

A Toeplitz property of ballot permutations and odd order permutations

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Abstract

We give a new semi-combinatorial proof for the equality of the number of ballot permutations of length n and the number of odd order permutations of length n , which was originally proven by Bernardi, Duplantier and Nadeau. Spiro conjectures that the descent number of ballot permutations and certain cyclic weights of odd order permutations of the same length are equi-distributed. We present a bijection to establish a Toeplitz property for ballot permutations with any fixed number of descents, and a Toeplitz property for odd order permutations with any fixed cyclic weight. This allows us to refine Spiro's conjecture by tracking the neighbors of the largest letter in permutations.

Mathematics Subject Classifications: 05A05, 05A15, 05A19, 15B05

1 Introduction

Let \mathfrak{S}_n be the symmetric group of permutations of the set $[n] = \{1, 2, \dots, n\}$. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$. The *signature* of π is defined to be the sequence $(q_1, q_2, \dots, q_{n-1})$ where

$$q_i = \begin{cases} -1, & \text{if } \pi_i > \pi_{i+1}; \\ 1, & \text{if } \pi_i < \pi_{i+1}. \end{cases}$$

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Niven [8] found a determinantal formula for the number of permutations of length n with a prescribed signature, and showed that this number attains its maximum if and only if the signature is for an André permutation [1]; see de Bruijn [5] for a recursive proof. A pair (π_i, π_{i+1}) of letters is a *descent* (resp., an *ascent*) if $\pi_i < \pi_{i+1}$ (resp., $\pi_i > \pi_{i+1}$). Denote the number of descents (resp., ascents) of π by $\text{des}(\pi)$ (resp., $\text{asc}(\pi)$). We call the number

$$h(\pi) = \text{asc}(\pi_1\pi_2 \cdots \pi_n) - \text{des}(\pi_1\pi_2 \cdots \pi_n)$$

the *height* of π . The permutation π is said to be a *ballot permutation* if the height of any prefix of π is nonnegative, namely, $h(\pi_1\pi_2 \cdots \pi_i) \geq 0$ for all $i \in [n]$. The number of ballot permutations of height 0 in \mathfrak{S}_{2n+1} , or *Dyck permutations* of length n , is the Eulerian-Catalan number; see Bidkhori and Sullivan [3]. A classical coin-tossing game problem concerning the descent-ascent structure in a sequence of independent random variables of values ± 1 was considered by Chung and Feller [4].

In this paper, we are concerned with the following beautiful result which is due to Bernardi, Duplantier and Nadeau [2].

Theorem 1 (Bernardi et al.). *The number of ballot permutations of length n is*

$$p_n = \begin{cases} (n-1)!!^2, & \text{if } n \text{ is even,} \\ n!! \cdot (n-2)!!, & \text{if } n \text{ is odd,} \end{cases}$$

where $(2m-1)!! = (2m-1)(2m-3) \cdots 3 \cdot 1$.

Bernardi et al. [2] obtained Theorem 1 by considering more generalized paths containing horizontal steps, called *well-labelled positive paths*. They constructed a bijection between well-labelled positive paths of size n with k horizontal steps and matchings on $[2n]$ having k pairs (i, j) with $i \in [n]$ and $j \in [n+1, \dots, 2n-1]$, and thus obtained an explicit formula for the number of well-labelled positive paths of size n having k horizontal steps. Taking $k = 0$ in their formula yields Theorem 1.

Denote by \mathcal{P}_n the set of odd order permutations of $[n]$, viz., the set of permutations of $[n]$ whose every cycle is of odd length. By considering the neighbours of the letter n , we see that

$$|\mathcal{P}_n| = |\mathcal{P}_{n-1}| + (n-1)(n-2)|\mathcal{P}_{n-2}|.$$

This recurrence gives $|\mathcal{P}_n| = p_n$ immediately, where p_n is the number defined in Theorem 1. In order to find an analogue for the descent statistic in the context of odd order permutations, Spiro [9] introduced the following interesting notion. For a permutation π , Spiro defines

$$M(\pi) = \sum_c \min(\text{cdes}(c), \text{casc}(c)),$$

where the sum runs over all cycles c of π , with the *cyclic descent*

$$\text{cdes}(c) = |\{i \in [k] : c_i > c_{i+1} \text{ where } c_{k+1} = c_1\}|,$$

and the *cyclic ascent*

$$\text{casc}(c) = |\{i \in [k]: c_i < c_{i+1} \text{ where } c_{k+1} = c_1\}| = |c| - \text{cdes}(c),$$

where $|c|$ is the length the cycle c . We call

$$w(c) = \min(\text{cdes}(c), \text{casc}(c))$$

the *cyclic weight* of c , and $w(\pi) = M(\pi)$ the *cyclic weight* of π .

