Periodic triangulations of $\mathbb{Z}^n$

Mathieu Dutour Sikirić*

Rudjer Bosković Institute
Bijenička 54, 10000 Zagreb, Croatia

mathieu.dutour@gmail.com

Alexey Garber

School of Mathematical & Statistical Sciences
The University of Texas Rio Grande Valley
1 West University Blvd, Brownsville, TX, 78520, U.S.A.

alexeygarber@gmail.com

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Abstract

We consider in this work triangulations of $\mathbb{Z}^n$ that are periodic along $\mathbb{Z}^n$. They generalize the triangulations obtained from Delaunay tessellations of lattices. In certain cases we impose additional restrictions on such triangulations such as regularity or invariance under central symmetry with respect to the origin; both properties hold for Delaunay tessellations of lattices.

Full enumeration of such periodic triangulations is obtained for dimension at most 4. In dimension 5 several new phenomena happen: there are centrally-symmetric triangulations that are not Delaunay, there are non-regular triangulations (it could happen in dimension 4) and a given simplex has a priori infinitely many possible adjacent simplices. We found 950 periodic triangulations in dimension 5 but finiteness of the whole family is unknown.

Mathematics Subject Classifications: 52C07, 52C22

1 Introduction

Given a positive definite quadratic form A we obtain a tessellation of $\mathbb{Z}^n$ by taking the projection of the facets of the convex hull of $\left\{(x, x^T Ax) \text{ for } x \in \mathbb{Z}^n\right\}$. This triangulation is $\mathbb{Z}^n$-periodic, centrally symmetric and is called the Delaunay tessellation [11]. For dimension at most 5 those tessellations are classified and there are 1, 2, 5, 52 and 110244

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for $1 \leqslant n \leqslant 5$ [19, 9] up to the action of $\mathrm{GL}_n(\mathbb{Z})$. If one limits oneself to the Delaunay triangulations formed of simplices only then the number of types is 1, 1, 1, 3 and 222 [3, 17, 12] for $1 \leqslant n \leqslant 5$, respectively. For $n = 6$ Baburin and Engel [2] reported more than 500'000'000 non-equivalent triangulations. A triangulation is called *regular* if it is obtained as projection of facets of an infinite convex body of vertices $(x, f(x))$ for f a function defined on $\mathbb{Z}^n$. This generalizes the Delaunay tessellations.

In this paper we consider general triangulations of the point set $\mathbb{Z}^n$ which are invariant under translations by $\mathbb{Z}^n$ and are face-to-face. Such triangulations can be viewed as decomposition of a torus into a cell complex with one vertex where all cells are simplices. Other triangulations of the torus into simplices were considered in [13, 14]. In Section 2 we consider general properties of periodic triangulations of $\mathbb{Z}^n$ in particular properties of their symmetry groups, simplices that may appear in such triangulation, and appearance of such triangulations as refinement of periodic tilings. In Section 3 we detail a number of computational tools for testing Delaunayness and regularity of triangulations that we use in this work.

In Section 4 we prove that for $n \leqslant 3$ all such triangulations are Delaunay. For $n = 4$ a non-centrally symmetric triangulation named "red-triangular" was described in [1, Example 5.13.1]. We also prove in Section 5 that this triangulation together with the Delaunay ones form all the set of triangulations up to the action of $\mathrm{GL}_4(\mathbb{Z})$.

In dimension $n \geqslant 5$ full enumeration of periodic triangulation appears to be difficult. First of all the finiteness is not proved and may not hold since we prove in Section 6 that given a simplex of volume 1 there are a priori infinitely many possibilities for an adjacent simplex.

In Section 7 we obtain 950 non-isomorphic periodic triangulations of $\mathbb{Z}^5$ but we do not know if this is the complete list. This list allows us to prove that there are centrally symmetric but not Delaunay triangulations and non-regular triangulations.

In 8 we list several open questions on enumeration, extensibility and regularity properties of periodic triangulations of $\mathbb{Z}^n$ that should be of general interest.

2 General properties of periodic triangulations

Definition 1. A *k-dimensional lattice* is a discrete subgroup of $\mathbb{R}^n$ of rank k, i.e. a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k$ with linearly independent vectors $v_1, \ldots, v_k$.

Throughout the paper we will work with the lattice $L = \mathbb{Z}^n$. The group of affine transformations preserving a n-dimensional lattice is isomorphic to $\mathrm{AGL}_n(\mathbb{Z})$.

Definition 2. A *partial triangulation* $\mathcal{PT}$ of $\mathbb{Z}^n$ is a packing in $\mathbb{R}^n$ by n-dimensional simplices with the vertex set $\mathbb{Z}^n$, i.e. representation of a subset of $\mathbb{R}^n$ as a union of countably many simplices with integer vertices such that the intersection of any pair of simplices is a face of both.

A *triangulation* $\mathcal{T}$ is a partial triangulation which is also a tiling.

Triangulation $\mathcal{T}$ is called $\mathbb{Z}^n$*-periodic*, or just *periodic* if $\mathcal{T} + v = \mathcal{T}$ for every $v \in \mathbb{Z}^n$.

Definition 3. The *symmetry group* $\text{Sym}(\mathcal{T})$ of a periodic triangulation $\mathcal{T}$ of $\mathbb{Z}^n$ is the group of affine transformations of $\mathbb{R}^n$ preserving $\mathcal{T}$.

The group $\text{Sym}(\mathcal{T})$ contains $\mathbb{Z}^n$ as a normal subgroup of finite index. The quotient $\text{Sym}(\mathcal{T})/\mathbb{Z}^n$ is called the *point group* $Pt(\mathcal{T})$.

The symmetry group is *split* if $\text{Sym}(\mathcal{T})$ is a semi-direct product $\mathbb{Z}^n \rtimes Pt(\mathcal{T})$. This is equivalent to having $Pt(\mathcal{T})$ being realized as a subgroup of $\text{Sym}(\mathcal{T})$, see [15].

