A lower bound on the average degree forcing a minor

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Abstract

We show that for sufficiently large d and for $t \geqslant d+1$, there is a graph G with average degree $(1-\varepsilon)\lambda t\sqrt{\ln d}$ such that almost every graph H with t vertices and average degree d is not a minor of G, where $\lambda = 0.63817\ldots$ is an explicitly defined constant. This generalises analogous results for complete graphs by Thomason (2001) and for general dense graphs by Myers and Thomason (2005). It also shows that an upper bound for sparse graphs by Reed and Wood (2016) is best possible up to a constant factor.

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1 Introduction

Mader [20] first proved that for every graph H, every graph with sufficiently large average degree contains H as a minor[1]. The natural extremal question arises: what is the least average degree that forces H as a minor? To formalise this question, let $f(H)$ be the infimum of all real numbers d such that every graph with average degree at least d contains H as a minor. This value has been extensively studied for numerous graphs H, including small complete graphs [6, 12, 21, 28, 29], the Petersen graph [11], general complete graphs [2, 5, 14, 15, 21, 23, 30, 31], complete bipartite graphs [3, 16–19, 24], general dense graphs [25], general sparse graphs [9, 26, 27], disjoint unions of graphs [4, 13, 33], and disjoint unions of cycles [8]; see [32] for a survey.

For complete graphs K_t, the above question was asymptotically answered in the following theorem of Thomason [31], where

$$\lambda := \max_{x>0} \frac{1-e^{-x}}{\sqrt{x}} = 0.63817\ldots.$$

Theorem 1 ([31]). *Every graph with average degree at least $(\lambda + o(1))\, t\sqrt{\ln t}$ contains K_t as a minor. Conversely, there is a graph with average degree at least $(\lambda + o(1))\, t\sqrt{\ln t}$ that contains no K_t minor. That is,*

$$f(K_t) = (\lambda + o(1))\, t\sqrt{\ln t}.$$

Myers and Thomason [25] generalised this result for all families of dense graphs as follows[2].

Theorem 2 ([25]). *For every $\tau \in (0,1)$, for all t and $d \geqslant t^\tau$, for almost every graph H with t vertices and average degree d (and for every d-regular graph with t vertices),*

$$f(H) = (\lambda + o(1))\, t\sqrt{\ln d}.$$

Theorem 2 determines $f(H)$ for most dense graphs H with $d \geqslant t^\tau$, but says nothing for sparse graphs H, where d can be much smaller than t. In this regime, Reed and Wood [26, 27] established the following upper bound on $f(H)$.

Theorem 3 ([26, 27]). *For sufficiently large d, and for every graph H with t vertices and average degree d, every graph with average degree at least $3.895\, t\sqrt{\ln d}$ contains H as a minor. That is,*

$$f(H) \leqslant 3.895\, t\sqrt{\ln d}.$$

[1]A graph H is *a minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

[2]As is standard, we write that *almost every* graph with t vertices and average degree d satisfies property P if the probability that a random graph with t vertices and average degree d satisfies property P tends to 1 as $t \to \infty$.

The purpose of this paper is to show that this result is best possible up to a constant factor. Indeed, we precisely match the lower bounds in the work of Thomason [31] and Myers and Thomason [25], strengthening the lower bound in Theorem 2 by eliminating the assumption that $d \geqslant t^\tau$. Informally, we prove that if d is large, then almost every H with $t \geqslant d+1$ vertices and average d satisfies $f(H) \geqslant (1-\varepsilon)\lambda\, t\sqrt{\ln d}$. To state the result precisely, let $\mathcal{G}(t,m)$ be the space of random graphs with vertex-set $\{1,2,\dots,t\}$ and m edges. Thus $\mathcal{G}(t,td/2)$ is the space of random graphs with t vertices and average degree d.