Conjecture 2 (Spiro). Let $n \geq 1$ and $0 \leq d \leq \lfloor (n-1)/2 \rfloor$. Then the number of ballot permutations of length n with d descents equals the number of odd order permutations of length n with cyclic weight d .

Spiro confirmed Conjecture 2 for $d \leq 3$ and for $d = \lfloor (n-1)/2 \rfloor$.

In the next section we give a new proof of Theorem 1 by using certain combinatorial decomposition of ballot permutations with respect to the neighbors of the letter n . In Section 3, we display a bijection to establish a Toeplitz property for the number of ballot permutations of length n with a fixed number of descents. A slight modification of the bijection gives the Toeplitz property for the number of odd order permutations of length n with a fixed cyclic weight. These Toeplitz properties lead us to a refinement conjecture of Conjecture 2; see Conjecture 8. In Section 4, we show some easy cases of Conjecture 8.

2 A new semi-bijective proof of Theorem 1

We give an overview of notation in combinatorics on words which will be of use; see [6, 7]. For any word w of length n , we denote its i th letter by w_i , denote its alphabet $\mathcal{A}(w) = \{w_1, \dots, w_n\}$, denote its length by $\ell(w) = n$, and write $w_{-1} = w_n$. Denote the *reversal* of w by $w' = w_n w_{n-1} \cdots w_1$. In particular, the reversal of the empty word ϵ is ϵ itself. We call a word u a *factor* (resp., *prefix*) of w if there exist words x and y such that $w = xuy$ (resp., $w = uy$). We say that w is *ballot* if $h(u) \geq 0$ for any prefix u of w .

The notions of reversal and factor have a cyclic version. Let $\pi = (c_1) \cdots (c_k) \in \mathfrak{S}_n$, where (c_i) are the cycles of π . We say that a word u is a *cyclic factor* of π if u is a factor of some word v such that (v) is a cycle of π . For example, the permutation $\pi = (145)(26837)$ has a cyclic factor (372) .

Denote by \mathcal{B}_n the set of ballot permutations of $[n]$. Denote

$$\mathcal{B}_{n,d} = \{\pi \in \mathcal{B}_n: \text{des}(\pi) = d\} \quad \text{and} \quad \mathcal{P}_{n,d} = \{\pi \in \mathcal{P}_n: w(\pi) = d\}.$$

For any letters i and j , denote by $\mathcal{B}_{n,d}(i, j)$ the set of permutations in $\mathcal{B}_{n,d}$ that contain the factor inj , by $\mathcal{P}_{n,d}(i, j)$ the set of permutations in $\mathcal{P}_{n,d}$ that contain the cyclic factor inj . For example, we have

$$\begin{aligned} \mathcal{B}_{4,1}(1, 3) &= \emptyset, & \mathcal{B}_{4,1}(3, 1) &= \{2341, 3412\}, \\ \mathcal{P}_{4,1}(1, 3) &= \{(143)\}, & \mathcal{P}_{4,1}(3, 1) &= \{(341)\}. \end{aligned}$$

We use lower-case letters b and p , in replace of \mathcal{B} and \mathcal{P} respectively, to denote the corresponding set cardinality, such as $p_{n,d}(i, j) = |\mathcal{P}_{n,d}(i, j)|$. It is clear that

$$b_{n,d} = p_{n,d} = \begin{cases} 1, & \text{if } d = 0; \\ 0, & \text{if } d > \lfloor (n-1)/2 \rfloor. \end{cases}$$

Lemma 3. For any $i, j \in [n]$ such that $|i - j| = 1$,

$$b_{n,d}(i, j) = b_{n-2,d-1} \quad \text{and} \quad p_{n,d}(i, j) = p_{n-2,d-1}.$$

Proof. Fix a letter i and suppose that $i < j$. Then $j = i + 1$. For $\pi \in \mathcal{B}_{n,d}(i, i + 1)$, define $\phi(\pi)$ to be the permutation obtained from π by removing the letters $i + 1$ and n and arranging the remaining letters in the order-preserving manner so that $\phi(\pi) \in \mathfrak{S}_{n-2}$. It is clear that ϕ is a bijection between the sets $\mathcal{B}_{n,d}(i, i + 1)$ and $\mathcal{B}_{n-2,d-1}$. The other case $i > j$ can be handled similarly. For odd order permutations, one obtains the desired equality by using the same operation of ϕ and keeping the cycle structure. \square

