{si}(M/e)$. By minimality, M' has a partition into two independent sets A and B. Suppose e is not in a triangle. Then $E(M') = E(M) - \{e\}$ and we have $r_M(A \cup \{e\}) = r_{M'}(A) + 1 = |A| + 1$ and $r_M(B \cup \{e\}) = |B| + 1$, so M is covered by the independent sets $A \cup \{e\}$ and $B \cup \{e\}$, which is a contradiction.

Now suppose e is in exactly one triangle $\{e, c, d\}$ of M. We may assume that $M' = M/e \backslash c$ and that $d \in A$. Then $r_M(A \cup \{c\}) = r_M(A \cup \{c, e\}) = r_{M'}(A) + 1 = |A| + 1$ and $r_M(B \cup \{e\}) = r_{M'}(B) + 1 = |B| + 1$, so M is covered by the independent sets $A \cup \{c\}$ and $B \cup \{e\}$. This contradiction implies that 12.1 holds.

12.2. $|E(M)| \leqslant 2r(M) + 1$ *and* $|A| \leqslant 2r(A)$ *for every proper subset* A *of* $E(M)$*.*

Suppose A is a proper subset of $E(M)$. By the minimality of M, we can cover $M|A$ by two independent sets, and so $|A| \leqslant 2r(A)$. It follows easily that $|E(M)| \leqslant 2r(M) + 1$. Thus 12.2 holds.

We construct a simple auxiliary graph G from M, the vertices of which are the elements of M; two such vertices are adjacent exactly when they share a triangle in M. Next, we show the following.

12.3. *Let* Z *be the vertex set of a component of* G*. Then* $M|Z$ *has a wheel or a whirl as a restriction.*

We may assume that $M|Z$ has no line with four or more points otherwise M has a rank-2 whirl as a restriction. For b_1 in Z, by 12.1, we can construct a maximal sequence $b_1, a_1, b_2, a_2, \ldots, b_n$ of distinct elements such that $\{b_1, b_2, \ldots, b_n\}$ is independent and $\{b_i, a_i, b_{i+1}\}$ is a triangle for all i in $\{1, 2, \ldots, n-1\}$. Then $n \geqslant 3$.

Now M has triangles $\{b_n, a_n, b_{n+1}\}$ and $\{b_0, a_0, b_1\}$ that differ from $\{b_{n-1}, a_{n-1}, b_n\}$ and $\{b_1, a_1, b_2\}$, respectively. Let $A' = \{b_1, a_1, b_2, a_2, \ldots, b_{n-1}, a_{n-1}, b_n\}$. Assume that both $\{a_n, b_{n+1}\}$ and $\{a_0, b_0\}$ avoid A'. Then $|A' \cup \{a_n, b_{n+1}\}| = 2n+1 = 2r(A' \cup \{a_n, b_{n+1}\}) + 1$. Thus, by 12.2, $A' \cup \{a_n, b_{n+1}\} = E(M)$. By symmetry, $A' \cup \{a_0, b_0\} = E(M)$. Hence $\{a_n, b_{n+1}\} = \{b_0, a_0\}$, so $\{b_n, a_n, b_{n+1}, b_1\}$ is a 4-point line, a contradiction.

We may now assume that b_{n+1} is a member c_i of $\{b_i, a_i\}$ for some i with $1 \leqslant i \leqslant n-1$. Then $\{c_i, b_{i+1}, b_{i+2}, \ldots, b_n\}$ is an independent set in $M|Z$ such that every two consecutive elements in the given cyclic order are in a triangle. Thus, by Lemma 11, $M|Z$ has a wheel or whirl of rank $n - i + 1$ as a restriction. Hence 12.3 holds.

12.4. *For some component of G having vertex set Z, the matroid $M|Z$ is not a wheel or a whirl.*

Assume that this fails. Then, by 12.1, the only components of G are rank-2 whirls or rank-3 wheels. Assume there are s of the former and t of the latter. Then $|E(M)| = 4s + 6t = 2(2s + 3t)$. Clearly $r(M) \leqslant 2s + 3t$. By 12.2, equality must hold here. Hence each component of G corresponds to a wheel or whirl component of M. As each wheel and each whirl can be covered by two independent sets, so too can M, a contradiction. Thus 12.4 holds.

Now take a component of G having vertex set Z such that $M|Z$ is not a wheel or a whirl. By 12.3, consider a wheel or whirl restriction of $M|Z$ with basis $B = \{b_1, b_2, \ldots, b_n\}$ and ground set $W = \{b_1, a_1, b_2, a_2, \ldots, b_n, a_n\}$. Let $\{b_i, a_i, b_{i+1}\}$ be a triangle for all i where $b_{n+1} = b_1$. As $W \neq Z$, there is a point β_1 in W that is contained in a triangle $\{\beta_1, \alpha_1, \beta_2\}$ that is not a triangle of $M|W$. If $M|W$ is a rank-2 whirl or a rank-3 wheel, then, by symmetry, we may assume that $\beta_1 = a_1$. If, instead, $M|W$ is neither a rank-2 whirl nor a rank-3 wheel, then 12.1 guarantees that such a triangle $\{\beta_1, \alpha_1, \beta_2\}$ exists with $\beta_1 = a_1$. By repeatedly using 12.1, we can construct a sequence $\beta_1, \alpha_1, \ldots, \beta_{m+1}$ where $\{\beta_i, \alpha_i, \beta_{i+1}\}$ is a triangle for all i in $\{1, 2, \ldots, m\}$ and $B \cup \{\beta_2, \ldots, \beta_{m+1}\}$ is dependent but $B \cup \{\beta_2, \ldots, \beta_m\}$ is independent. By potentially interchanging α_m and β_{m+1}, we may assume that $\alpha_m \notin W$. Let $Q = \{\beta_1, \alpha_1, \ldots, \beta_{m+1}\}$. Then