Theorem 4. *For every $\varepsilon, c \in (0,1)$ there exists d_0 such that for every integer $d \geqslant d_0$ and for every integer $t \geqslant d+1$, there is a graph G with average degree at least $(1-\varepsilon)\lambda\, t\sqrt{\ln d}$ such that if $H \in \mathcal{G}(t,td/2)$ then $\mathbb{P}(H$ is a minor of $G) < c^t$, and in particular, $\mathbb{P}(f(H) \geqslant (1-\varepsilon)\lambda\, t\sqrt{\ln d}) > 1-c^t$.*

Note that in the proofs of Theorems 1 and 2 the host graph G is a random graph of appropriately chosen constant density. Indeed, every such extremal graph is essentially a disjoint union of pseudo-random graphs [23, 25]. However, random graphs themselves are not extremal when d is small compared to t. Indeed, Alon and Füredi [1] showed that if $d \leqslant \log_2 t$ then, for every graph H with t vertices and maximum degree d, a random graph on t vertices (with edge probability $\frac{1}{2}$) will almost certainly contain a *spanning* copy of H. To prove Theorem 4, we take G to be a blowup of a suitably chosen small random graph. Note that Fox [7] also considers minors of blowups of random graphs. On the face of it, such blowups might not to be pseudo-random, thus contradicting the fact that in many cases the extremal graphs are known to be pseudo-random. But the notion of pseudo-randomness involved is weak, asserting only that induced subgraphs of constant proportion have roughly the same density, and the blowups used here have this property.

Note that Reed and Wood [26] claimed that a lower bound analogous to Theorem 4 followed from the work of Myers and Thomason [25]. However, this claim is invalid. The error occurs in the footnote on page 302 of [26], where Theorem 4.8 and Corollary 4.9 of Myers and Thomason [25] are applied. The assumptions in these results mean that they are only applicable if the average degree of H is at least $|V(H)|^\varepsilon$ for some fixed $\varepsilon > 0$, which is not the case here. Also note that Reed and Wood [26] claimed that a $ct\sqrt{\log d}$ lower bound holds for every d-regular graph (also as a corollary of the work of Myers and Thomason [25]). This is false, for example, when H is the d-dimensional hypercube [10].

2 The Proof

We will need the following Chernoff Bound.

Lemma 5 ([22]). *Let $X_1, X_2, \dots, X_n$ be independent random variables, where each $X_i = 1$ with probability p and $X_i = 0$ with probability $1-p$. Let $X := \sum_{i=1}^n X_i$. Then for $\delta \in (0,1)$,*

$$\mathbb{P}(X \leqslant (1-\delta)pn) \leqslant \exp(-\tfrac{\delta^2}{2}pn).$$

Let G be a graph. For $\ell \in \mathbb{R}^+$, a non-empty set of at most ℓ vertices in G is called an *ℓ-set*. Two sets A and B of vertices in G are *non-adjacent* if there is no edge in G between A and B.

Our first lemma gives properties about a random graph.

Lemma 6. *Fix $p, \varepsilon, \alpha \in (0,1)$ and $\beta \in (\alpha, 1)$, and let $b := (1-p)^{-1}$. Then there exists d_0 such that for every integer $d \geqslant d_0$, if $s := \lceil d^\beta \rceil$ and $\ell := \sqrt{\alpha \log_b d}$, then there exists a graph G with exactly d vertices and more than $(\frac{1}{2} - \varepsilon)pd^2$ edges, such that for every set S of s pairwise disjoint ℓ-sets in G, more than $\frac{1}{2}d^{-\alpha}\binom{s}{2}$ pairs of ℓ-sets in S are non-adjacent.*

Proof. Let G be a graph on d vertices, where each edge is chosen independently at random with probability p. By Lemma 5, the probability that $|E(G)| \leqslant (\frac{1}{2} - \varepsilon)pd^2$ is less than $\frac{1}{2}$.