For any permutation π on a set of n positive integers, we define the *standard form* of π to be the permutation $\sigma \in \mathfrak{S}_n$ such that $\pi_i < \pi_j$ if and only if $\sigma_i < \sigma_j$ for all pairs (i, j) . Denote by \mathcal{B} the set of finite permutations on positive integers whose standard forms are ballot permutations. Let $\omega \in \mathcal{B} \setminus \{\epsilon\}$. We say that a ballot permutation $\pi \in \mathcal{B}_n$ is ω -decomposable if π is the concatenation $\alpha\omega\gamma\delta$ of factors $\alpha, \omega, \gamma, \delta$ such that

$$h(\alpha\omega\gamma) = h(\omega), \tag{2.1}$$

where α, γ and δ are allowed to be the empty word ϵ . Let $X_n(\omega)$ be the set of ω -decomposable ballot permutations $\pi \in \mathfrak{S}_n$. Define the ω -decomposition of a permutation $\pi \in X_n(\omega)$ to be the 4-tuple $(\alpha, \omega, \gamma, \delta)$ such that $\pi = \alpha\omega\gamma\delta$ and that γ is the longest word satisfying

$$\gamma'\omega_{-1} \in \mathcal{B}.$$

From the definitions we see that the ω -decomposition of any permutation in $X_n(\omega)$ uniquely exists. In this case, we write $\pi = (\alpha, \omega, \gamma, \delta)$. In addition, we will use the convenience $\gamma_{-1} = \omega_{-1}$ if $\gamma = \epsilon$.

Lemma 4. Suppose that $\pi = (\alpha, \omega, \gamma, \delta)$. If $\delta \notin \mathcal{B}$, then $\gamma_{-1} > \delta_1$ and $h(\omega) \neq 1$.

Proof. Denote by χ the characteristic function defined by $\chi(P) = 1$ if a proposition P is true, and $\chi(P) = 0$ if P is false. Suppose that $\delta \notin \mathcal{B}$. Then $\ell(\delta) \geq 2$. Let j be the minimum index such that

$$h(\delta_1\delta_2 \cdots \delta_j) = -1.$$

Then the factor $\eta = \gamma\delta_1\delta_2 \cdots \delta_j$, which is longer than γ , satisfies

$$\begin{aligned} h(\alpha\omega\eta) &= h(\alpha\omega\gamma) + (-1)^{\chi(\gamma_{-1} > \delta_1)} + h(\delta_1\delta_2 \cdots \delta_j) \\ &= h(\omega) + (-1)^{\chi(\gamma_{-1} > \delta_1)} - 1. \end{aligned} \tag{2.2}$$

On the other hand, note that

$$h(\delta_{j-1} \cdots \delta_2 \delta_1) = h(\delta_1 \delta_2 \cdots \delta_{j-1}) = 0 \quad \text{and} \quad \delta_{j-1} \cdots \delta_2 \delta_1 \in \mathcal{B}.$$

If $\gamma_{-1} < \delta_1$, then

$$h(\delta_j \delta_{j-1} \cdots \delta_1 \gamma_{-1}) = 0 \quad \text{and} \quad \delta_j \delta_{j-1} \cdots \delta_1 \gamma_{-1} \in \mathcal{B}.$$

Since $\gamma' \omega_{-1} \in \mathcal{B}$, we find $\eta' \omega_{-1} \in \mathcal{B}$. In this case, Eq. (2.2) reduces to $h(\alpha \omega \eta) = h(\omega)$, contradicting the choice of γ . This proves $\gamma_{-1} > \delta_1$.

Now assume that $h(\omega) = 1$. Then Eq. (2.2) reduces to

$$h(\alpha \omega \eta) = h(\omega) - 2 = -1,$$

contradicting the premise $\pi \in \mathcal{B}$. This completes the proof. \square

As will be seen, Theorem 5 is the key in the new proof of Theorem 1.

Theorem 5. *Let $\lambda = in(j-1)j$ and $\mu = (j-1)jni$ where $i+2 \leq j \leq n-1$. The map $(\alpha, \lambda, \gamma, \delta) \mapsto (\gamma', \mu, \alpha', \delta)$ is a bijection between the sets $X_n(\lambda)$ and $X_n(\mu)$, with the inverse map $(\alpha, \mu, \gamma, \delta) \mapsto (\gamma', \lambda, \alpha', \delta)$.*

Proof. For $\pi = (\alpha, \lambda, \gamma, \delta) \in X_n(\lambda)$, define $f(\pi) = \gamma' \mu \alpha' \delta$. We shall show that $f(\pi) \in X_n(\mu)$ and the μ -decomposition of $f(\pi)$ is $(\gamma', \mu, \alpha', \delta)$, i.e.,

- (1) $f(\pi) \in \mathcal{B}$;
- (2) $h(\gamma' \mu \alpha') = h(\mu)$;
- (3) $\alpha \mu_{-1} \in \mathcal{B}$; and
- (4) α' is the longest factor of $f(\pi)$ satisfying (2) and (3).

Note that (1) to (3) are independent of each other. We show (2) first for it will be of use in showing (1).