$$r(W \cup Q) = r(W \cup (Q - \{\beta_{m+1}\})) = n + m - 1. \tag{1}$$

As $|W \cup (Q - \{\beta_{m+1}\})| = 2(n + m - 1) + 1 = 2r(W \cup (Q - \{\beta_{m+1}\})) + 1$, we deduce, by 12.2, that

$$W \cup (Q - \{\beta_{m+1}\}) = E(M). \tag{2}$$

Hence

$$\beta_{m+1} \in W \cup (Q - \{\beta_{m+1}\}). \tag{3}$$

Assume that the theorem fails. We now show that

12.5. *$M|Z$ has no wheel-restriction of rank exceeding three and no whirl-restriction of rank exceeding two.*

Assume that this fails. Then we may assume that $M|W$ is a wheel of rank at least four or a whirl of rank at least three. Now $r(W) = n$ and $r(Q) \leqslant m + 1$. By (1) and submodularity, $r(\mathrm{cl}(W) \cap \mathrm{cl}(Q)) \leqslant 2$. Assume W does not span M. Then, by (1) and (2), we see that $m > 1$ and the only possible elements of W that can lie in triangles with elements of $Q - W$ are β_1 and β_{m+1}. But a wheel of rank at least four and a whirl of rank at least three have at least three elements that are in unique triangles. Hence one of these elements will violate 12.1.

We now know that W spans M, so the unique element of $Q - W$ is α_1. Each of $a_1, a_2, \ldots, a_n$ must be in a triangle with α_1, the other element of which is in W. Assume both $\{a_1, \alpha_1, a_3\}$ and $\{a_1, \alpha_1, a_{n-1}\}$ are triangles. Then $n = 4$. Suppose $\{a_2, \alpha_1, a_4\}$ is also a triangle. Then, by Lemma 11, for each i in $\{2, 4\}$, deleting a_i from $M|(W \cup Q)$ gives a wheel or whirl of rank four. As $\{b_1, b_4, \alpha_1, a_2\}$ and $\{b_2, b_3, \alpha_1, a_4\}$ are circuits, both of these deletions are wheels. It follows that $M|(W \cup Q) \cong M^*(K_{3,3})$, so $M \cong M^*(K_{3,3})$, a contradiction. Thus, we may assume that $\{a_2, \alpha_1, a_4\}$ is not a triangle. Since $\alpha_1 \notin \mathrm{cl}(\{b_1, b_2, b_3\}) \cup \mathrm{cl}(\{b_2, b_3, b_4\})$, there is no triangle containing $\{a_2, \alpha_1\}$, a contradiction.

We may now assume that $\{a_1, \alpha_1, a_3\}$ is not a triangle. Then, by 12.1, W has distinct elements x and y such that $\{a_1, \alpha_1, x\}$ and $\{a_3, \alpha_1, y\}$ are triangles. Thus $\{a_1, a_3, x, y\}$ contains a circuit. Now $\{a_1, a_3\}$ is not in a triangle of $M|W$. Moreover, if $\{a_1, x, y\}$ is a triangle, then $\{x, y\} = \{b_1, b_2\}$. Using the triangles, $\{a_1, \alpha_1, x\}$ and $\{a_3, \alpha_1, y\}$, we deduce that $a_3 \in \mathrm{cl}(\{b_1, b_2\})$, a contradiction. It follows that $\{a_1, a_3, x, y\}$ is a circuit of M. Thus $M|W$ is either a rank-3 whirl or a rank-4 wheel.

Suppose $M|W$ is a rank-3 whirl. Then M is an extension of this matroid by α_1 in which every element is in at least two triangles. If $\{a_1, a_2, \alpha_1\}$ or $\{a_2, a_3, \alpha_1\}$ is a triangle, then one easily checks that $M \cong O_7$ or $M \cong P_7$, a contradiction. Hence we may assume that none of $\{a_1, a_2, \alpha_1\}$, $\{a_2, a_3, \alpha_1\}$, or $\{a_3, a_1, \alpha_1\}$ is a triangle. Then, to avoid having $U_{2,5}$ as a minor of M, we must have $\{a_1, b_3, \alpha_1\}$, $\{a_2, b_1, \alpha_1\}$, and $\{a_3, b_2, \alpha_1\}$ as triangles, that is, $M \cong F_7^-$, a contradiction.

We are left with the possibility that $M|W$ is a rank-4 wheel. Since it has $\{a_1, a_3, x, y\}$ as a circuit, it follows that $\{x, y\} = \{a_2, a_4\}$. Then M has either $\{a_1, a_2, \alpha_1\}$ and $\{a_3, a_4, \alpha_1\}$ as triangles or $\{a_1, a_4, \alpha_1\}$ and $\{a_2, a_3, \alpha_1\}$ as triangles. By symmetry, we may assume that we are in the second case. Then, by submodularity using the sets $\{b_1, b_2, a_1, a_4, b_4, \alpha_1\}$ and $\{b_2, b_3, a_2, a_3, b_4, \alpha_1\}$, we deduce that $r(\{b_2, b_4, \alpha_1\}) = 2$. It follows that $M \cong M(K_5 \backslash e)$, a contradiction. We conclude that 12.5 holds.