Proposition 4. *The symmetry group of a periodic triangulation $\mathcal{T}$ is split.*

Proof. Let $v_0 = 0$, $v_1 = e_1$, $\dots$, $v_n = e_n$ with (e_i) the standard basis of $\mathbb{Z}^n$. Let f be a symmetry of the triangulation $\mathcal{T}$. Define $v'_i = f(v_i)$ for $0 \leqslant i \leqslant n$ and write the transformation f in matrix form as $Ax + b$. Then

$$\begin{pmatrix} 1 & \dots & 1 \\ v'_0 & \dots & v'_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ v_0 & \dots & v_n \end{pmatrix}.$$

This equation implies that A and b are integral. Thus f is a composition of a transformation preserving the origin and an integral translation. This implies that the symmetry group is split. □

Definition 5. Let Λ be a d-dimensional lattice with fundamental volume V (so the volume of d-dimensional parallelepiped spanned by basis vectors of Λ is V), and let S be a d-dimensional simplex with vertices from Λ. In this case volume of S is $k \cdot \frac{V}{d!}$ for some integer k, and we will say that *relative volume* of S is k.

In the following we will refer to relative volume of S as just volume of S, or $\text{vol}(S)$, unless we need to emphasize the dimension.

Proposition 6. *Let S be a simplex of an n-dimensional periodic triangulation.*
(i) We have the inequality $\text{vol}(S) \leqslant n!$.
(ii) If the triangulation is centrally symmetric then

$$\text{vol}(S) \leqslant \frac{2^n}{\binom{2n}{n}} n!.$$

Proof. If the periodic triangulation is formed by the simplices S_1, $\dots$, S_p and their $\mathbb{Z}^n$ translations then we have the equality

$$\sum_{i=1}^{p} \text{vol}(S_i) = n!$$

from which (i) obviously follows.

The proof of (ii) follows exactly the same arguments as [6, Proposition 14.2.4]. □

Definition 7. Let $\mathcal{T}$ be a periodic tiling of $\mathbb{Z}^n$ by polytopes having only integer points as vertices. Let A be a positive definite quadratic form on $\mathbb{Z}^n$. Then A induces another tiling $Ref_A(\mathcal{T})$ of $\mathbb{Z}^n$ defined on each polytope $P \in T$ as projection of the lower facets of

$$Scal(P) = \text{conv}\left\{(x, A[x]) \text{ for } x \in \text{vert}(P)\right\}.$$

Lemma 8. *For a n-dimensional periodic tiling $\mathcal{T}$ and a positive definite quadratic form A the following properties hold:*

(i) $Ref_A(\mathcal{T})$ is a periodic tiling of $\mathbb{Z}^n$ which is a refinement of $\mathcal{T}$.

(ii) If $g \in \mathrm{GL}_n(\mathbb{Z})$ preserves A and belongs to the point group of $\mathcal{T}$ then it belongs to the point group of $Ref_A(\mathcal{T})$.

(iii) If A is generic then $Ref_A(\mathcal{T})$ is a triangulation; one way to get generic matrices is to force the coefficients of A to be independent over the rationals.

Proof. (i) Let us consider for each polytope P of $\mathcal{T}$ the scaling map used to describe the Delaunay polytopes.

$$Scal(P) = \mathrm{conv}\left\{(x, A[x]) \text{ for } x \in \mathrm{vert}(P)\right\}.$$

The lower facets of $Scal(P)$ define a tiling of P into polytopes. It also defines a tiling of the faces of P. If a face F of $\mathcal{T}$ is contained into polytopes $P_1, \ldots, P_m$ then the induced tilings are the same. The tiling is $\mathbb{Z}^n$-periodic since $A[x]$ differ on two different translate of a tile by an affine term which gives the same lower facets.

(ii) is trivial.

(iii) If a face of the tessellation is not a simplex then there are at least $n + 2$ points of the form $(x, A[x])$ lying on an hyperplane. Since the points x are integral this implies some non-trivial rational relations between the coefficients of A, which is not possible for a generic matrix. □

Definition 9. A triangulation T of $\mathbb{Z}^n$ is called *regular* if there exists a function $f : \mathbb{Z}^n \to \mathbb{R}$ such that:

- The points $(x, f(x))$ are vertices of the convex polyhedron
$$H(f) = \mathrm{conv}\{(x, f(x)) | x \in \mathbb{Z}^n\}$$
in $\mathbb{R}^{n+1}$.
- The simplices of T are orthogonal projections of the lower facets of $H(f)$ onto $\mathbb{R}^n$; here the sets of lower facets is naturally defined by the direction of the last coordinate axis in $\mathbb{R}^{n+1}$.

Obviously Delaunay triangulations are regular as we introduced them as projections of the facets of the convex hull $\left\{(x, x^T A x) \text{ for } x \in \mathbb{Z}^n\right\}$.

3 Computational tools

3.1 Testing Delaunay property

Suppose we are given a Delaunay triangulation for some positive definite quadratic form $A[x] = x^T A x$; this triangulation is the projection of the facets of the infinite body $\{(x, A[x]) \text{ for } x \in \mathbb{Z}^n\}$. For each simplex $S = \mathrm{conv}(v_0, \ldots, v_n)$ of this triangulation there is a sphere of center c and radius r for the matrix A such that $A[x-c] \geqslant r^2$ for all $x \in \mathbb{Z}^n$ and $A[x-c] = r^2$ for $x \in \mathbb{Z}^n$ only for vertices of S. A fundamental property of Delaunay triangulations is that for a point $v \in \mathbb{Z}^n$ the condition $A[v-c] \geqslant r^2$ translates into a linear condition on the entries of the matrix A. If one writes $v = \sum_{i=0}^n \lambda_i v_i$ with $1 = \sum_{i=0}^n \lambda_i$ then

$$A[v-c] > r^2 \Leftrightarrow Tr\,(AN_{S,v}) > 0 \text{ with } N_{S,v} = vv^T - \sum_{i=0}^n \lambda_i v_i v_i^T$$

with Tr being the matrix trace. The matrices $N_{S,v}$ are called *Voronoi regulator* (see [20, 10] for details).

For a given Delaunay triangulation, this construction gives an infinite set of defining inequalities. Another fundamental result of Voronoi gives a finite set of defining inequalities. For each Delaunay simplex S and facet F of S there is an adjacent Delaunay simplex $\mathrm{conv}(F, v')$. The inequalities $Tr\,(AN_{S,v'}) \geqslant 0$ for S iterating over the translation classes of Delaunay simplices and facets F of S are sufficient to imply all other inequalities. This theorem is proved in [20, Section 77], see a generalization in [10, Theorem 3.1].