(2) Since the integers $\lambda_{-1} = j$ and $\mu_1 = j-1 \notin \mathcal{A}(\gamma)$ are adjacent, we obtain

$$h(\gamma' \mu_1) = h(\gamma' \lambda_{-1}) = -h(\lambda_{-1} \gamma).$$

One may similarly show that $h(\mu_{-1} \alpha') = -h(\alpha \lambda_1)$. Since $h(\lambda) = h(\mu) = 1$, we can verify that

$$\begin{aligned} h(\gamma' \mu \alpha') &= h(\gamma' \mu_1) + h(\mu) + h(\mu_{-1} \alpha') \\ &= -h(\lambda_{-1} \gamma) + (2 - h(\lambda)) - h(\alpha \lambda_1) \\ &= 2 - h(\alpha \lambda \gamma) = 2 - h(\lambda) = 1. \end{aligned} \tag{2.3}$$

(1) Since $\pi = (\alpha, \lambda, \gamma, \delta)$, we have $\gamma' \lambda_{-1} \in \mathcal{B}$. Replacing $\lambda_{-1} = j$ by the adjacent integer $j-1$ gives $\gamma' \mu_1 \in \mathcal{B}$. Since $\mu \in \mathcal{B}$, we find $\gamma' \mu \in \mathcal{B}$ immediately. Now, since $\alpha \in \mathcal{B}$, for

any factorization $\alpha' = \rho\tau$, we have $h(\tau) = -h(\tau') \leq 0$. Using Eq. (2.3), we can deduce that

$$h(\gamma'\mu\rho) \geq h(\gamma'\mu\rho) + h(\tau) \geq h(\gamma'\mu\alpha') - 1 = 0.$$

Hence $\gamma'\mu\alpha' \in \mathcal{B}$. Since $h(\lambda) = 1$, Lemma 4 implies $\delta \in \mathcal{B}$. Together with Eq. (2.3), we conclude $\gamma'\mu\alpha'\delta \in \mathcal{B}$.

(3) It is true since the word $\alpha\mu_{-1} = \alpha\lambda_1$ is a prefix of the ballot permutation π , and thus a ballot one.

(4) Assume that α' is not the longest factor of $f(\pi)$ satisfying (2) and (3). Then δ has a nonempty prefix ρ such that $\rho'\alpha\mu_{-1} \in \mathcal{B}$ and

$$h(\gamma'\mu\alpha'\rho) = 1.$$

Together with Eq. (2.3), we find

$$h(\alpha'_{-1}\rho) = h(\gamma'\mu\alpha'\rho) - h(\gamma'\mu\alpha') = 0. \tag{2.4}$$

On the other hand, since $\rho'\alpha\mu_{-1}$ is ballot, so is its prefix ρ' . Since δ is ballot, so is its prefix ρ . Therefore, the word ρ must be of height 0, and $h(\alpha'_{-1}\rho) \in \{\pm 1\}$, contradicting Eq. (2.4).

It remains to show that $f: X_n(\lambda) \rightarrow X_n(\mu)$ is a bijection. For $\sigma = (\alpha, \mu, \gamma, \delta) \in X_n(\mu)$, define $g(\sigma) = \gamma'\lambda\alpha'\delta$. Similar to the above argument, one may show that $g(\sigma) \in X_n(\lambda)$ and the λ -decomposition of $g(\sigma)$ is $(\gamma', \lambda, \alpha', \delta)$. By definition, we can derive that

$$gf(\pi) = g((\gamma', \mu, \alpha', \delta)) = ((\alpha')', \lambda, (\gamma')', \delta) = \pi.$$

Thus the composition gf is the identity on $X_n(\lambda)$. In the same fashion, one may show that fg is the identity on $X_n(\mu)$. Hence f is a bijection, with the inverse g . \square

Now we are in a position to give the new proof of Theorem 1.

Proof of Theorem 1. Let $n \geq 4$, $i \geq 1$ and $i + 2 \leq j \leq n - 1$. Let $b_n(i, j)$ be the number of ballot permutations in \mathcal{B}_n containing the factor inj . Define $\lambda = in(j - 1)j$ and $\mu = (j - 1)jni$. We claim that

$$|X_n(\lambda)| = b_n(i, j - 1) - b_n(i, j). \tag{2.5}$$

In fact, consider the involution Φ of exchanging the letters $j - 1$ and j on the set

$$Y_n = \mathcal{B}_n(i, j - 1) \setminus X_n(\lambda).$$

We shall show that $\Phi(Y_n) = \mathcal{B}_n(i, j)$, which implies Eq. (2.5) immediately because $X_n(\lambda) \subseteq \mathcal{B}_n(i, j - 1)$. Let $\pi \in Y_n$. If the letters $j - 1$ and j are not adjacent in π , then $\Phi(\pi) \in \mathcal{B}_n(i, j)$ and the letters $j - 1$ and j are not adjacent in $\Phi(\pi)$. Suppose that $j - 1$ and j are adjacent in π . Since $\pi \notin X_n(\lambda)$, there is no factor γ satisfying Eq. (2.1). In other words, the height of any prefix of π that is longer than $\alpha\lambda$ is at least 2, where $\alpha\lambda$

is the prefix of π ending at λ . Therefore $\Phi(\pi) \in \mathcal{B}_n(i, j)$. It is clear that the preimage of every permutation $\sigma \in \mathcal{B}_n(i, j)$ lies in Y_n . This proves Eq. (2.5).