Now suppose that W spans Z. If $M|W$ is a rank-2 whirl, then $M|Z \cong U_{2,5}$, a contradiction. If $M|W$ is a rank-3 wheel, then one easily checks that $M|Z$ is isomorphic to one of O_7, F_7^-, or F_7, a contradiction.

We may now assume that W does not span Z. Then $m > 1$. By (3), $\beta_{m+1} \in W \cup (Q - \{\beta_{m+1}\})$. We will first suppose that $\beta_{m+1} = \beta_i$ for some i in $\{1, 2, \ldots, m\}$. Then $\{\beta_i, \beta_{i+1}, \ldots, \beta_m\}$ is an independent set and $\{\beta_j, \alpha_j, \beta_{j+1}\}$ is a triangle for all j in $\{i, i+1, \ldots, m\}$. By 12.5 and Lemma 11, for $R = \{\beta_i, \alpha_i, \beta_{i+1}, \alpha_{i+1}, \ldots, \beta_m, \alpha_m\}$, the matroid $M|R$ is a rank-3 wheel or a rank-2 whirl. Then the matroid obtained from $M|Z$ by contracting $\{\alpha_2, \alpha_3, \ldots, \alpha_{i-1}\}$ and simplifying is the parallel connection of $M|W$ and $M|R$, that is, $M|Z$ has as a minor one of $P(U_{2,4}, U_{2,4})$, $P(U_{2,4}, M(K_4))$, and $P(M(K_4), M(K_4))$, a contradiction.

Finally, suppose that $\beta_{m+1} \notin \{\beta_1, \beta_2, \ldots, \beta_m\}$. Then β_{m+1} is α_i for some $i \geqslant 1$, or $\beta_{m+1} \in W$. Consider the first case and take $\alpha_{m+1} = \beta_i$. Then, by 12.5 and Lemma 11, with $R = \{\beta_{i+1}, \alpha_{i+1}, \ldots, \beta_{m+1}, \alpha_{m+1}\}$, we have that $M|R$ is a rank-3 wheel or a rank-2 whirl. Contracting $\{\alpha_2, \alpha_3, \ldots, \alpha_{i-1}\}$ from $M|Z$ and simplifying, we obtain one of $P(U_{2,4}, U_{2,4})$, $P(U_{2,4}, M(K_4))$, and $P(M(K_4), M(K_4))$, a contradiction. In the second case, when $\beta_{m+1} \in W$, we recall that $\beta_1 = a_1$. Suppose that $\{\beta_1, \beta_{m+1}\}$ is not in a triangle of $M|W$. Then $M|W \cong M(K_4)$ and $\beta_{m+1} = b_3$. By assumption, $\{b_1, b_2, b_3\} \cup \{\beta_2, \ldots, \beta_m\}$ is independent. By Lemma 11, the triangles $\{b_1, b_2, a_1\}$, $\{a_1, \alpha_1, \beta_2\}, \ldots, \{\beta_m, \alpha_m, b_3\}$, $\{b_3, a_3, b_1\}$ imply that $M|Z$ has a wheel or whirl of rank at least four as a restriction, a contradiction. We deduce that $\{\beta_1, \beta_{m+1}\}$ is in a triangle of $M|W$. Then, by symmetry, we may assume that $\beta_{m+1} = b_1$. We let $\alpha_{m+1} = b_2$. Then, for $R = \{\beta_1, \alpha_1, \ldots, \beta_{m+1}, \alpha_{m+1}\}$, we have that $M|R$ is a rank-3 wheel or a rank-2 whirl. But $\alpha_1 \notin \text{cl}(W)$, so $M|R$ is a rank-3 wheel. If $M|W$ is a rank-2 whirl, then O_7 is a restriction of $M|Z$, a contradiction. If $M|W$ is a rank-3 wheel, then $M|(W \cup R)$ has rank four and consists of two copies of $M(K_4)$ sharing a triangle. This matroid is $M(K_5 \backslash e)$, a contradiction. $\square$

4 The density-critical matroids of small density

In this section, we prove Theorem 9. The following result [6] (see also [7, Lemma 4.3.10]) will be used repeatedly in this proof.

Lemma 13. *In a connected matroid M with at least two elements, let $\{e_1, e_2, \dots, e_m\}$ be a cocircuit of M such that M/e_i is disconnected for all i in $\{1, 2, \dots, m-1\}$. Then $\{e_1, e_2, \dots, e_{m-1}\}$ contains a 2-circuit of M.*

We shall make repeated use of the following consequence of this lemma.

Corollary 14. *Let M be a simple connected matroid and Z be a non-empty subset of $E(M)$. Then M has a simple connected minor N such that $N|Z = M|Z$ and $r(N) = r_M(Z)$.*

Proof. We may assume that Z is non-spanning, otherwise we can take N to be M. Let C^* be a cocircuit of M that is disjoint from $\text{cl}(Z)$. As M is simple, it follows by Lemma 13 that there is an element e of C^* such that M/e is connected. Since $e \notin \text{cl}(Z)$, we see that $(M/e)|Z = M|Z$. Clearly we can label $\text{si}(M/e)$ so that its ground set contains Z. If $r(M) - r(Z) = 1$, then we take $N = \text{si}(M/e)$. Otherwise we repeat the above process using $\text{si}(M/e)$ in place of M. After $r(M) - r(Z)$ applications of this process, we obtain the desired minor N. $\square$

The next result, which was proved by Dirac [2], follows easily by induction after recalling that a connected matroid with no minor isomorphic to $U_{2,4}$ or $M(K_4)$ is isomorphic to the cycle matroid of a series-parallel network.

