

Deformation of Cayley's hyperdeterminants

Tommy Wuxing Cai

Department of Mathematics
University of Manitoba
Winnipeg, MB R3T 2N2 Canada

cait@myumanitoba.ca

Naihuan Jing*

Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205, U.S.A

jing@ncsu.edu

Submitted: Aug 14, 2018; Accepted: Jun 2, 2020; Published: Jun 12, 2020

© The authors. Released under the CC BY-ND license (International 4.0).

Abstract

We introduce a deformation of Cayley's second hyperdeterminant for even-dimensional hypermatrices. As an application, we formulate a generalization of the Jacobi-Trudi formula for Macdonald functions of rectangular shapes generalizing Matsumoto's formula for Jack functions.

Mathematics Subject Classifications: Primary: 05E05; Secondary: 17B69, 05E10

1 Introduction

For a $2m$ dimensional hypermatrix $A = (A(i_1, \dots, i_{2m}))_{1 \leq i_1, \dots, i_{2m} \leq n}$ Cayley's second hyperdeterminant [5] of A is a generalization of the usual determinant:

$$\det^{[2m]}(A) = \frac{1}{n!} \sum_{\sigma_1, \dots, \sigma_{2m} \in \mathfrak{S}_n} \prod_{i=1}^{2m} \operatorname{sgn}(\sigma_i) \prod_{i=1}^n A(\sigma_1(i), \dots, \sigma_{2m}(i)). \quad (1)$$

In [7] properties of hyperdeterminants and discriminants are studied in general. In particular, it is easy to see that $\det^{[2m]}(A)$ is invariant when A is replaced by $B \circ_k A$, where the action is defined for example, $(B \circ_1 A)(i_1, \dots, i_{2m}) = \sum_{j=1}^n B_{i_1, j} A(j, i_1, \dots, i_{2m})$ for any $B \in \operatorname{GL}_n$. This means that the hyperdeterminant is an invariant under the action of $\operatorname{GL}_n^{\otimes 2m}$. In general, it is a challenging problem to come up with relative invariants under the (non-diagonal) action of $\operatorname{GL}_n \times \dots \times \operatorname{GL}_n$.

*Supported by Simons Foundation grant no. 523868, Humbolt Foundation, and NSFC grant no. 11531004.

A q -analog Hankel determinant has been studied in [8] (see also [1]) and q -analog of the hyperdeterminant for non-commuting matrices has been introduced in the context of quantum groups [9].

In this paper, we introduce a λ -hyperdeterminant (of commuting entries) by replacing permutations with alternating sign matrices in a natural and nontrivial manner. We will show that the new hyperdeterminant deforms Cayley's hyperdeterminant. As an application, we give a Jacobi-Trudi type formula for Macdonald polynomials to generalize the previous formula of Matsumoto for Jack polynomials of rectangular shapes. It would be interesting to uncover the relation with q -Hankel determinants. Also the relationship with q -deformations remains mysterious.

2 λ -hyperdeterminants

To introduce the λ -hyperdeterminant, we first consider generalization of the permutation sign in the usual determinant, and then generalize Cayley's hyperdeterminants.

A permutation matrix is any square matrix obtained by permuting the rows (or columns) of the identity matrix, therefore its determinant is equal to the sign of the corresponding permutation. Each row or column of a fixed permutation matrix has only one nonzero entry along each row and each column. Generalizing the notion of permutation matrices, an alternating sign matrix is a square matrix with entries 0's, 1's and (-1) 's such that the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign. Therefore, each permutation matrix is an alternating sign matrix. However, there are more alternative sign matrices than permutation matrices of the same size. For example, the unique alternating sign 3×3 matrix which is not a permutation matrix is

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (2)$$

Clearly the nonzero entries in each row or column of an alternating sign matrix must start with 1. We denote by Alt_n the set of n by n alternating sign matrices and P_n the subset of permutation matrices of size n . The set P_n , endowed with the usual matrix multiplication, is a group isomorphic to the symmetric group \mathfrak{S}_n . As n increases, Alt_n contains more and more non-permutation matrices. In fact, it is known [17, 10] that Alt_n has a cardinality of $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$.