Similarly, one may show that

$$|X_n(\mu)| = b_n(j, i) - b_n(j - 1, i). \tag{2.6}$$

By Theorem 5, Eqs. (2.5) and (2.6), and Lemma 3, for $i + 2 \leq j$, we have

$$\begin{aligned} b_n(i, j) + b_n(j, i) &= b_n(i, j - 1) + b_n(j - 1, i) \\ &= \dots = b_n(i, i + 1) + b_n(i + 1, i) = 2b_{n-2}. \end{aligned}$$

We also have $b_n(i, i + 1) + b_n(i + 1, i) = 2b_{n-2}$. Since the number of ballot permutations in \mathcal{B}_n ending with the letter n is b_{n-1} , we obtain

$$b_n = b_{n-1} + \sum_{i, j} b_n(i, j) = b_{n-1} + (n - 1)(n - 2)b_{n-2}.$$

It is trivial to check that $b_1 = b_2 = 1$. Since the sequence p_n admits the same recurrence and initial values, we conclude that $b_n = p_n$. \square

3 A Toeplitz property

Computer calculus gives that the matrices $(b_n(i, j))_{i, j}$ for $3 \leq n \leq 8$ are respectively

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 3 & 2 \\ 4 & 3 & 0 & 3 \\ 5 & 4 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & 6 & 3 & 1 \\ 9 & 0 & 9 & 6 & 3 \\ 12 & 9 & 0 & 9 & 6 \\ 15 & 12 & 9 & 0 & 9 \\ 17 & 15 & 12 & 9 & 0 \end{bmatrix}, \\ &\begin{bmatrix} 0 & 45 & 36 & 27 & 19 & 13 \\ 45 & 0 & 45 & 36 & 27 & 19 \\ 54 & 45 & 0 & 45 & 36 & 27 \\ 63 & 54 & 45 & 0 & 45 & 36 \\ 71 & 63 & 54 & 45 & 0 & 45 \\ 77 & 71 & 63 & 54 & 45 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 225 & 182 & 139 & 99 & 65 & 38 \\ 225 & 0 & 225 & 182 & 139 & 99 & 65 \\ 268 & 225 & 0 & 225 & 182 & 139 & 99 \\ 311 & 268 & 225 & 0 & 225 & 182 & 139 \\ 351 & 311 & 268 & 225 & 0 & 225 & 182 \\ 385 & 351 & 311 & 268 & 225 & 0 & 225 \\ 412 & 385 & 351 & 311 & 268 & 225 & 0 \end{bmatrix}. \end{aligned}$$

A square matrix (a_{ij}) is said to be *Toeplitz* if $a_{i+1, j+1} = a_{i, j}$ for all well defined entries $a_{i+1, j+1}$ and $a_{i, j}$. In this section, we manage to show that the matrices

$$B(n, d) = (b_{n, d}(i, j))_{i, j=1}^{n-1} \quad \text{and} \quad P(n, d) = (p_{n, d}(i, j))_{i, j=1}^{n-1}$$

are Toeplitz for any number d .

Theorem 6. *The matrix $B(n, d)$ is Toeplitz for all n and d .*

Proof. Suppose that $n \geq 4$, $1 \leq d \leq \lfloor (n-1)/2 \rfloor$, and $i, j \in [n-2]$. This proof is organized as follows. First we give necessary notation to define a map T between the sets $\mathcal{B}_{n,d}(i, j)$ and $\mathcal{B}_{n,d}(i+1, j+1)$. Second we interpret some facts implicitly contained in the definition. We then make some efforts to show that T is well defined. Finally we prove that T is bijective, which will complete the proof.

Let $m = \min(i, j)$ and $M = \max(i, j)$. We introduce the notation

$$\bar{x} = \begin{cases} M+1, & \text{if } x = M, \\ x, & \text{otherwise,} \end{cases} \quad \text{and} \quad \underline{x} = \begin{cases} m, & \text{if } x = m+1, \\ x, & \text{otherwise,} \end{cases}$$

and define the words

$$\overline{(k; l)} = \overline{k(k+1) \cdots (k+l-1)} \quad \text{and} \quad \underline{(k+1; l)} = \underline{(k+1)k \cdots (k-l+2)}$$

for any k . In particular, $\overline{(k; 0)} = \underline{(k+1; 0)} = \epsilon$. For $\pi \in \mathcal{B}_{n,d}(i, j)$, define the *lower width* l_π to be the length l of the longest factor of the form $\overline{(m; l)}$ or $\underline{(m; l)}$ if the letters M and $M+1$ are not adjacent in π ; and define $l_\pi = 0$ if M and $M+1$ are adjacent. Note that $l_\pi \leq M - m + 1$ since the letter $M+1$ occurs twice in the word $\underline{(m; M-m+2)}$. Define the *lower core* of π to be the word

$$\kappa_\pi = \begin{cases} \overline{(i; l_\pi)}' n j, & \text{if } i < j, \\ i n \underline{(j; l_\pi)}, & \text{if } i > j. \end{cases} \quad (3.1)$$