Lemma 15. *Let M be a simple matroid having no minor isomorphic to $U_{2,4}$ or $M(K_4)$. Then*

$$|E(M)| \leqslant 2r(M) - 1.$$

We omit the elementary proof of the next result a consequence of which is that every density-critical matroid is connected.

Lemma 16. *Let M_1 and M_2 be matroids of rank at least one. Then*

$$d(M_1 \oplus M_2) \leqslant \max\{d(M_1), d(M_2)\}.$$

Moreover, equality holds here if and only if $d(M_1) = d(M_2)$.

The next result will be useful in identifying the density-critical matroids of density at most two.

Lemma 17. *Let M be a density-critical matroid with $d(M) \leqslant 2$. If (X_1, X_2) is a 2-separation of M, then there is an element p in $\mathrm{cl}(X_1) \cap \mathrm{cl}(X_2)$, and $M = P(M|(X_1 \cup \{p\}), M|(X_2 \cup \{p\}))$.*

Proof. As (X_1, X_2) is a 2-separation of M, for some element q not in $E(M)$, we can write M as $M_1 \oplus_2 M_2$ where each M_i has ground set $X_i \cup \{q\}$. Let $|E(M_i)| = n_i$ and $r(M_i) = r_i$. Assume that both M_1 and M_2 are simple. Then $\frac{|E(M)|}{r(M)} > \frac{|E(M_1)|}{r(M_1)}$, so

$$\frac{n_1 + n_2 - 2}{r_1 + r_2 - 1} > \frac{n_1}{r_1}.$$

Hence

$$r_1 n_2 - 2r_1 > r_2 n_1 - n_1.$$

By symmetry,

$$r_2 n_1 - 2r_2 > r_1 n_2 - n_2.$$

Adding the last two inequalities gives $n_1 + n_2 > 2(r_1 + r_2)$, so $n_i > 2r_i$ for some i. Thus $d(M_i) > 2$. Since M is density-critical with density at most two, this is a contradiction. We conclude that M_1 or M_2, say M_1, is non-simple. Thus it has an element p in parallel with the basepoint q of the 2-sum. Hence $M = P(M|(X_1 \cup \{p\}), M|(X_2 \cup \{p\}))$. □

Lemma 18. *Let N be a simple connected matroid in which all but at most one element is in at least two triangles. Then N has no 2-cocircuits. Moreover, if N has $\{a, b, c\}$ as a triad, then either*

- *(i) $\{a, b, c\}$ is contained in a 4-point line and $N = P(U_{2,4}, N \setminus \{a, b, c\})$; or*
- *(ii) N has a triangle $\{x, y, z\}$ such that $N|\{a, b, c, x, y, z\} \cong M(K_4)$ and N is the generalized parallel connection of $N|\{a, b, c, x, y, z\}$ and $N \setminus \{a, b, c\}$ across the triangle $\{x, y, z\}$.*

Proof. As N has at most one element that is not in at least two triangles, N has no 2-cocircuits. Suppose $\{a, b, c\}$ is a triad of N. If $\{a, b, c\}$ is also a triangle, then $\{a, b, c\}$ is 2-separating in N. Moreover, $\{a, b, c\}$ is contained in a 4-point line $\{a, b, c, d\}$ and (i) holds.

We may now assume that $\{a, b, c\}$ is not a triangle of N. Then, because at least two of a, b, and c are in at least two triangles, the hyperplane $E(N) - \{a, b, c\}$ of N contains distinct elements x, y, and z such that $\{a, b, z\}$, $\{a, y, c\}$, and $\{x, b, c\}$ are triangles. Now

$$\begin{aligned} r(\{x, y, z\}) &\leqslant r(E(N) - \{a, b, c\}) + r(\mathrm{cl}(\{a, b, c\})) - r(N) \\ &= r(N) - 1 + 3 - r(N) = 2. \end{aligned}$$

Thus $\{x, y, z\}$ is a triangle of N and $N|\{a, b, c, x, y, z\} \cong M(K_4)$. It follows by a result of Brylawski [1] (see also [7, Proposition 11.4.15]) that (ii) holds. □

Corollary 19. *Let N be a simple connected matroid in which all but at most one element is in at least two triangles and $d(N) \leqslant \frac{9}{4}$. If $r(N) = 2$, then $N \cong U_{2,4}$. If $r(N) = 3$, then $N \cong M(K_4)$. If $r(N) = 4$, then $N \cong P(U_{2,4}, M(K_4)), M(K_5 \setminus e)$, or $M^*(K_{3,3})$.*

Proof. We omit the straightforward proof for the case when $r(N) \in \{2,3\}$. Assume $r(N) = 4$. By Lemma 18, N has no 2-cocircuits. Now suppose N has $\{a,b,c\}$ as a triad. If (i) of Lemma 18 holds, then $N = P(U_{2,4}, N \setminus \{a,b,c\})$. By the result in the rank-3 case, $N \setminus \{a,b,c\} \cong M(K_4)$, so $N \cong P(U_{2,4}, M(K_4))$. If, instead, (ii) of Lemma 18 holds, then N is the generalized parallel connection across a triangle $\{x,y,z\}$ of $M(K_4)$ and $N \setminus \{a,b,c\}$. In the latter, $E(N \setminus \{a,b,c,x,y,z\})$ must be a triad of N, so $N \setminus \{a,b,c\} \cong M(K_4)$. Hence N is the generalized parallel connection across a triangle of two copies of $M(K_4)$, so $N \cong M(K_5 \setminus e)$.