Given a periodic triangulation $\mathcal{T}$ of $\mathbb{Z}^n$ the above description gives us an algorithm to test if a periodic triangulation is Delaunay. We determine all facets F of simplices S of $\mathcal{T}$ up to translations. Any such facet F is contained in exactly two simplices $S_1 = \mathrm{conv}(F \cup \{v_1\})$ and $S_2 = \mathrm{conv}(F \cup \{v_2\})$ of $\mathcal{T}$. We define the polyhedral cone

$$\mathcal{P} = \left\{A \text{ s.t. } Tr\,(AN_{S_1,v_2}) \geqslant 0 \text{ for all } F\right\}.$$

If $\mathcal{P}$ is full dimensional then every quadratic form in its interior satisfies $Tr\,(AN_{S_1,v_2}) > 0$ which by [20, Section 77] implies the inequalities $Tr\,(AN_{S,v}) > 0$ for all simplices S and $v \in \mathbb{Z}^n - S$ meaning that $\mathcal{T}$ is a Delaunay triangulation. If $\mathcal{P}$ is not full dimensional then some of the inequalities defining $\mathcal{P}$ cannot be satisfied strictly which means that $\mathcal{T}$ is not a Delaunay triangulation.

3.2 Adjacency of simplices

Suppose we have two simplices Δ_1 and Δ_2 and we want to check if the $\mathbb{Z}^n$ translates of Δ_1 and Δ_2 are a priori admissible as parts of a periodic triangulation of $\mathbb{Z}^n$. That is we want to check that the translates do not intersect in their interior and that the intersection is always a face of both. If F is a facet of Δ_1 represented by an inequality $f(x) \geqslant 0$ with an affine function f then if we have $f(\Delta_2 + v) < 0$ then there is no intersection. That is for some facet f we have $\max_{x\in T_2} f(x+v) < 0$ then there is no intersection. So, the feasible vectors v are the ones that satisfies $\max_{x\in T_2} f(x+v) \geqslant 0$ for all facet inequalities f of Δ_1.

This defines a convex body $\mathcal{C}$ and the integer points can be obtained by using exhaustive enumeration. Then for each integer point $v \in \mathcal{C}$ we check if $\Delta_1 \cap (\Delta_2 + v)$ is n-dimensional or not. If it is not then we check that the intersection is a face of both.

With this method we can find the possible simplices adjacent to a given simplex Δ. That is for a facet F of Δ and a point $v \in \mathbb{Z}^n$ we consider whether the pair Δ and $\mathrm{conv}(F \cup \{v\})$ is admissible for a periodic tessellation. By iterating over the facets and vectors v we have a list of possible candidates. However, we have no method for restricting the set of possible vectors v and in Section 6 we show that the set of such vectors can be infinite.

3.3 Testing regularity

Given a periodic triangulation $\mathcal{T}$ we want to check if it is regular. According to Definition 9 the condition that the simplices correspond to facets of the convex body $H(f)$ translates into linear inequalities on the values of $f(v)$ for $v \in \mathbb{Z}^n$. Thus testing regularity is equivalent to checking if an infinite dimensional linear program has a feasible solution.

We are not aware of a general method for working with infinite dimensional linear programs and thus we cannot check regularity of triangulations easily. What we can do instead is prove in some cases that a periodic triangulation is not regular. Let us take a triangulation $\mathcal{T}$ of $\mathbb{Z}^n$ and select a finite set $\mathcal{S}$ of simplices from $\mathcal{T}$. Suppose we are searching for a function f on the set of vertices $\mathcal{V}$ corresponding to the simplices of $\mathcal{S}$ such that $\mathcal{S}$ can be obtained as regular using f. If we have two adjacent simplices S_1 and S_2 then denote by ϕ_{S_1} the affine linear function coinciding with f on the vertices of S_1. For a vertex v of S_2 which is not in S_1 we must have the inequality

$$f(v) > \phi_{S_1}(v).$$

Now if the function f does exist, and therefore $\mathcal{T}$ is regular, then by rescaling there exists a function f on $\mathcal{V}$ satisfying

$$f(v) \geqslant \phi_{S_1}(v) + 1.$$

The set of strengthened inequalities define a polyhedral convex body $\mathcal{Q}$. If $\mathcal{Q}$ is proven to be empty by linear programming, then we have proved that $\mathcal{T}$ is not regular.

3.4 Equivalence and stabilizer

For enumeration purposes, we need to be able to check that two triangulations are equivalent and to compute the stabilizer of a triangulation in the group $\mathrm{GL}_n(\mathbb{Z})$. The method is simply to take one simplex in a triangulation and to consider all ways in which it may be mapped in a simplex of another or the same triangulation. While computationally expensive, this method is adequate for the low-dimensional cases that we consider in this work.

4 Enumeration of periodic triangulations in dimension 3

The main goal of this and the next section is to give complete enumeration of periodic triangulations in dimensions 3 and 4.

Let S be a simplex in a triangulation (not necessary full-dimensional), denote by $tr(S)$ the translation class of S, i.e. the set of all translations of S by vectors of $\mathbb{Z}^n$.

The following lemma is true for arbitrary dimension.

Lemma 10. *If a facet F is common for full-dimensional simplices S_1 and S_2, then facets in the intersection of $tr(S_1)$ and $tr(S_2)$ are only facets in $tr(F)$.*

Proof. Assume contrary, then there is one more facet F' of S_1 which is parallel to some facet of S_2. If F and F' are in the same copy of S_2 in $tr(S_2)$, then $S_1{=}S_2$. Otherwise, facets F and F' have a common ridge, so S_2 should have two parallel ridges which is impossible. □

In dimension 3, it appears to be known that there is only one periodic triangulation. See work of Alexeev, [1, Sect. 5.13]. Here we give another straightforward approach.

In order to obtain the full classification, we show how one can find the upper bound on the relative volume of a simplex in a periodic triangulation. We apply it for dimensions 3 here and for dimension 4 in the following section. For dimension 2 it is clear that each simplex should have volume 1 (for example, from Pick's formula).

Let $\mathcal{T}$ denote an arbitrary periodic triangulation of $\mathbb{Z}^n$. We will use a more careful approach compared to Proposition 6 in order to show, that if the relative volume of a simplex exceeds a certain number (actually it is 1 for dimensions 3 and 4), then this simplex can not be included in a periodic triangulation, and in $\mathcal{T}$ particularly.

Proposition 11. *If $\mathcal{T}$ is three-dimensional, then the relative volume of each three-dimensional simplex is* 1.