Then κ_π is a factor of π of length $l_\pi + 2$. For example, if $\pi = 382549671 \in \mathcal{B}_{9,4}(4, 6)$, we have $l_\pi = 0$ since $M = 6$ is adjacent to $M+1$, and $\kappa_\pi = 96$. Similarly for $\sigma = 134875962$, we have $\kappa_\sigma = 7596$. Define a map

$$T: \mathcal{B}_{n,d}(i, j) \rightarrow \mathcal{B}_{n,d}(i+1, j+1)$$

by firstly replacing the lower core κ_π by the word

$$T|_{\kappa_\pi} = \begin{cases} (i+1)n \underline{(j+1; l_\pi)}, & \text{if } i < j, \\ (i+1; l_\pi)' n (j+1), & \text{if } i > j, \end{cases} \quad (3.2)$$

and secondly substituting the letter \bar{x} by the letter $\underline{x - l_\pi + 1}$ for each element $x \in [m + l_\pi, M] \setminus \mathcal{A}(\kappa_\pi)$. For example, for the permutation π , we first replace the word 96 with 59, then because $5 \in [4+0, 6] \setminus \{6, 9\}$ we replace it by $\overline{5-0+1} = \overline{6} = 6$. Note that we do nothing to the other 5 letters since T restricted to that part of the word is defined by the above process. In other words, $T(\pi) = 382645971$. Similarly, we have $T(\sigma) = 134869752$.

We call the first operation the *core replacement*, and the second the *straightening*. We will show that T is well defined, i.e., $T(\pi) \in \mathcal{B}_{n,d}(i+1, j+1)$ for any $\pi \in \mathcal{B}_{n,d}(i, j)$, and that T is bijective. By definition, we observe that the map T operates in the following 3 steps:

1. The core replacement is length-preserving and position-preserving, namely the words $T|_{\kappa_\pi}$ and κ_π have the same length and the same positions. Every letter in the set

$$\mathcal{A}(\kappa_\pi) = \{\overline{m}, \overline{m+1}, \dots, \overline{m+l_\pi-1}, M, n\}$$

is replaced by a letter in the set

$$\mathcal{A}(T|_{\kappa_\pi}) = \{m+1, \underline{M-l_\pi+2}, \underline{M-l_\pi+3}, \dots, \underline{M+1}, n\}. \quad (3.3)$$

In particular, the letter n is contained in both words κ_π and $T|_{\kappa_\pi}$.

2. The straightening operation maps the letters in the set

$$[m, M+1] \setminus \mathcal{A}(\kappa_\pi) = \{\overline{m+l_\pi}, \overline{m+l_\pi+1}, \dots, \overline{M}\}$$

to the letters in the set

$$[m, M+1] \setminus \mathcal{A}(T|_{\kappa_\pi}) = \{\underline{m+1}, \underline{m+2}, \dots, \underline{M-l_\pi+1}\}$$

in the order-preserving manner, where $\overline{M} = M+1$ and $\underline{m+1} = m$.

3. The letters that are not dealt with in the first two steps constitute the union $[m-1] \cup [M+2, n-1]$, and are unchanged by T .

From the above observations we can infer that $T(\pi) \in \mathfrak{S}_n$. Fix $\pi \in \mathcal{B}_{n,d}(i, j)$. Then π induces a unique bijection $T_\pi: [n] \rightarrow [n]$ which maps the letter π_i to the letter $T(\pi)_i$ for all $i \in [n]$. Note that the preimage of every letter in the interval $[m, M+1]$ under T_π must be in the same set $[m, M+1]$.

Now, we show that $T(\pi) \in \mathcal{B}_{n,d}(i+1, j+1)$. More precisely, we need to show

- (i) the word $T(\pi)$ contains the factor $(i+1)n(j+1)$;
- (ii) $\text{des}(T(\pi)) = \text{des}(\pi)$; and
- (iii) $T(\pi) \in \mathcal{B}$.