We may now assume that N has no triads. Then every cocircuit of N has at least four elements. As N certainly has a plane that contains two intersecting triangles, $\{x, f_1, g_1\}$ and $\{x, f_2, g_2\}$, we deduce that $|E(N)| \geqslant 9$, so $|E(N)| = 9$. Let $\{a,b,c,d\}$ be the cocircuit $E(N) - \{x, f_1, f_2, g_1, g_2\}$. Because N has no plane with more than five points and has all but at most one element in two triangles, we may assume that $\{a, b, g_1\}$ and $\{a, c, g_2\}$ are triangles of N. Then $N \setminus d$ has $\{x, f_1, g_1\}, \{g_1, b, a\}, \{a, c, g_2\}, \{g_2, f_2, x\}$ as triangles. By Lemma 11, $N \setminus d$ is a rank-4 wheel or whirl. In this matroid, f_1, b, c, and f_2 are in unique triangles. It follows that N must have $\{d, f_1, c\}$ and $\{d, b, f_2\}$ as triangles. Thus $N \setminus d$ is a rank-4 wheel. Likewise, $N \setminus f_1$ and $N \setminus c$ are also rank-4 wheels, so $N \cong M^*(K_{3,3})$. □

Lemma 20. *Let N be a simple matroid of rank at least three in which every element is in at least two triangles. Suppose $e \in E(N)$. Then*

(i) e is in a plane of N having at least seven points; or

(ii) every element of $\mathrm{si}(N/e)$ *is in at least two triangles; or*

(iii) N has a $U_{2,4}$- or $M(K_4)$-restriction using e.

Proof. Assume that neither (i) nor (iii) holds. We show that every element of $\mathrm{si}(N/e)$ is in at least two triangles. First consider a triangle $\{e, c_1, c_2\}$ of N containing e. Let $\{c_1, d_1, f_1\}$ and $\{c_2, d_2, f_2\}$ be triangles of N where neither contains e. If $r(\{e, c_1, d_1, f_1, c_2, d_2, f_2\}) = 4$, then, in $\mathrm{si}(N/e)$, the element c corresponding to c_1 and c_2 is in at least two triangles. Now suppose $r(\{e, c_1, d_1, f_1, c_2, d_2, f_2\}) = 3$. Since N has no plane with more than six points, we may assume that $f_1 = f_2$. Rename this element f. If $\{e, d_1, d_2\}$ is not a triangle, then $\mathrm{si}(N/e)$ has a 4-point line containing c, so c is in at least two triangles of this matroid. If $\{e, d_1, d_2\}$ is a triangle of N, then $N|\{e, c_1, c_2, d_1, d_2, f\}) \cong M(K_4)$, a contradiction.

Now let f be an element of N that is not in a triangle with e. Let $\{f, g_1, h_1\}$ and $\{f, g_2, h_2\}$ be triangles of N. Then $\mathrm{si}(N/e)$ has at least two triangles containing f otherwise $N|\{e, f, g_1, g_2, h_1, h_2\}) \cong M(K_4)$, a contradiction. □

Recall that M_{18} is the 18-element matroid that is obtained by attaching, via parallel connection, a copy of $M(K_4)$ at each element of an $M(K_3)$.

Lemma 21. *Let N be a simple connected non-empty matroid in which every element is in a $U_{2,4}$- or $M(K_4)$-restriction. Assume that $d(N) \leqslant \frac{9}{4}$ but $d(N') < \frac{9}{4}$ for all proper minors N' of N. Then N is isomorphic to $U_{2,4}, M(K_4), P(U_{2,4}, M(K_4)), P(M(K_4), M(K_4))$, $M(K_5 \setminus e)$, or M_{18}.*

Proof. Since $d(N') \leqslant \frac{9}{4}$ for all minors N' of N, we see that, in any such N', no line has more than four points and no plane has more than six points. Next we show the following.

21.1. *If N has a 4-point line, then N is isomorphic to $U_{2,4}$ or $P(U_{2,4}, M(K_4))$.*

This is immediate if $r(N) = 2$. Because N has no plane with more than six points, $r(N) \neq 3$. Let L be a 4-point line of N and let Z be a subset of $E(N)$ not containing L such that $N|Z$ is isomorphic to $U_{2,4}$ or $M(K_4)$. If $L \cap Z \neq \emptyset$, then again, since N has no plane with more than six points, we deduce that $N \cong P(U_{2,4}, M(K_4))$. We may now assume that $L \cap Z = \emptyset$. If $r(L \cup Z) \leqslant r(Z) + 1$, then N has a rank-3 or rank-4 restriction of density exceeding $\frac{9}{4}$, a contradiction. We deduce that $r(L \cup Z) = r(Z) + 2$.