Proof. Let $ABCD$ be an arbitrary simplex of $\mathcal{T}$ with volume at least 2. It is clear, that relative volume of every facet of $ABCD$ in the corresponding sublattice is 1. So, we can choose a coordinate system (with matrix transformation from $\mathrm{GL}_3(\mathbb{Z})$) such that vertices of $ABCD$ will have coordinates represented by columns of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & c \end{pmatrix},$$

where a, b, c are non-negative, $c \geqslant 2$, $a < c$, and $b < c$.

If a is non-zero (similarly for non-zero b), then the point

$$\frac{(a-1)A + (c-a)B + D}{c} = \frac{(c-b)A + bC}{c} \pmod 1$$

belongs to translations of two faces of $ABCD$, namely to the two-dimensional face ABD (the left-hand side of the formula) and to the edge AC (the right-hand side of the formula), and is not an integer point. Thus this is a contradiction with the face-to-face property of the tiling $\mathcal{T}$.

If $a = b = 0$ then we have an integer point $(0, 0, 1)$ in the interior of the edge AD which is impossible. □

Remark 12. From this proof we can see, that each 3-dimensional face of $\mathcal{T}$ should have the relative volume 1, otherwise we will find a contradiction in 3-dimensional affine space spanned by this face. Indeed, if a relative volume of a 3-dimensional simplex is more than 1, then according to the proof of the previous proposition, its lattice translates will intersect in a non-face-to-face manner.

Next we establish all possible neighbors of a given simplex in a periodic triangulation $\mathcal{T}$ of $\mathbb{Z}^3$.

Lemma 13. *If $n = 3$, then given a simplex S_1 of $\mathcal{T}$ and its facet F, we have 3 options for a simplex S_2 of $\mathcal{T}$ adjacent to S_1 by F. More precisely, if $S_1 = ABCD$ and $S_2 = BCDE$, then $ABEC$, $ABED$, or $ACED$ is a parallelogram. These options are summarized in Figure 1*

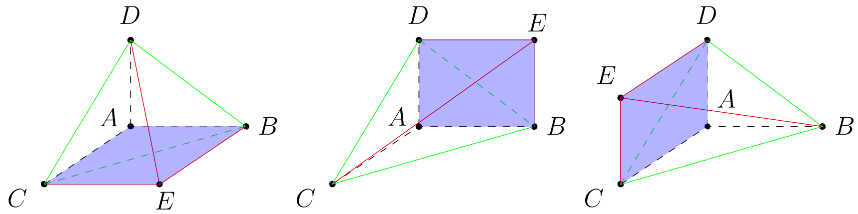

Figure 1: Tetrahedron $S_1 = ABCD$ and three options for the neighbor tetrahedron $S_2 = BCDE$. Edges AB, AC and AD are colored with black; edges BC, BD, and CD of common face of S_1 and S_2 are colored with green; edges BE, CE, and DE are colored with red. Two faces of S_1 and S_2 that comprise a parallelogram are shaded with blue.

Proof. Without loss of generality we assume, that vertices of S_1 are: $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 1, 0)$, $D = (0, 0, 1)$, and BCD is the common facet. The fourth vertex E of S_2 has coordinates (x, y, z) with $x + y + z = 2$.

At least one of numbers x, y, z is even, assume z. Then the midpoint of EB has coordinates $\left(\frac{x+1}{2}, \frac{y}{2}, \frac{z}{2}\right)$. Among numbers $x+1$ and y one is even, so this midpoint has two integer coordinates, and one half-integer. Therefore, this midpoint is a translation of one of midpoints: AB or AC. Thus, the edge EB is a translation of AB or AC. Similarly, the edge EC is a translation of AB or AC.

There are two options remaining for point E: $E = (0, 0, 0) = A$, or $E = (1, 1, 0)$. In the first case simplices S_1 and S_2 coincide, which is impossible. In the second case faces ABC and EBC form a parallelogram. □

We proceed with the classification of periodic triangulations of $\mathbb{Z}^3$. We continue to use all notations of Lemma 13 and its proof.

Theorem 14. *There is unique periodic triangulation of $\mathbb{Z}^3$ up to $\mathrm{GL}_3(\mathbb{Z})$ equivalence.*

Proof. From the proof of Lemma 13 we have a pair of simplices S_1 and S_2, and a parallelogram $ABEC$ (actually a unit square). With translations of this parallelogram we can tile an arbitrary plane $z = k$ for integer k, so any simplex of the triangulation should be between a pair of consecutive planes parallel to $z = 0$.

Currently we have six "unpaired" facets of tiling (i.e. facets that belong to only one simplex currently determined): ABC, ABD, ACD, EBC, EBD, ECD. No translational class of full-dimensional simplex can contain more than two of these facets, because otherwise it will have two common facets with one of simplices S_1 or S_2. The second simplex S_3 incident to the facet ABC has the fourth vertex on the plane $z = -1$, because S_3 has three vertices on $z = 0$, and $ABCD$ has fourth vertex on $z = 1$). Similarly, the simplex S_4 incident to EBC has the fourth vertex on the plane $z = -1$. So, S_3 can not have a facet which is a translation of any of five remaining "unpaired" facets, since four of these facets (except EBC) have two vertices on lower plane and one on the upper, and the facet EBC is parallel to facet ABC. Therefore, classes $tr(S_3)$ and $tr(S_4)$ contain only facets ABC and EBC from these six classes.

The remaining unpaired facets are: ABD, ACD, EBD, ECD. These classes should be contained in two translational classes of simplices (we already found four classes generated by S_1, S_2, S_3, and S_4, and we must have six translational classes in total due to volume argument). No class can cover more than two, so these four facets should be divided in pairs, and each pair should belong to one translational class. Pairs can not be from one simplex S_1 or S_2. So ABD should be paired with EBD or ECD. The first case is impossible, because the edge BD can belong only to one simplex from this class, so this class should be $tr(ABDE)$, but $ABDE$ intersects with interior of $ABCD$.

Therefore, the class $tr(S_5)$ contains facets ABD and ECD, and the class $tr(S_6)$ contains remaining facets ACD and EBD.