Suppose that $l_\pi = 0$. Then $\kappa_\pi \in \{nM, Mn\}$, and π contains the factor

$$\iota = mnM(M+1)$$

or its reversal. From the definition of the core replacement, $T|_{\kappa_\pi}$ is the factor $(m+1)n$ or its reversal. Since the letter m (resp., $M+1$) is the smallest (resp., largest) one in the interval $[m, M+1]$, it is invariant in the order-preserving straightening. Therefore, the image $T(\pi)$ contains the factor

$$T|_\iota = m(m+1)n(M+1)$$

or its reversal. This proves (i). Note that each of the words ι and $T|_\iota$ has a unique descent, and the reversal operation exchanges descents and ascents. Since the straightening is

order-preserving, the map T preserves all descents and ascents that has empty intersection with the cores. As a result, we obtain (ii). Moreover, the unique descent in $T|_\iota$ appears later than the unique descent appears in ι , and the unique ascent in $T|'_\iota$ appears earlier than that in ι . Note that the height of π is sufficiently large for the existence of the factor ι in π . Since π is ballot, so is $T(\pi)$. This proves (iii) and $T(\pi) \in \mathcal{B}_{n,d}(i+1, j+1)$ for $l_\pi = 0$.

From now on we can assume that $l_\pi \geq 1$. Then (i) is clear from the definition of the core replacement. In order to show (ii), we claim the equivalence

$$u < v \iff T_\pi(u) < T_\pi(v) \tag{3.4}$$

for any adjacent letters u and v such that $\{u, v\} \not\subseteq \mathcal{A}(\kappa_\pi)$. Since the straightening is order-preserving, Equivalence (3.4) is true if $\{u, v\} \cap \mathcal{A}(\kappa_\pi) = \emptyset$. Below we can suppose that

$$v \in \{M, \overline{m + l_\pi - 1}\} \subseteq \mathcal{A}(\kappa_\pi) \quad \text{and} \quad u \notin \mathcal{A}(\kappa_\pi),$$

because if (3.4) holds for the smallest and largest letters in $\mathcal{A}(\kappa_\pi)$, then it holds for all of the letters.

If $v = \overline{m + l_\pi - 1}$, then $T_\pi(v) = m + 1$ by the core replacement. Since $u \notin \mathcal{A}(\kappa_\pi)$, the letter u is not mapped in the core replacement. If $T_\pi(u) = m$, then the letter u is mapped in the straightening operation. Since m is the smallest image in the straightening, its preimage u must be the smallest element in the straightening, that is, $u = T_\pi^{-1}(m) = \overline{m + l_\pi}$. Now, the elements $\overline{m + l_\pi} = u$ and $\overline{m + l_\pi - 1} = v$ are adjacent, contradicting the definition of the lower width l_π . Therefore, $T_\pi(u) \neq m$. We then obtain the equivalence

$$T_\pi(u) < T_\pi(v) \iff T_\pi(u) \leq m - 1.$$

On the other hand, by the premise $u \notin \mathcal{A}(\kappa_\pi)$ and $v \in \mathcal{A}(\kappa_\pi)$, we deduce the equivalence

$$u < v = \overline{m + l_\pi - 1} \iff u \leq \overline{m} - 1 = m - 1.$$

Since all letters in the set $[m - 1]$ are fixed under T_π , we obtain the desired Equivalence (3.4) by combining the above two equivalences.

If $v = M$, then $T_\pi(v) = \overline{M - l_\pi + 2}$. Since $l_\pi \neq 0$, we find $u \neq M + 1$. Since $u \notin \mathcal{A}(\kappa_\pi)$, we obtain the equivalence

$$u > v \iff u \in [M + 2, n - 1].$$

On the other hand, since $u \notin \mathcal{A}(\kappa_\pi)$, we deduce $T_\pi(u) \notin \mathcal{A}(T(\kappa_\pi))$. In view of Eq. (3.3), we obtain the equivalence

$$T_\pi(u) > T_\pi(v) \iff T_\pi(u) \in [M + 2, n - 1].$$

Since the map T_π restricted on the set $[M + 2, n - 1]$ is the identity, we obtain Equivalence (3.4) as desired.

By Equivalence (3.4), the map T preserves all descents and ascents that are not entirely contributed by the lower core. For those entirely contained in the lower core, we note both the numbers of descents of the words κ_π and $T|_{\kappa_\pi}$ equal the lower width l_π . Therefore, T preserves the total number of descents. This proves (ii).

For (iii), we observe that in the core replacement, either

- both the preimage κ_π and its image $T|_{\kappa_\pi}$ contain a unique ascent, and the ascent in $T|_{\kappa_\pi}$ appears earlier than the ascent in κ_π appears, or
- both of them contain a unique descent, and the descent in $T|_{\kappa_\pi}$ appears later than the descent in κ_π appears.

Since T preserves all other descents and ascents, the premise $\pi \in \mathcal{B}$ implies (iii). This confirms that $T(\pi) \in \mathcal{B}_{n,d}(i+1, j+1)$.