By Corollary 14, N has a simple connected minor N' such that $N'|(L \cup Z) = N|(L \cup Z)$ and $r(N') = r(Z) + 2$. As N' is connected, it has an element x' that is not in the closure of L or of Z. Then N'/x' has $N|L$ and $N|Z$ as restrictions and has rank $r(Z) + 1$. Thus $\mathrm{si}(N'/x')$ has either a plane with more than six points or has $P(U_{2,4}, M(K_4))$ as a restriction. Each possibility yields a contradiction, so 21.1 holds.

We may now assume that every element of N is in an $M(K_4)$-restriction. We may also assume that N is not isomorphic to $M(K_4)$ or $P(M(K_4), M(K_4))$. Next we show the following.

21.2. *Let X and Y be distinct subsets of $E(N)$ such that both $N|X$ and $N|Y$ are isomorphic to $M(K_4)$. If $|X \cap Y| \geqslant 2$, then $N \cong M(K_5 \setminus e)$.*

Since N has no plane with more than six points, $r(X \cup Y) > 3$. As $|X \cap Y| \geqslant 2$, it follows by submodularity that $r(X \cup Y) = 4$ and $r(X \cap Y) = 2$. As $d(N|(X \cup Y)) \leqslant \frac{9}{4}$, we deduce that $|X \cup Y| = 9$, so $|X \cap Y| = 3$ and $N = N|(X \cup Y)$. Moreover, $N|X$ and $N|Y$ meet in a triangle Δ. By Lemma 18, N is the generalized parallel connection of $N|X$ and $N|Y$ across Δ. Thus $N \cong M(K_5 \setminus e)$ as each of $N|X$ and $N|Y$ is isomorphic to $M(K_4)$, so 21.2 holds.

We may now assume that $E(N)$ has at least three distinct subsets X with $N|X \cong M(K_4)$ and that no two such subsets meet in more than one element.

21.3. *N does not have $P(M(K_4), M(K_4))$ as a restriction.*

Assume that $N|X \cong P(M(K_4), M(K_4))$ and $N|Y \cong M(K_4)$ where $Y \not\subseteq X$. Suppose $|X \cap Y| = k$ where $k \in \{1, 2\}$. Then $r(X \cup Y) \leqslant 8 - k$ and $|X \cup Y| = 17 - k$, so

$$\frac{9}{4} \geqslant d(N|(X \cup Y)) \geqslant \frac{17 - k}{8 - k}.$$

Simplifying we obtain the contradiction that $4 \geqslant 5k \geqslant 5$. We deduce using 21.2 that $|X \cap Y| = 0$. Then $r(X \cup Y) = 8$ otherwise $d(N|(X \cup Y)) > \frac{9}{4}$.

By Corollary 14, N has a simple connected minor N' such that $N'|(X \cup Y) = N|(X \cup Y)$ and $r(N') = 8$. As $N|(X \cup Y)$ is disconnected, N' must contain an element that is not in $X \cup Y$. Hence $|E(N')| \geqslant 18$, so $d(N') \geqslant \frac{9}{4}$. Thus $N' = N$ and $|E(N)| = 18$, so N has a single element z that is not in $X \cup Y$. The $M(K_4)$-restriction of N that contains z is forced to have more than one element in common with Y or one of the $M(K_4)$-restrictions of $N|X$. This contradiction to 21.2 completes the proof of 21.3.

We now know that any two $M(K_4)$-restrictions of N have disjoint ground sets. Let X, Y, and Z be distinct subsets of $E(N)$ such that each of $N|X$, $N|Y$, and $N|Z$ is isomorphic to $M(K_4)$. Next we show the following.

21.4. $r(X \cup Y) = 6$. *Moreover,* $r(X \cup Y \cup Z) = 9$ *unless* $N \cong M_{18}$.

As $|X \cup Y| = 12$ and $d(N|(X \cup Y)) < \frac{9}{4}$, we deduce that $r(X \cup Y) = 6$. The density constraint also means that $r(X \cup Y \cup Z) \geqslant 8$. Suppose $r(X \cup Y \cup Z) = 8$. Then $d(N|(X \cup Y \cup Z)) = \frac{9}{4}$, so $N = N|(X \cup Y \cup Z)$. Now $r(N/Z) = 5$. As $\frac{12}{5} > \frac{9}{4}$, we must have some parallel elements in N/Z. As Z is skew to each of X and Y, we know that $(N/Z)|X = N|X$ and $(N/Z)|Y = N|Y$. Thus there must be elements x of X and y of Y that are parallel in N/Z. If there is a second such parallel pair, then $r(N/Z) \leqslant 4$, a contradiction. In N, we see that $r(Z \cup \{x, y\}) = 4$. Hence, in N/x, we obtain a 7-point plane $Z \cup y$ unless $\{x, y, z\}$ is a triangle of N for some z in Z. Observe that each of N/x, N/y, and N/z is disconnected, so N is obtained from $M(K_3)$ by attaching a copy of $M(K_4)$ via parallel connection at each element. Thus $N \cong M_{18}$ and 21.4 holds.