The edges AB and EC are equal and parallel, so if we translate ECD so that EC will coincide with AB (by the vector $(0, -1, 0)$), we will get the second facet of the same simplex from this class. Thus, we can reconstruct a representative S_5 of this class with vertices: $A = (0, 0, 0)$ (translation of C by $(0, -1, 0)$), $B = (1, 0, 0)$ (translation of E by $(0, -1, 0)$), $D = (0, 0, 1)$, $F = (0, -1, 1)$ (translation of D by $(0, -1, 0)$).

Similarly as S_6 we can take the simplex with vertices: $A = (0, 0, 0)$ (translation of B by $(-1, 0, 0)$), $C = (0, 1, 0)$ (translation of E by $(-1, 0, 0)$), $D = (0, 0, 1)$, $G = (-1, 0, 1)$, (translation of D by $(-1, 0, 0)$).

From four completely defined classes $tr(S_1)$, $tr(S_2)$, $tr(S_5)$, and $tr(S_6)$ we have the following facets that do not belong to the second full-dimensional simplex so far: ABC, EBC, ADF, BDF, ADG, CDG. No class $tr(S_3)$ or $tr(S_4)$ can cover more than three of these facets, otherwise it will cover two facets from one simplex (facets ABC and EBC cannot be covered simultaneously because they are parallel). So each class covers exactly three.

We apply Lemma 13 for simplices $ABCD$ and S_3 with common facet ABC. We have three options for the fourth vertex of S_3 (this vertex forms a parallelogram with three vertices of $ABCD$): $H_1 = (1, 1, -1)$ (parallelogram with BCD), $H_2 = (1, 0, -1)$ (parallelogram with ABD), $H_3 = (0, 1, -1)$ (parallelogram with ACD).

Assume that $S_3 = ABCH_2$ ($S_3 = ABCH_3$ is similar). It contains facets ABC and ADG (ADG translated by $(1, 0, -1)$ is H_2BA), but does not contain other facets. For example, if it contains ADF, then lower point of S_3 (vertex with smallest z-coordinate, i.e. H_2) should be translated into lower point of this facet, i.e. A. But this translation does not match any facet of S_3 with ADF. Similarly with other "unpaired" facets, except EBC, but S_3 already has a facet parallel to EBC, which is ABC.

So, there is only one possible case for S_3 which is $ABCH_1$ (the translation class contains ABC, BDF, CDG). Similarly, there is only one case for $S_4 = EBCH_1$ (the translation class contains EBC, ADF, ADG).

We reconstructed the whole triangulation which is unique up to $\mathrm{GL}_3(\mathbb{Z})$-transformation. □

5 Enumeration of periodic triangulations in dimension 4

As with dimension 3, we first bound the relative volume of a four-dimensional simplices.

Proposition 15. *If $\mathcal{T}$ is a periodic triangulation of $\mathbb{Z}^4$, then volume of each four-dimensional simplex is* 1.

Proof. Let $ABCDE$ be an arbitrary simplex of $\mathcal{T}$ with volume at least 2. We can choose a coordinate system (with matrix transformation from $\mathrm{GL}_4(\mathbb{Z})$) such that vertices of $ABCDE$ will have coordinates represented by columns of the following matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & a \\ 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 & d \end{pmatrix},$$

where a, b, c, d are non-negative, $d \geqslant 2$, $a \leqslant b \leqslant c < d$.

If $c = 0$, then the point $(0,0,0,1)$ lies in the interior of AE which is impossible, so $c \geqslant 1$.

If $a + b \leqslant d$, then

$$\frac{(c-1)A + (d-c)D + E}{d} = \frac{(d-a-b)A + aB + bC}{d} \pmod 1,$$

but convex combinations in the left-hand side and the right-hand side represent points that lie on different faces of $ABCDE$ and the tiling will be non face-to-face.

If $b + c > d$, then

$$\frac{(b+c-d-1)A + (d-b)C + (d-c)D + E}{d} = \frac{(d-a)A + aB}{d} \pmod 1$$

which is again a contradiction.

So, $a + b > d \geqslant b + c$, which contradicts with the inequality $a \leqslant c$. □

Note that the proofs of this proposition and of the similar Proposition 11 for dimension 3 can be combined in the following corollary.

Corollary 16. *If $\mathcal{T}$ is an n-dimensional periodic triangulation, then all* 3- *and* 4-*dimensional faces have relative volume* 1.

This corollary allows us to formulate a local approach to enumeration of all periodic tilings. We used this approach in dimension 3 in the previous section and now we are going to use it in dimension 4. We can analyze local structure of the tiling $\mathcal{T}$ and show that given a simplex S and its facet F, there are only finitely many options to attach another simplex T at F without violating the face-to-face property. Unfortunately this method doesn't work if dimension $n \geqslant 5$ as shown in Section 6 below.

Theorem 17. *For a fixed four-dimensional simplex S and its facet F, there are at most 10 options for another simplex adjacent to S by F in a periodic triangulation of $\mathbb{Z}^4$.*

Proof. We already know, that all simplices have volume 1. We fix one simplex S_1 with vertices $A = (0,0,0,0)$, $B = (1,0,0,0)$, $C = (0,1,0,0)$, $D = (0,0,1,0)$, $E = (0,0,0,1)$, and find all possibilities for the vertex F of the simplex $S_2 = BCDEF$ adjacent to S_1 by facet $BCDE$. We know that F has coordinates (x,y,z,t) with $x+y+z+t = 2$. We will show that there are only 10 options for the vertex F.

We will do that by analyzing all possible remainders of coordinates of F modulo powers of 2. First, assume t is even, then at least one more number among x, y, z is even, say z. Then midpoint of BF has coordinates

$$\left(\frac{x+1}{2}, \frac{y}{2}, 0, 0\right) \pmod 1$$

and it is an integer translation of the midpoint of AB (if x and y are odd) or AC (if x and y are even). The only case that will not contradict that $\mathcal{T}$ is face-to-face is when BF is parallel and equal to AC. Similarly we get that CF is parallel and equal to AB, so $F = (1,1,0,0)$. Also we can get five more coordinate permutations of this point in the case F has an even coordinate, in all other cases x, y, z, t are odd.

We know that x, y, z, t are odd and their sum is 2, so possible cases for modulo 4 remainders are $(3,3,3,1)$ and $(1,1,1,3)$ (x, y, z, t are equivalent, so we will treat these cases as coordinates for (x,y,z,t) modulo 4). In the first case

$$\frac{B+C+D+F}{4} = \frac{3A+E}{4} \pmod 1,$$

and the tiling is non face-to-face, so only the case $(1,1,1,3)$ of remainders modulo 4 is possible.