For $X = (x_{ij}) \in \text{Alt}_n$, we define the inversion number of X by [14]

$$i(X) = \sum_{r>i, s<j} x_{ij}x_{rs}.$$

If $P = (\delta_{i, \pi_i})$ is the permutation matrix associated to the permutation $\pi \in \mathfrak{S}_n$, Then $i(P)$ counts the number of pairs (i, j) such that $i < j$ but $\pi_i > \pi_j$, i.e., $i(P)$ is the inversion number of π , thus explains the name for alternating sign matrices.

Let $n(X)$ be the number of negative entries in X . For any $n \times n$ matrix $A = (a_{ij})$, we write $A^X = \prod_{i,j} a_{ij}^{x_{ij}}$. As usual if $x_{ij} = 0$, $a_{ij}^{x_{ij}} = 1$. In general A^X is a quotient of two monomials and the degree of the denominator being $n(X)$.

Let λ be a parameter and A an $n \times n$ matrix. Following Robbins and Rumsey [15], we introduce the λ -determinant as follows.

$$\begin{aligned} \det_\lambda(A) &= \sum_{X \in \text{Alt}_n} (-\lambda)^{i(X)} (1 - \lambda^{-1})^{n(X)} A^X \\ &= \sum_{\pi \in \mathfrak{S}_n} (-\lambda)^{i(\pi)} a_{1\pi_1} \cdots a_{n\pi_n} + \sum_{X \in \text{Alt}_n \setminus P_n} (-\lambda)^{i(X)} (1 - \lambda^{-1})^{n(X)} A^X. \end{aligned} \quad (3)$$

Note that the second equality holds even for $\lambda = 1$ due to $0^0 = 1$. We remark that the original λ -determinant [15] corresponds to a sign change of ours, while our definition deforms the usual determinant at $\lambda = 1$. In fact when $\lambda = 1$, the second summand vanishes, therefore $\det_1(A)$ reduces to the usual determinant. In general, $\det_\lambda(A)$ is a rational function or rather a polynomial function in the variable $a_{ij}^{\pm 1}$ as long as $a_{ij} \neq 0$ whenever $X_{ij} = -1$ for some $X \in \text{Alt}_n$. For any 3×3 matrix $A = (a_{ij})$

$$\begin{aligned} \det_\lambda(A) &= a_{11}a_{22}a_{33} - \lambda a_{12}a_{21}a_{33} - \lambda^3 a_{13}a_{22}a_{31} - \lambda a_{11}a_{23}a_{32} \\ &\quad + \lambda^2 a_{12}a_{23}a_{31} + \lambda^2 a_{13}a_{21}a_{32} + \lambda(1 - \lambda) \frac{a_{12}a_{21}a_{23}a_{32}}{a_{22}}, \end{aligned}$$

where the last term is due to the alternating sign matrix Q in (2).

Another example is a λ -deformation of the Vandermonde determinant [15]: For commuting variables x_i , $1 \leq i \leq n$, one has that

$$\det_\lambda(x_i^{j-1}) = \prod_{1 \leq i < j \leq n} (x_j - \lambda x_i), \quad (4)$$

which is a continuous deformation of the Vandermonde determinant at $\lambda = 1$. When $\lambda = -1$, this formula has an interesting connection with the Pfaffian Pf :

$$\frac{\det_1(x_i^{j-1})}{\det_{-1}(x_i^{j-1})} = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i} = \text{Pf}\left(\frac{x_j - x_i}{x_j + x_i}\right), \quad (5)$$

where the last equality is Schur's formula for Pfaffian [13]. Here the Pfaffian $\text{Pf}(M)$ of an antisymmetric matrix M is defined by $\text{Pf}(M) = \sqrt{\det(M)}$.