If A and B are disjoint ℓ-sets, then the probability that A and B are non-adjacent equals $(1-p)^{|A||B|} \geqslant (1-p)^{\ell^2} = d^{-\alpha}$. Consider a set S of s pairwise disjoint ℓ-sets in G. Let X_S be the number of pairs of elements of S that are non-adjacent. Since the elements of X_S are pairwise disjoint, Lemma 5 is applicable and implies that the probability that $X_S \leqslant \frac{1}{2}d^{-\alpha}\binom{s}{2}$ is at most $\exp(-\frac{1}{8}d^{-\alpha}\binom{s}{2}) \leqslant \exp(-\frac{1}{16}d^{\beta-\alpha}(s-1))$, which is at most $\frac{1}{2}(2d^\ell)^{-s}$ since d is sufficiently large.

The number of ℓ-sets is $\sum_{i=1}^{\ell}\binom{d}{i} \leqslant 2d^\ell$. Thus the number of sets of s pairwise disjoint ℓ-sets is at most $\binom{2d^\ell}{s} \leqslant (2d^\ell)^s$. By the union bound, the probability that $X_S \leqslant \frac{1}{2}d^{-\alpha}\binom{s}{2}$, for some set S of s pairwise disjoint ℓ-sets, is less than $\frac{1}{2}$.

Hence with positive probability, $|E(G)| > (\frac{1}{2} - \varepsilon)pd^2$ edges, and $X_S > \frac{1}{2}d^{-\alpha}\binom{s}{2}$ for every set S of s pairwise disjoint ℓ-sets. The result follows. □

The next lemma is the heart of our proof.

Lemma 7. *Fix $p, \varepsilon, \alpha, c \in (0,1)$ and let $b := (1-p)^{-1}$. Then there exists d_0 such that for every integer $d \geqslant d_0$ and for every integer $t \geqslant d+1$, there is a graph G with average degree at least $(1-\varepsilon)pt\sqrt{\alpha \log_b d}$ such that such that if $H \in \mathcal{G}(t, td/2)$ then $\mathbb{P}(H \text{ is a minor of } G) < c^t$.*

Proof. Let $\ell := \sqrt{\alpha \log_b d}$. Choose $\beta \in (\alpha, 1)$ and let $s := \lceil d^\beta \rceil$. We assume that d is sufficiently large as a function of α, β and ε to satisfy the inequalities occurring throughout the proof.

Let G_0 be the graph from Lemma 6 applied with $\frac{\varepsilon}{4}$ in place of ε. Thus $|V(G_0)| = d$ and $|E(G_0)| > (\frac{1}{2}\frac{\varepsilon}{4})pd^2 = \frac{1}{2}(1 - \frac{\varepsilon}{2})pd^2$, and for every set S of s pairwise disjoint ℓ-sets in G_0, more than $\frac{1}{2}d^{-\alpha}\binom{s}{2}$ pairs of ℓ-sets in S are non-adjacent. Call this property $(\star)$.

Let G be obtained from G_0 by replacing each vertex x by an independent set I_x of size

$$r := \left\lceil \left(1 - \frac{\varepsilon}{2}\right)\frac{t\ell}{d} \right\rceil,$$

and replacing each edge xy of G_0 by a complete bipartite graph between I_x and I_y. Note that

$$|V(G)| = dr = d\left\lceil \left(1 - \frac{\varepsilon}{2}\right)\frac{t\ell}{d} \right\rceil < \left(1 - \frac{\varepsilon}{4}\right)\ell t,$$

and

$$|E(G)| = |E(G_0)|\, r^2 > \left(1 - \frac{\varepsilon}{2}\right)\frac{pr^2d^2}{2} = \left(1 - \frac{\varepsilon}{2}\right)\frac{prd}{2}|V(G)| \geqslant \left(1 - \frac{\varepsilon}{2}\right)^2\frac{pt\ell}{2}|V(G)|.$$

Hence G has average degree $2\frac{|E(G)|}{|V(G)|} \geqslant (1-\frac{\varepsilon}{2})^2 pt\ell \geqslant (1-\varepsilon)pt\sqrt{\alpha \log_b d}$, as claimed. It remains to show that almost every graph H with t vertices and average degree d is not a minor of G.