Lemma 18. *For any $k \geqslant 2$ the remainders of the coordinates of F modulo 2^k are $(1,1,a,2^k-a)$ for some odd $a \in [0, 2^k-1]$, probably permuted.*

Proof. We prove the statement by induction on k. The basis of induction is true for $k = 2$ and $a = 1$. Suppose the lemma is true for k and we will prove it for $k+1$. All the coordinates in our proof could be permuted, and when we consider a coordinate modulo n we usually take a representative from the interval $[0, n)$.

The point F has coordinates $(1,1,a,2^k-a)$ modulo 2^k with odd $a < 2^k$. Taking in account that sum of all coordinates is 2 there are five options for the remainders of coordinates modulo 2^{k+1}:

- $F = (1, 1, a, 2^{k+1} - a) \pmod{2^{k+1}}$. This case satisfies requirements of the induction step.

- $F = (1, 1, 2^k + a, 2^k - a) \pmod{2^{k+1}}$. This case satisfies requirements of the induction step.

- $F = (1, 2^k + 1, a, 2^k - a) \pmod{2^{k+1}}$. One of numbers a or $2^k - a$ is less than 2^{k-1}, without loss of generality we can assume that $0 < a < 2^{k-1}$. Then the interval $(2^k, 2^{k+1})$ contains at least two multiples of a, so there is a positive odd number $b < 2^{k+1}$ such that $2^k < ab < 2^{k+1}$. Then $bF = (b, 2^k + b, ab, 2^{k+1} + 2^k - ab)$ modulo 2^{k+1}, and

$$\frac{bF + (2^k - b)C + (2^{k+1} - ab)D + (ab - 2^k)E}{2^{k+1}} = \\ = \left(\frac{b}{2^{k+1}}, 0, 0, 0\right) = \frac{(2^{k+1} - b)A + bB}{2^{k+1}} \pmod{1}$$

and the tiling is not face-to-face.

- $F = (1, 2^k + 1, 2^k + a, 2^{k+1} - a) \pmod{2^{k+1}}$. Then $(2^k + 1)F = (2^k + 1, 1, a, 2^k - a)$ modulo 2^{k+1}, and

$$\frac{(2^k + 1)F + (2^k - 1)B}{2^{k+1}} = \\ = \left(0, \frac{1}{2^{k+1}}, \frac{a}{2^{k+1}}, \frac{2^k - a}{2^{k+1}}\right) = \frac{(2^k - 1)A + C + aD + (2^k - a)E}{2^{k+1}} \pmod{1}$$

and the tiling is not face-to-face.

- $F = (2^k + 1, 2^k + 1, 2^k + a, 2^k - a) \pmod{2^{k+1}}$. Then $(2^k + 1)F = (1, 1, a, 2^{k+1} - a)$ modulo 2^{k+1}, and

$$\frac{(2^k + 1)F + (2^k - a - 1)D + aE}{2^{k+1}} = \\ = \left(\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}}, \frac{2^k - 1}{2^{k+1}}, 0\right) = \frac{(2^k - 1)A + B + C + (2^k - 1)D}{2^{k+1}} \pmod{1}$$

and the tiling is not face-to-face.

- $F = (2^k + 1, 2^k + 1, a, 2^{k+1} - a) \pmod{2^{k+1}}$. Then $(2^k + 1)F = (1, 1, 2^k + a, 2^k - a)$, and

$$\frac{(2^k + 1)F + (2^k - a)D + (a - 1)E}{2^{k+1}} = \\ = \left(\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}}, 0, \frac{2^k - 1}{2^{k+1}}\right) = \frac{(2^k - 1)A + B + C + (2^k - 1)E}{2^{k+1}} \pmod{1}$$

and the tiling is not face-to-face.

Thus, the induction step is proved. ☐

We proceed with the proof of the theorem.

We can take k such that $2^k > 2\max(|x|,|y|,|z|,|t|)$, then the only possibility for coordinates with remainder 1 modulo 2^k is 1, so two coordinates of F are 1's and two other add up to 0, so $F = (1, 1, a, -a)$ for some positive odd number a (or permutation). If $a \geqslant 3$, then

$$\frac{F + (a-1)B}{a} = \left(0, \frac{1}{a}, 0, 0\right) = \frac{(a-1)A + B}{a} \pmod 1,$$

and if the tiling is face-to-face then the edge BF is a translation of the edge AB. In that case $F = (0, 0, 0, 2)$, so it doesn't have all odd coordinates. Therefore $a = 1$ and $F = (1, 1, 1, -1)$.

In total we get 10 options for the point F: $(1, 1, 0, 0)$ (all six permutations), and $(1, 1, 1, -1)$ (all four permutations). ☐

```
Data: Dimension n, list L_n of possible simplices in dimension n, and for each
      simplex Δ ∈ L_n and facet F of Δ the list Adj(Δ, F) of possible adjacent
      simplices
Result: The list R of periodic triangulation in dimension n
R ← ∅
W ← {(D)} for D ∈ L_n
while W ≠ ∅ do
    N ← ∅
    for T ∈ W do
        if there is a facet F of a simplex Δ of T without adjacent simplex then
            for Δ' ∈ Adj(Δ, F) do
                if Δ' does not overlap with any simplex in T then
                    T' ← T ∪ {Δ'}
                    if T' is not equivalent to an element of N then
                        N ← N ∪ {T'}
                    end
                end
            end
        else
            R ← R ∪ {T}
        end
    end
    W ← N
end
```

Algorithm 1: Algorithm for enumerating the periodic triangulations in a fixed dimension

Theorem 19. *There are exactly four periodic triangulations of* $\mathbb{Z}^4$ *up to* $\mathrm{GL}_4(\mathbb{Z})$ *equivalence.*

Proof. We use Theorem 17 for an exhaustive computer-assisted search (see [8] for the software). We start from one simplex of volume 1 and add adjacent simplices one by one by considering all possibilities. The number of cases to consider is kept down by keeping only non-isomorphic partial tilings in memory. In Algorithm 1 the procedure is detailed. For $n = 4$ the input L_4 is just the 1 dimensional simplex via Corollary 16 and the adjacency are obtained in Theorem 17. In the end we get four non-equivalent triangulations three of which are Delaunay triangulations and the "red-triangular" triangulation [1, Example 5.13.1] which concludes the enumeration. □

6 Local approach in higher dimensions

In this section we show that local approach we used in Lemma 13 and Theorem 17 can not prove finiteness of non-equivalent triangulations in dimension at least 5.