A permutation matrix P of size n sends the row vector $(1, 2, \dots, n)$ to the vector $(\pi(1), \pi(2), \dots, \pi(n))$ of its corresponding permutation $\pi \in \mathfrak{S}_n$ by right matrix multiplication, i.e.

$$(1, 2, \dots, n)P = (\pi(1), \pi(2), \dots, \pi(n)), \quad (6)$$

where π_i are distinct integers from $[1, n]$. Similarly P also sends the column vector by left matrix multiplication:

$$P(1, 2, \dots,)^t = (\pi^{-1}(1), \pi^{-1}(2), \dots, \pi^{-1}(n))^t.$$

Suppose $X = (x_{ij})$ is an alternating sign matrix of size n , we claim that X sends the vector $(1, 2, \dots, n)^t$ to the vector $(X(1), X(2), \dots, X(n))^t$ by right matrix multiplication and $1 \leq X(i) \leq n$. So for each alternating sign matrix X , we define the generalized permutation $(X(1), X(2), \dots, X(n))$ by

$$X \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} := \begin{bmatrix} X(1) \\ X(2) \\ \vdots \\ X(n) \end{bmatrix}, \quad (7)$$

where $X(i)$ may not be distinct (cf. (6)). Note that $(X(1), X(2), \dots, X(n))$ descends to the permutation $(\pi(1), \pi(2), \dots, \pi(n))$ when X is the permutation matrix corresponding to $\pi \in \mathfrak{S}_n$. For example, $(1, 2, 3)Q = (2, 2, 2)$ for Q in (2). Suppose there is $x_{ij} = -1$, then there are odd number of nonzero entries along the i th row and they are alternatively ± 1 located at the j_1, \dots, j_{2k+1} th columns such that $1 \leq j_1 < j_2 < \dots < j_{2k+1} \leq n$ and $k \geq 1$. Therefore

$$\begin{aligned} X(i) &= \sum_{j=1}^n jx_{ij} = j_1 - j_2 + j_3 - j_4 + \dots - j_{2k} + j_{2k+1} \\ &= j_1 + (j_3 - j_2) + \dots + (j_{2k+1} - j_{2k}) \geq k + 1 \end{aligned}$$

and

$$\sum_{j=1}^n jx_{ij} = (j_1 - j_2) + \dots + (j_{2k-1} - j_{2k}) + j_{2k+1} \leq j_{2k+1} \leq n.$$

Hence one always has that $1 \leq X(i) = \sum_{j=1}^n jx_{ij} \leq n$ for any alternating sign matrix $X = (x_{ij})$, and X is a non-permutation alternative sign matrix iff $X(i) > 1$ for all i , i.e., $(X(1), X(2), \dots, X(n))$ is not a permutation of $12 \dots n$.

For fixed real numbers q and a , we use the conventional q -analogue of a :

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}). \quad (8)$$

With this preparation, we now introduce the notion of λ -hyperdeterminants.

Definition 1. Let $M = (M(i_1, i_2, \dots, i_{2m}))_{1 \leq i_1, i_2, \dots, i_{2m} \leq n}$ be a $2m$ dimensional hypermatrix, the λ -hyperdeterminant of M is defined by

$$\begin{aligned} \det_{\lambda}^{[2m]}(M) &= \frac{(1 - \lambda)^n}{(\lambda; \lambda)_n} \sum_{X_1, X_2, \dots, X_{2m} \in \text{Alt}_n} \phi_{\lambda}(X_1, X_2, \dots, X_{2m}) \\ &\quad \times \prod_{i=1}^n M(X_1(i), X_2(i), \dots, X_{2m}(i)), \end{aligned} \quad (9)$$

where the generalized sign factor $\phi_\lambda(X_1, \dots, X_{2m})$ is the following expression

$$\prod_{r=1}^m (-\lambda^{r-1})^{i(X_r)} (1 - \lambda^{r-1})^{n(X_r)} (-\lambda^r)^{i(X_{m+r})} (1 - \lambda^r)^{n(X_{m+r})}, \quad (10)$$

and the generalized permutation $(X(1), \dots, X(n))$ for $X \in \text{Alt}_n$ was defined in (7).

Note that the λ -hyperdeterminant is different from the q -hyperdeterminant defined on quantum linear group [9].