A *blob* is a non-empty subset of $V(G_0)$. A *blobbing* $(B_1, B_2, \ldots, B_t)$ is an ordered sequence of t blobs with total size at most $|V(G)|$, such that each vertex of G_0 is in at most r blobs.

The motivation for these definitions is as follows: Suppose that a graph H is a minor of G and $V(H) = \{1, 2, \ldots, t\}$. Then for each vertex v of H there is a set $X_v \subseteq V(G)$, such that $X_v \cap X_w = \emptyset$ for distinct $v, w \in V(H)$, and for every edge vw of H, there is an edge in G between X_v and X_w. For each vertex v of H, let $B_v := \{x \in V(G_0) : X_v \cap I_x \neq \emptyset\}$, called the *projection* of X_v to G_0. Note that $\sum_v |B_v| \leqslant \sum_v |X_v| \leqslant |V(G)|$, and each vertex of G_0 is in at most r of $B_1, B_2, \ldots, B_t$. Thus $(B_1, B_2, \ldots, B_t)$ is a blobbing. Also note that by the construction of G, if $B_v \cap B_w = \emptyset$, then there is an edge of G between X_v and X_w if and only if there is an edge of G_0 between B_v and B_w.

Claim 1. *The number of blobbings is at most* $(4d)^{t\ell}$.

Proof. For positive integers d, t and for each positive integer $n \geqslant d$, let $g(d, t, n)$ be the number of t-tuples $(X_1, X_2, \ldots, X_t)$ such that X_i is a non-empty subset of $\{1, 2, \ldots, d\}$ for all $i \in \{1, 2, \ldots, t\}$, and $\sum_{i=1}^{t} |X_i| \leqslant n$. Below we prove that $g(d, t, n) \leqslant (4d)^n$ by induction on t. The result follows, since the number of blobbings is at most $g(d, t, |V(G)|) \leqslant g(d, t, \lfloor t\ell \rfloor)$.

In the base case, $g(d, 1, n) \leqslant 2^d \leqslant (4d)^n$, as desired. Now assume the claim for $t-1$. Observe that

$$g(d, t, n) = \sum_{i=1}^{d} \binom{d}{i} g(d, t-1, n-i).$$

By induction,

$$g(d, t, n) \leqslant \sum_{i=1}^{d} \binom{d}{i} (4d)^{n-i} \leqslant \sum_{i=1}^{d} \left(\frac{ed}{i}\right)^i (4d)^{n-i} = (4d)^n \sum_{i=1}^{d} \left(\frac{e}{4i}\right)^i < (4d)^n.$$

This completes the proof. □

Two blobs are a *good pair* if they are disjoint and non-adjacent ℓ-sets in G_0.

Claim 2. *Every blobbing has at least* $\frac{\varepsilon^2}{400} d^{-\alpha} t^2$ *good pairs.*

Proof. Suppose for a contradiction that some blobbing $(B_1, B_2, \ldots, B_t)$ has less than $\frac{\varepsilon^2}{400} d^{-\alpha} t^2$ good pairs. Let X be the set of blobs B_i such that $|B_i| \leqslant \ell$. Then $\ell(t - |X|) < |V(G)| < \left(1 - \frac{\varepsilon}{4}\right) \ell t$, implying $|X| > \frac{\varepsilon}{4} t$. Let Y be the set of blobs in X that belong to at most $\frac{\varepsilon}{20} d^{-\alpha} t$ good pairs. Thus the total number of good pairs is at least $\frac{\varepsilon}{40} d^{-\alpha} t |X \setminus Y|$, implying that $|X \setminus Y| < \frac{\varepsilon}{10} t$ and $|Y| > \frac{3\varepsilon}{20} t$. Let Z be a maximal subset of Y such that the blobs in Z are pairwise disjoint and contain at most $\frac{1}{2} d^{-\alpha} \binom{|Z|}{2}$ good pairs. Then $1 \leqslant |Z| < s$ by property $(\star)$ of G_0. Let Z' be the set of blobs in Y that are disjoint from every blob in Z. Since each blob in Z intersects at most ℓr other blobs,