By Corollary 14, N has a simple connected minor N' of rank 9 such that $N'|(X \cup Y \cup Z) = N|(X \cup Y \cup Z)$. As N' is connected, there is an element g of $E(N') - (X \cup Y \cup Z)$. Since N' has no plane with more than six points, g is not in the closure of any of X, Y, or Z in N'. As N'/g has rank 8 but has density less than $\frac{9}{4}$, the eighteen elements of $X \cup Y \cup Z$ cannot all be in distinct parallel classes of N'/g. Thus N' has a triangle $\{x, y, g\}$ where we may assume that $x \in X$ and $y \in Y$. Since $N'|(X \cup Y \cup Z \cup g)$ has Z as a component, there is an element h of $E(N')$ that is in neither $\mathrm{cl}_{N'}(X \cup Y)$ nor $\mathrm{cl}_{N'}(Z)$. As above, N' has a triangle $\{h, z, t\}$ where $t \in X \cup Y$ and $z \in Z$. Contracting g and h from $N'|(X \cup Y \cup Z \cup \{g, h\})$ and simplifying, we get a rank-7 matroid with 16 elements. As $\frac{16}{7} > \frac{9}{4}$, we have a contradiction that completes the proof of Lemma 21. $\square$

Lemma 22. *Let N be a simple connected matroid having an element z such that each of N and $\mathrm{si}(N/z)$ has every element in at least two triangles. If $d(N) \leqslant \frac{9}{4}$ and $d(N') < \frac{9}{4}$ for all proper minors N' of N, then N is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$.*

Proof. We argue by induction on $r(N)$, which must be at least three. Suppose it is exactly three. Since $\mathrm{si}(N/z)$ has density less than $\frac{9}{4}$, it is isomorphic to $U_{2,4}$. As $d(N) \leqslant \frac{9}{4}$, we see that $|E(N)| \leqslant 6$. By Lemma 18, N has no 2-cocircuits. Thus N has a triangle whose complement is a triad. By Lemma 18 again, $N \cong M(K_4)$ and we get a contradiction. Hence $r(N) \geqslant 4$. If $r(N) = 4$, then, by Corollary 19, N is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$.

Now assume the result holds for $r(N) < k$ and let $r(N) = k \geqslant 5$. Let $N_1 = \mathrm{si}(N/z)$. Every element of N_1 is in at least two triangles. Let N_2 be a component of N_1. By Lemma 20, either every element of N_2 is in a $U_{2,4}$- or $M(K_4)$-restriction, or N_2 has an element z_2 such that every element of $\mathrm{si}(N_2/z_2)$ is in at least two triangles. If the latter occurs, then, by the induction assumption, N_2 is isomorphic to $P(U_{2,4}, M(K_4))$, $M(K_5 \setminus e)$, or $M^*(K_{3,3})$. Each of these matroids has density $\frac{9}{4}$, a contradiction. Thus every element of N_2 is in a $U_{2,4}$- or $M(K_4)$-restriction. As $d(N_2) < \frac{9}{4}$, Lemma 21 implies that N_2, and hence each component of N_1, is isomorphic to one of $U_{2,4}$, $M(K_4)$, or $P(M(K_4), M(K_4))$.

Suppose that $N_2 = N_1$. Then, as $r(N) \geqslant 5$, we deduce that $N_1 \cong P(M(K_4), M(K_4))$. As $N_1 = \mathrm{si}(N/z)$, we see that $r(N) = 6$. Because $d(N) \leqslant \frac{9}{4}$, it follows that $|E(N)| \leqslant 13$. Since z is in at least two triangles of N, we deduce that $|E(N)| \geqslant |E(N_1)| + 3 = 14$, a contradiction.

We may now assume that N_1 has more than one component. Hence, for some $k \geqslant 2$, there is a collection $N^1, N^2, \ldots, N^k$ of connected matroids such that $E(N^i) \cap E(N^j) = \{z\}$ for all $i \neq j$, the matroid N^i/z is connected for all i, and N is the parallel connection of $N^1, N^2, \ldots, N^k$ across the common basepoint z. As noted above, each $\mathrm{si}(N^i/z)$ is isomorphic to one of $U_{2,4}$, $M(K_4)$, or $P(M(K_4), M(K_4))$. As every element of N is in at least two triangles, every element of each N^i except possibly z is in at least two triangles of N^i. Thus, by Corollary 19, $N^i \cong M(K_4)$; or $r(N^i) = 4$ and $|E(N^i)| = 9$; or $r(N^i) > 4$. In the first case, $\mathrm{si}(N^i/z) \not\cong U_{2,4}$; in the second case, $d(N^i) = \frac{9}{4}$. Both of these possibilities give contradictions, so $\mathrm{si}(N^i/z) \cong P(M(K_4), M(K_4))$ for each i. As z is in at least two triangles of N, we may assume the elements of two such triangles lie in $E(N^1) \cup E(N^2)$. As $|E(\mathrm{si}(N^i/z))| = 11$ and $r(N^i/z) = 5$, we see that $|E(N^1) \cup E(N^2)| \geqslant 25$ and $r(E(N^1) \cup E(N^2)) = 11$. But $\frac{25}{11} > \frac{9}{4}$, a contradiction. □

We conclude the paper by proving Theorem 9. In this proof, we will make extensive use of the Cunningham-Edmonds canonical tree decomposition of a connected matroid. The definition and properties of this decomposition may be found in [7, Section 8.3]. In brief, associated with each connected matroid M, there is a tree T that is unique up to the labelling of its edges. Each vertex of T is labelled by a circuit, a cocircuit, or a 3-connected matroid with at least four elements. Moreover, no two adjacent vertices of T are labelled by circuits and no two adjacent vertices are labelled by cocircuits. For an edge e of T whose endpoints are labelled by matroids M_1 and M_2, the ground sets of these two matroids meet in $\{e\}$. When we contract e from T, the composite vertex that results by identifying the endpoints of e is labelled by the 2-sum of M_1 and M_2. By repeating this process, contracting all of the remaining edges of T one by one, we eventually obtain a single-vertex tree. Its vertex is labelled by M.