We also remark that alternating sign matrices with $n(X_1) > 0$ do not contribute to the sum, the first summation variable X_1 is therefore always taken as a permutation matrix. Moreover, one can even fix the first indices to be $1, 2, \dots, n$ and drop the quantum factor.

We now explain how the λ -hyperdeterminant deforms the Cayley hyperdeterminant.

Proposition 2. *For any $2m$ -dimensional hypermatrix*

$$A = (a_{i_1, \dots, i_{2m}})_{1 \leq i_1, \dots, i_{2m} \leq n}$$

one has that

$$\lim_{\lambda \rightarrow 1} \det_\lambda^{[2m]}(A) = \det^{[2m]}(A). \quad (11)$$

Proof. When $\lambda = 1$ the summands in the right side of (9) (omitting the factor) vanish if one of X_i 's has at least one negative entry. Thus only permutation matrices contribute to the sum, so we can let $X_r = P_{\sigma_r} = (\delta_{j\sigma_r(i)}) \in \mathfrak{S}_n$ for $r = 1, \dots, 2m$. Subsequently $X_r(i) = \sum_{j=1}^n j X_r(i, j) = \sum_{j=1}^n j \delta_{j\sigma_r(i)} = \sigma_r(i)$, and we see that the right side of (9) matches exactly with that of (1). \square

3 Hyperdeterminant formula for Macdonald functions

We now generalize Matsumoto's hyperdeterminant formula for Jack polynomials [13] as an application of the λ -hyperdeterminant.

Recall that the classical Jacobi-Trudi formula expresses the Schur function associated to partition μ as a determinant of simpler Schur functions of row shapes:

$$s_\mu(x) = \det(s_{\mu_i - i + j}(x)). \quad (12)$$

In [13], Matsumoto gave a simple formula for Jack functions of rectangular shapes using the hyperdeterminant (see also [1]).

Proposition 3 ([13]). *Let k, s, m be positive integers. The rectangular Jack functions $Q_{(k^s)}(m^{-1})$ can be expressed compactly as follows.*

$$\frac{(sm)!}{(m!)^s} Q_{(k^s)}(m^{-1}) = s! \det^{[2m]}(Q_{k + \sum_{j=1}^m (i_{m+j} - i_j)})_{1 \leq i_1, \dots, i_{2m} \leq s}. \quad (13)$$

As the Schur function s_μ is the specialization $Q_\mu(1)$ of the Jack function, one sees immediately that this formula specializes to the Jacobi-Trudi formula for rectangular Schur functions.

Macdonald symmetric functions $Q_\mu(q, t)$ [12] are a family of orthogonal symmetric functions with two parameters q, t and labeled by partition μ . In this paper we consider the case that $t = q^m$, m any positive integer. When q approaches to 1, $Q_\mu(q, q^m)$ specializes to the Jack symmetric function $Q_\mu(m^{-1})$ [12] which has been used in many applications (eg. [2]). When the partition $\mu = (k^s)$, the symmetric function $Q_\mu(q, t)$ or $Q_\mu(m^{-1})$ is referred as the rectangular (shaped) Macdonald function.

We now state our main result.

Theorem 4. *Let k, s, m be positive integers. Up to a scalar factor, the rectangular Macdonald function $Q_{(k^s)}(q, q^m)$ is a $2m$ -dimensional λ -hyperdeterminant:*

$$\begin{aligned} & \frac{(q; q)_{sm}}{(q; q)_m^s} Q_{(k^s)}(q, q^m) \\ &= \frac{(q; q)_s}{(1-q)^s} \det_q^{[2m]}(Q_{k+\sum_{r=1}^m(i_{m+r}-i_r)}(q; q^m))_{1 \leq i_1, \dots, i_{2m} \leq s}, \end{aligned} \tag{14}$$

where $\det_q^{[2m]}$ is the λ -hyperdeterminant with $\lambda = q$.

Remark 5. When $q = 1$, one recovers the hyperdeterminant formula (13) for Jack functions of rectangular shapes. The formula also gives another expression of the general formula of Lassalle-Schlosser [11] (see also [16]) in the special shape.