$|Y| \leqslant |Z'| + \ell r|Z| < |Z'| + \ell rs$, and $|Z'| > \frac{3\varepsilon}{20}t - \ell rs \geqslant \frac{\varepsilon}{10}t$ for sufficiently large d. By the maximality of Z, every blob in Z' is in a good pair with at least $\frac{1}{2}d^{-\alpha}|Z|$ blobs in Z. So in total there are at least $\frac{1}{2}d^{-\alpha}|Z||Z'|$ good pairs $\{B_i, B_j\}$ with $B_i \in Z$ and $B_j \in Z'$. So some $B_i \in Z$ is in more than $\frac{1}{2}d^{-\alpha}|Z'| > d^{-\alpha}\frac{\varepsilon}{20}t$ good pairs, contradicting the definition of Y. □

Let H be a graph with $V(H) = \{1, \dots, t\}$. We say that a blobbing $(B_1, B_2, \dots, B_t)$ is *H-compatible* if for every $ij \in E(H)$ the blobs B_i and B_j intersect or are adjacent, implying that $\{B_i, B_j\}$ is not good. As explained above, if H is a minor of G, then there exists an H-compatible blobbing. By Theorem 2, if $H \in \mathcal{G}(t, m)$ then the probability that a given blobbing is H-compatible is at most

$$\binom{\binom{t}{2} - \frac{\varepsilon^2}{400}d^{-\alpha}t^2}{m} \Big/ \binom{\binom{t}{2}}{m} \leqslant \left(1 - \frac{\varepsilon^2 d^{-\alpha}}{200}\right)^m \leqslant \exp\left(-\frac{\varepsilon^2 m d^{-\alpha}}{200}\right).$$

Combining this inequality, Theorem 1 and the union bound, if $H \in \mathcal{G}(t, td/2)$ then

$$\mathbb{P}(H \text{ is a minor of } G) \leqslant (4d)^{t\ell} \exp\left(-\frac{\varepsilon^2 t d^{1-\alpha}}{400}\right),$$

which is less than c^t for sufficiently large d. □

Proof of Theorem 4. Choose $x > 0$ so that $\lambda = \frac{1-e^{-x}}{\sqrt{x}}$, let $b := e^x$, $p := 1 - e^{-x}$. Let $\alpha := (\frac{1-\varepsilon}{1-\varepsilon/2})^2$, implying $(1 - \frac{\varepsilon}{2})\sqrt{\alpha} = 1 - \varepsilon$. By Lemma 7, there exists d_0 such that for every integer $d \geqslant d_0$ and for every integer $t \geqslant d+1$, there is a graph G with average degree at least $(1 - \frac{\varepsilon}{2})pt\sqrt{\alpha \log_b d}$ such that if $H \in \mathcal{G}(t, td/2)$ then $\mathbb{P}(H$ is a minor of $G) < c^t$. Since

$$\left(1 - \frac{\varepsilon}{2}\right) pt\sqrt{\alpha \log_b d} = \left(1 - \frac{\varepsilon}{2}\right)\sqrt{\alpha}\left(\frac{1 - e^{-x}}{\sqrt{x}}\right) t\sqrt{\ln d} = (1 - \varepsilon)\lambda t\sqrt{\ln d},$$

the graph G satisfies the conditions of the theorem. □

We finish with the natural open problem that arises from this work: Can the constant in the upper bound of Reed and Wood [26] be improved to match the lower bound in the present paper? That is, is $f(H) \leqslant (\lambda + o(1))t\sqrt{\ln d}$ for every graph H with t vertices and average degree d?

Note Added in Proof

Following the initial release of this paper, Thomason and Wales [34] announced a solution to the above open problem.

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