Each edge f of T induces a partition of $E(M)$. This partition is a 2-separation of M *displayed by* f. The remaining 2-separations of M coincide with those that are displayed by those vertices of T that are labelled by circuits or cocircuits. For such a vertex v having label N, there is a partition $\{X_1, X_2, \ldots, X_k\}$ of $E(M) - E(N)$ induced by the components of $T - v$. A partition (X, Y) of $E(M)$ is *displayed by the vertex* v if each X_i is contained in X or Y. Every such partition of $E(M)$ with both X and Y having at least two elements is a 2-separation of M and these 2-separations along with those displayed by the edges of T are all of the 2-separations of M. Recall that, for all $n \geqslant 2$, we denote by P_n any matroid that can be constructed from n copies of $M(K_3)$ via a sequence of parallel connections.

Proof of Theorem 9. Let M be a density-critical matroid with $d(M) \leqslant \frac{9}{4}$. Suppose $d(M) \geqslant 2$. By Lemma 10, every element of M is in at least two triangles. By Corollary 19, if $r(M) \in \{2, 3\}$, then M is $U_{2,4}$ or $M(K_4)$. We may now assume that $r(M) \geqslant 4$. By Lemma 20, either every element of M is in a $U_{2,4}$- or $M(K_4)$-restriction, or, for some element z of M, every element of $\mathrm{si}(M/z)$ is in at least two triangles. In the first case, by Lemma 21, M is isomorphic to $P(U_{2,4}, M(K_4)), P(M(K_4), M(K_4)), M(K_5 \setminus e)$, or M_{18}. In the second case, by Lemma 22, M is isomorphic to $P(U_{2,4}, M(K_4)), M(K_5 \setminus e)$, or $M^*(K_{3,3})$. Thus the theorem identifies all possible density-critical matroids with density in $[2, \frac{9}{4}]$ and one easily checks that each of the matroids identified is indeed density-critical.

Now suppose that $d(M) < 2$. By Lemma 16, M is connected. Clearly, if $r(M)$ is 1 or 2, then M is isomorphic to $U_{1,1}$ or $U_{2,3}$. As $U_{2,4}$ and $M(K_4)$ both have density 2, M is a series-parallel network (see, for example, [7, Corollary 12.2.14]). Thus, in the Cunningham-Edmonds canonical tree decomposition T of M, every vertex is labelled by a circuit or a cocircuit. Since M is simple, for every vertex of T that is labelled by a cocircuit C^*, at most one element of C^* is in $E(M)$. Let e be an edge of T that meets the vertex labelled by C^*. Then, for the 2-separation (X, Y) of M that is displayed by e, Lemma 17 implies that M has an element p in $\mathrm{cl}(X) \cap \mathrm{cl}(Y)$. Thus $p \in C^*$, so C^* contains exactly one element of M.

Now take a vertex of T that is labelled by a circuit C where $C = \{e_1, e_2, \ldots, e_k\}$ and suppose that $k \geqslant 4$. Suppose $e_1 \in E(M)$. Then M/e_1 is simple having rank $r(M) - 1$. As $\frac{|E(M)|-1}{r(M)-1} < \frac{|E(M)|}{r(M)}$, we obtain the contradiction that $|E(M)| < r(M)$. We deduce that $C \cap E(M) = \emptyset$. Now $T \setminus e_1, e_2$ has exactly three components. Let T' be the one containing e_3 and let X be the subset of $E(M)$ corresponding to T'. Then $(X, E(M) - X)$ is a 2-separation of M. By Lemma 17, there is an element p of M that is in $\mathrm{cl}(X) \cap \mathrm{cl}(E(M) - X)$. But the tree decomposition implies that there is no such element. We deduce that C has exactly three elements. Thus every vertex of T that is labelled by a circuit is labelled by a triangle. Since every vertex of T that is labelled by a cocircuit has exactly one element of $E(M)$ in that cocircuit, a straightforward induction argument establishes that, for some $n \geqslant 2$, the matroid M is obtained from n copies of $M(K_3)$ by a sequence of $n-1$ parallel connections. Thus $M \cong P_n$.

Finally, we show by induction that P_n is density-critical. This is true for $n = 1$. Assume it true for $n < m$ and let $n = m \geqslant 2$. Take x in $E(P_n)$. Assume first that x is in exactly one triangle $\{x, y, z\}$. Then $\mathrm{si}(P_n/x) \cong P_n/x \setminus z$. As the last matroid is easily seen to be isomorphic to the density-critical matroid P_{n-1} and $d(P_{n-1}) < d(P_n)$, every minor of P_n/x has density less that $d(P_n)$. Now assume x is in at least two triangles of P_n. Then $\mathrm{si}(P_n/x)$ is easily seen to be the direct sum of a collection of matroids each of which is isomorphic to some P_k with $k < n$ or to $U_{1,1}$. By Lemma 16 and the induction assumption, every minor of P_n/x has density less that $d(P_n)$. We conclude that P_n is density-critical, so the theorem is proved. □

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