To prepare its proof, we first recall the following result from [4].

Proposition 6. *Let $\rho = (k^s)$ be an rectangular partition with $k, s > 0$. Let m be a positive integer. Then $\frac{(q; q)_{sm}}{(q; q)_m^s} Q_\rho(q, q^m)$ is the coefficient of $z_1^k z_2^k \cdots z_s^k$ in the Laurent polynomial FG , where*

$$F = \prod_{1 \leq i < j \leq s} \left(z_i z_j^{-1}; q \right)_m \left(q z_i^{-1} z_j; q \right)_m \tag{15}$$

$$G = \prod_{i=1}^s \left(\sum_{j=0}^{\infty} Q_j(q, q^m) z_i^j \right). \tag{16}$$

We can now prove Theorem 4.

Proof. Recall that the q -Dyson Laurent polynomial (15) is a generalization of the Vandermonde polynomial. Using (4), we can rewrite the q -Dyson Laurent polynomial in the z_i

using λ -determinants.

$$\begin{aligned}
F &= \prod_{1 \leq i < j \leq s} \left(z_i z_j^{-1}; q \right)_m \left(q z_j z_i^{-1}; q \right)_m \\
&= \prod_{1 \leq i < j \leq s} \prod_{r=1}^m (z_j - q^{r-1} z_i)(z_j^{-1} - q^r z_i^{-1}) \\
&= \prod_{r=1}^m \det_{q^{r-1}}(z_i^{j-1}) \det_{q^r}(z_i^{1-j}) \quad (\text{using (4) for } s \text{ by } s \text{ matrices}) \\
&= \prod_{r=1}^m \left(\sum_{X \in \text{Alt}_s} (-q^{r-1})^{i(X)} (1 - q^{r-1})^{n(X)} \prod_{i,j} z_i^{(j-1)X(i,j)} \right) \\
&\quad \times \left(\sum_{X \in \text{Alt}_s} (-q^r)^{i(X)} (1 - q^r)^{n(X)} \prod_{i,j} z_i^{(1-j)X(i,j)} \right) \\
&= \sum_{X_1, \dots, X_{2m} \in \text{Alt}_s} \prod_{r=1}^m (-q^{r-1})^{i(X_r)} (-q^r)^{i(X_{m+r})} (1 - q^{r-1})^{n(X_r)} (1 - q^r)^{n(X_{m+r})} \\
&\quad \times \prod_{i=1}^s z_i^{\sum_{j=1}^s \sum_{r=1}^m j(X_r(i,j) - X_{m+r}(i,j))} \\
&= \sum_{X_1, \dots, X_{2m} \in \text{Alt}_s} \phi_q(X_1, \dots, X_{2m}) \prod_{i=1}^s z_i^{\sum_{j=1}^s \sum_{r=1}^m j(X_r(i,j) - X_{m+r}(i,j))}
\end{aligned}$$

where in the last equality we have used the fact that the sum of entries in each row of an alternating sign matrix is one. By Proposition 6, $\frac{(q; q)_{sm}}{(q; q)_m^s} Q_{(k^s)}(q, q^m)$ is the coefficient of $z_1^k z_2^k \dots z_s^k$ in

$$\begin{aligned}
FG &= \sum_{X_1, \dots, X_{2m} \in \text{Alt}_s} [\phi_q(X_1, \dots, X_{2m}) \prod_{i=1}^s z_i^{\sum_{j=1}^s \sum_{r=1}^m j(X_r(i,j) - X_{m+r}(i,j))} \\
&\quad \prod_{i=1}^s \left(\sum_{j=0}^{\infty} Q_j(q, q^m) z_i^j \right)],
\end{aligned}$$

and this coefficient is seen as

$$\sum_{X_1, \dots, X_{2m} \in \text{Alt}_s} \phi_q(X_1, \dots, X_{2m}) \prod_{i=1}^s Q_{k - \sum_{j=1}^s \sum_{r=1}^m j(X_r(i,j) - X_{m+r}(i,j))}(q, q^m).$$