Theorem 20. *For* $n \geqslant 5$ *there exist a simplex* S *of volume* 1 *and an infinite sequence* S_k *of simplices of volume* 1 *such that* $S \cap S_k$ *is a facet and the intersection of translations of* S *and* S_k *is a facet or a vertex of both or empty.*

Proof. We first consider the case $n = 5$.

We fix simplex $S = OABCDX$ where $O = (0, 0, 0, 0, 0)$, $X = (-1, 0, 0, 0, 0)$, $A = (0, 1, 0, 0, 0)$, $B = (0, 0, 1, 0, 0)$, $C = (0, 0, 0, 1, 0)$, and $D = (0, 0, 0, 0, 1)$. We will show that there are infinitely many options to choose a neighbor T of S adjacent by the facet $x_1 = 0$ such that $\mathbb{Z}^5$ translations of S and T do not violate the face-to-face property.

Let $X' = (1, 1, 1, 1, k + 1)$ for any $k \geqslant 0$, then $T = OABCDX'$ will satisfy this condition.

For any k both simplices S and T have volume 1, so S doesn't intersect translations of S, and T doesn't intersect translations of T. It is enough to show that an arbitrary integer translation of S doesn't intersect T other than by vertices or by the facet $x_1 = 0$.

Consider the translation S' of S by the integer vector (a, b, c, d, e). Assume the intersection $S' \cap T$ contains a point $\mathbf{x}$ which is not a vertex of T and has non-zero first coordinate. Since S satisfies the inequality $-1 \leqslant x_1 \leqslant 0$ and T satisfies the inequality $0 \leqslant x_1 \leqslant 1$ we must have $a = 1$. Since $\mathbf{x}$ is a point of intersection of S' and T, then $\mathbf{x}$ is in the cone with vertex $(0, b, c, d, e)$ generated by the vectors $(1, 0, 0, 0, 0)$, $(1, 1, 0, 0, 0)$, $(1, 0, 1, 0, 0)$, $(1, 0, 0, 1, 0)$, and $(1, 0, 0, 0, 1)$, the edges of S from the vertex X.

Then the point $(0, b, c, d, e)$ is in the cone with vertex $\mathbf{x}$ generated by negatives of these vectors. Since $\mathbf{x}$ is in T, so $(0, b, c, d, e)$ is in the convex hull of the 6 cones with vertices at vertices of T generated by the vectors $(-1, 0, 0, 0, 0)$, $(-1, -1, 0, 0, 0)$, $(-1, 0, -1, 0, 0)$, $(-1, 0, 0, -1, 0)$, and $(-1, 0, 0, 0, -1)$. We are interested only in the part of the convex hull of these 6 cones in the plane $x_1 = 0$, and the extremal points of these cones are 5 vertices of T (except $(1, 1, 1, 1, k + 1)$) and the points $(0, 1, 1, 1, k + 1)$, $(0, 0, 1, 1, k + 1)$,

$(0,1,0,1,k+1)$, $(0,1,1,0,k+1)$, and $(0,1,1,1,k)$. These are 5 points of intersection of the edges of the cone with the vertex at $(1,1,1,1,k+1)$ with the hyperplane $x_1 = 0$.

Thus the point $(0,b,c,d,e)$ is in the convex hull of the 10 points $(0,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$, $(0,0,0,0,1)$, $(0,1,1,1,k+1)$, $(0,0,1,1,k+1)$, $(0,1,0,1,k+1)$, $(0,1,1,0,k+1)$, and $(0,1,1,1,k)$.

Now we can see that b,c,d must be 0's or 1's. Since the convex hull is centrally symmetric with respect to the point $\left(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{k+1}{2}\right)$ we can assume that $b=c=0$. Then the point $(0,b,c,d,e)$ must be in the convex hull of only three points $(0,0,0,0,0)$, $(0,0,0,1,0)$, and $(0,0,0,0,1)$, and there is no such point except themselves. None of these points is in the interior of the convex hull of the 10 points above, so we have found a contradiction with existence of such a point $\mathbf{x}$.

For $n > 5$, we can take five-dimensional simplices S and S_k and add $n-5$ common vertices to each simplex. If the resulted n-dimensional simplices have volume 1, and this can be achieved, then their translations have desired intersection. □

7 Flipping and five-dimensional partial enumeration

Let us consider a periodic triangulation $\mathcal{T}$ of $\mathbb{Z}^n$. Given a simplex S and a facet F of S, we can consider the adjacent simplex $S(F)$ to S. The union of the vertex sets of S and $S(F)$ is a set of $n+2$ points and we call the convex hull of those $Cv(S,F)$.

Lemma 21. *([16]) Any n-dimensional convex polytope with $n+2$ vertices (called* repartitioning polytope*) admits exactly 2 triangulations.*

Proof. Suppose that the vertices are $v_1, \ldots, v_{n+2}$ then there is unique (up to a non-zero multiple) non-trivial linear relation of the form

$$a_1 v_1 + \cdots + a_{n+2} v_{n+2} = 0$$

satisfying $a_1 + \ldots + a_{n+2} = 0$. For $1 \leqslant j \leqslant n+2$ we define S_j the simplex formed by v_i for $i \in \{1,\ldots,n+2\} \setminus \{j\}$. The first triangulation is formed by the simplices S_j for j such that $a_j \geqslant 0$ and the second triangulation by the simplices S_j for j such that $a_j \leqslant 0$. □

Suppose now that the simplices of T contained in $Cv(S,F)$ have determined a tiling of it. Then the simplices in $Cv(S,F)$ form a triangulation and we can swap it into another triangulation. Unfortunately things are not always so simple. If we have $a_j = 0$ for some j then the vertices v_i for $i \in \{1,\ldots,n+2\} - \{j\}$ define a $(n-1)$-dimensional polytope with $n+1$ vertices. Therefore flipping the triangulation of $Cv(S,F)$ also flips the triangulation of the facets of $Cv(S,F)$. The set $Irr(S,F) = \mathrm{conv}\{v_i | a_i \neq 0\}$ defines a face of $Cv(S,F)$. Thus if one flips the triangulation in $Cv(S,F)$, then one needs to flip it in all repartitioning polytopes containing $Irr(S,F)$ as well. We call a family of such flips *coherent*. This kind of flip is sometimes called *bistellar flip* in the literature.

Note that in the case of Delaunay triangulations the flips that are considered are formed by several bistellar flips done at the same time.

Theorem 22. *There are at least* 950 *periodic triangulations of* $\mathbb{Z}^5$ *up to* $\mathrm{GL}_5(\mathbb{Z})$ *equivalence.*

Proof. Given a periodic triangulation of $\mathbb{Z}^5$ we consider all ways to do a coherent flipping on it. We thus obtain a set of new periodic triangulations. We insert element of this list into the list of known periodic triangulations if they are not isomorphic to a triangulation already known. We start from one arbitrary Delaunay triangulation of $\mathbb{Z}^5$. We finish when all periodic triangulations in the list have been treated. Since the finiteness of the set of periodic triangulation is not proved in dimension 5 this process was not guaranteed to terminate. But it did and yielded 950 periodic triangulations. The code is available at [8]. □

The list of 950 periodic triangulations (222 of them Delaunay) is interesting in its own right and is available at [8]. The volumes of the simplices in the list of 950 triangulations are 1 or 2 which corresponds to the possible volume of simplices in Delaunay tessellations. Given a simplex S of volume 1 and vertices $v_0, \ldots, v_5$ we can consider which simplices S' can be adjacent to S. Their vertex set will be of the form

$$\{w\} \cup \{v_j\}_{0 \leqslant j \leqslant 5, j \neq i} \text{ with } w = \sum_{k=0}^{5} b_k v_k \text{ and } 1 = \sum_{k=0}^{5} b_k.$$

Thus we can encode them by a pair $\{(b_0, \ldots, b_5), i\}$. Up to permutation with the list of 950 possible tilings we found following possibilities for the pairs:

$$\{(-1,1,1,0,0,0),0\} \quad \{(-1,-1,1,1,1,0),0\} \quad \{(-1,-1,-1,1,1,2),0\}$$
$$\{(-2,-1,1,1,1,1),1\} \quad \{(-1,-1,-1,-1,2,3),0\}.$$

The last possibility $\{(-1,-1,-1,-1,2,3),0\}$ does not show up in the case of Delaunay triangulations.

The symmetry of the tiling varies widely with one of the periodic tiling having a point group symmetry isomorphic to the symmetric group Sym(6).

Theorem 23. *Periodic triangulations of* $\mathbb{Z}^n$ *which are not Delaunay but are centrally symmetric exist for* $n \geqslant 5$.

Proof. For $n = 5$ it suffices to take one of the 23 triangulations out of 950 known in dimension 5 that are not Delaunay but are centrally symmetric. For $n > 5$ this tiling $\mathcal{T}$ can be extended with tiles of the form $\Delta \times [0,1]^{n-5}$ for Δ a 5-dimensional simplex of $\mathcal{T}$. By applying Lemma 8 (iii) for an arbitrary generic quadratic form we obtain a $\mathbb{Z}^n$-periodic triangulation. This triangulation is centrally symmetric since $x \mapsto -x$ is a symmetry of the original tiling but also of the quadratic form. □

Note that existence of a periodic centrally symmetric non-Delaunay triangulation for $n = 8$ was established in [18].

Theorem 24. *There exist non-regular periodic triangulations for* $n \geqslant 5$.

Proof. For $n = 5$ we apply the method of subsection 3.3 to one of the 950 triangulations of Theorem 22. The list of 3264 simplices of the triangulation number 430 that cannot be part of a regular triangulation is available at [8]. For $n > 5$ this tiling $\mathcal{T}$ can be extended with tiles of the form $\Delta \times [0, 1]^{n-5}$ with Δ a 5-dimensional simplex of $\mathcal{T}$. By applying Lemma 8 (iii) for an arbitrary generic quadratic form we obtain a $\mathbb{Z}^n$-periodic triangulation which is necessarily non-regular. □

8 Open problems

In this section we list a number of interesting questions that showed up in the course of this research.

8.1 Finiteness and enumeration

A natural question that we were unable to resolve is whether there are finitely many $\mathbb{Z}^n$-periodic triangulations of $\mathbb{Z}^n$ up to the action of $\mathrm{GL}_n(\mathbb{Z})$? Theorem 20 shows that a local approach considering only pairs of simplices will not work.

There are many related question. For example in a fixed dimension n, is the set of all periodic triangulations of $\mathbb{Z}^n$ connected by flipping? The resolution of such questions is certainly very hard since analogue questions about triangulations of the hypercube are still unsolved [5]. The resolution of the above connectedness would imply that the number of triangulations in dimension 5 is exactly 950.

A proof of finiteness in dimension 5 would not a priori give an algorithm for the enumeration since we do not know the possible volume of simplices nor the adjacencies between them.

8.2 Extensibility of partial triangulations

In a lot of contexts of this search we reach a point where we had a partial triangulation of $\mathbb{Z}^n$ and we wanted to extend it to a full triangulation. Is this always possible? If so what would be a process for obtaining such a triangulation? If this extensibility were true then we would have an infinity of types of periodic triangulation in dimension 5.

One possible way to consider the problem for arbitrary dimension would be following [4] to consider *constrained Delaunay triangulations* and see if the relevant notion could be extended to our case. It would require a twofold generalization: a generalization from dimension 2 to any dimension and a generalization to the periodic case.

8.3 Regularity

Is every periodic regular triangulation also Delaunay? The answer is not known. As we saw in Section 3 we can test regularity on finite subsets of $\mathbb{Z}^n$ by linear programming. But we need actually to define the height function all over $\mathbb{Z}^n$ and the corresponding "linear program" will be infinite-dimensional.

Is the "red-triangular" [1, Example 5.13.1] $\mathbb{Z}^4$-periodic triangulation regular? If this triangulation is restricted to a set of 12864 simplices containing 1224 points then we can find a corresponding function f which indicates that this triangulation is likely to be regular.

8.4 Volume of simplices

What is the maximum volume of a simplex in a periodic triangulation? So far in all cases considered, we found that the volumes of the simplices occurring was not higher than the volume of the simplices of the Delaunay triangulations in the same dimension which are 1, 2, 3 and 5, respectively in dimension $n \leqslant 4$, 5, 6 and 7, see [7]. We see no reason why this should always be the case.

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