We now show formula (14) holds. The entries of matrix M in Definition 1 is now given

by $M(i_1, i_2, \dots, i_{2m}) = Q_{k+\sum_{r=1}^m (i_{m+r}-i_r)}(q, q^m)$. Then we have that

$$\begin{aligned} & \prod_{i=1}^s M(X_1(i), X_2(i), \dots, X_{2m}(i)) \\ &= \prod_{i=1}^s M\left(\sum_{j=1}^s jX_1(i, j), \dots, \sum_{j=1}^s jX_{2m}(i, j)\right) \\ &= \prod_{i=1}^s Q_{k+\sum_{r=1}^m (\sum_{j=1}^s jX_{m+r}(i, j) - \sum_{j=1}^s X_r(i, j))}(q, q^m) \\ &= \prod_{i=1}^s Q_{k+\sum_{r=1}^m \sum_{j=1}^s j(X_{m+r}(i, j) - X_r(i, j))}(q, q^m), \end{aligned}$$

which then implies the formula (14). □

Acknowledgements

Both authors thank South China University of Technology at the initial stage of work. Jing also thanks Max-Planck Institute for Mathematics in the Sciences in Leipzig during the work.

References

- [1] H. Belbachir, A. Boussicault, and J.-G. Luque. Hankel hyperdeterminants, rectangular Jack polynomial and even powers of the Vandermonde. *J. Algebra*, 320:3911–3925, 2008.
- [2] W. Cai and N. Jing. On vertex operator realizations of Jack functions. *J. Algebraic Combinatorics*, 32:579–595, 2010.
- [3] T. W. Cai and N. Jing. Jack vertex operators and realization of Jack functions. *J. Algebraic Combinatorics*, 39:53–74, 2014.
- [4] T. W. Cai. Macdonald symmetric functions of rectangular shapes. *J. Combinatorial Theory, Ser. A*, 128:162–179, 2014.
- [5] A. Cayley. On the theory of determinants. Collected Papers, volume 1, Cambridge University Press, Cambridge, pages 63–80, 1889.
- [6] F. J. Dyson. Statistical theory of the energy levels of complex systems I. *J. Math. Phys.*, 3:140–156, 1962.
- [7] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. Discriminants, resultants and multidimensional determinants. Birkhäuser, Boston, 1994.
- [8] M. Ishikawa, H. Tagawa, and J. Zeng. A q-analogue of Catalan Hankel determinants. New trends in combinatorial representation theory, pages 19–41, RIMS Kôkyûroku Bessatsu, B11, Research Institute of Mathematical Sciences (RIMS), Kyoto, 2009.

- [9] N. Jing and J. Zhang. Quantum hyperdeterminants and hyper-Pfaffians. *Math. Zeit.*, 287(3-4):897-914, 2017.
- [10] G. Kuperberg. Another proof of the alternating-sign matrix conjecture. *International Math. Research Notices*, No. 3:139–150, 1996.
- [11] M. Lassalle and M. J. Schlosser. Inversion of the Pieri formula for Macdonald polynomials. *Adv. Math.*, 202:289–325, 2006.
- [12] I. G. Macdonald. Symmetric functions and Hall polynomials. 2nd edition, Oxford University Press, New York, 1995.
- [13] S. Matsumoto. Hyperdeterminantal expressions for Jack functions of rectangular shapes. *J. Algebra*, 320:612–632, 2008.
- [14] W. H. Mills, D. Robbins, and H. Rumsey. Alternating sign matrices and descending plane partitions. *J. Combinatorial Theory A, Ser. A*, 34:340-359, 1983.
- [15] D. Robbins and H. Rumsey. Determinants and alternating sign matrices. *Adv. Math.*, 62:169–184, 1986.
- [16] E.-A. Williams. A formula for N-row Macdonald polynomials. *J. Algebraic Combinatorics*, 21:111–130, 2005.
- [17] D. Zeilberger. Proof of the alternating sign matrix conjecture. *Electronic J. Combinatorics*, 3:#P13, 84 pp, 1996.