It remains to show that T is bijective. We define the *upper width* l'_σ to be the length l of the longest factor of the form $(M+1; l)$ or $(M+1; l)'$, if the letters m and $m+1$ are not adjacent in σ ; and define $l'_\sigma = 0$ otherwise. Define the *upper core* of σ to be the word

$$\rho_\sigma = \begin{cases} (i+1)n(j+1; l'_\sigma), & \text{if } i < j, \\ \overline{(i+1; l'_\sigma)'}n(j+1), & \text{if } i > j. \end{cases}$$

Define a map

$$T': \mathcal{B}_{n,d}(i+1, j+1) \rightarrow B_{n,d}(i, j)$$

as follows. For $\sigma \in \mathcal{B}_{n,d}(i+1, j+1)$, define $T'(\sigma)$ to be the permutation obtained from σ by firstly replacing its upper core by

$$T'|_{\rho_\sigma} = \begin{cases} \overline{(i; l'_\sigma)'}nj, & \text{if } i < j, \\ in(j; l'_\sigma), & \text{if } i > j, \end{cases}$$

and then replacing each letter in the set $[m, M+1] \setminus \mathcal{A}(\rho_\sigma)$ to a letter in the set $[m, M+1] \setminus \mathcal{A}(T'|_{\rho_\sigma})$ in the order-preserving manner.

We shall show that

$$l_\pi = l'_{T(\pi)}. \tag{3.5}$$

In fact, if $l_\pi = 0$, then π contains the factor ι or its reversal. It follows that σ contains the factor $T|_\iota$ or its reversal. Thus $l'_\sigma = 0 = l_\pi$. Suppose that $l_\pi \geq 1$. Then $m+1$ and m are not adjacent in σ , since otherwise the letters

$$\overline{m+l_\pi-1} = T_\pi^{-1}(m+1) \quad \text{and} \quad \overline{m+l_\pi} = T_\pi^{-1}(m)$$

would be adjacent in π , which contradicts the definition of l_π . Thus $l'_\sigma \geq l_\pi$ from the definition of the core replacement. Furthermore, the letters $\overline{M-l_\pi+2}$ and $\overline{M-l_\pi+1}$ are not adjacent in σ , since otherwise the letters

$$M = T_\pi^{-1}(\overline{M-l_\pi+2}) \quad \text{and} \quad M+1 = T_\pi^{-1}(\overline{M-l_\pi+1})$$

would be adjacent in π , which contradicts $l_\pi \geq 1$. This proves Eq. (3.5).

In the same fashion one may prove that $l'_\sigma = l_{T'(\sigma)}$. As a consequence, both compositions TT' and $T'T$ are identities. This completes the whole proof. \square

We remark that the map T_π can be represented letter-wise using a piecewise function, with the aid of the number l_π .

Note that the key notion of lower and upper core widths, and that of lower and upper cores, are defined in a local structure of a permutation. In fact, the map T looks for the longest factor consisting of discretely continuous numbers starting from m . This localness perspective inspires us to translate these ideas from permutations to certain cyclic words, especially to each cycle of a permutation in its cycle representation, and obtain Theorem 7.

Theorem 7. *The matrix $P(n, d)$ is symmetric and Toeplitz for all n and d .*

Proof. In this proof, we keep in mind that permutations are considered to be unions of cycles, and forget their nature as maps on the set $[n]$. Since taking reversal cycle-wise is an involution between the sets $\mathcal{P}_{n,d}(i, j)$ and $\mathcal{P}_{n,d}(j, i)$, the matrix $P(n, d)$ is symmetric. Let $n \geq 4$ and $1 \leq d \leq \lfloor (n-1)/2 \rfloor$. In order to show the Toeplitz property, namely $P(n, d)_{i,j} = P(n, d)_{i+1, j+1}$ for $1 \leq i < j \leq n-2$, let $M = \max(i, j)$ and $m = \min(i, j)$.

For $\pi \in \mathcal{P}_{n,d}(i, j)$, let C_π be the cycle of π containing the maximum letter n . Paralleling the notion of lower width by considering cyclic factors instead of factors, we define the lower width l_π to be the length l of the longest *cyclic factor* of C_π that is of the form $\overline{(m; l)}$ or $\overline{(m; l)'}$, if the letters M and $M+1$ are not adjacent in C_π ; and define $l_\pi = 0$ otherwise. The subsequent notion of the lower core and core replacement are exactly the same as those in the proof of Theorem 6, using the cyclic version of the lower width. Define the map

$$T: \mathcal{P}_{n,d}(i, j) \rightarrow \mathcal{P}_{n,d}(i+1, j+1)$$

by running firstly the core replacement and then the straightening.

For example, if $\pi = (1, 6, 8, 2, 10)(3, 12, 9, 11, 7, 5, 4) \in \mathcal{P}_{12,4}(3, 9)$, then

$$C_\pi = (3, 12, 9, 11, 7, 5, 4), \quad l_\pi = 3, \quad \kappa_\pi = (5, 4, 3, 12, 9) \text{ and } T|_{\kappa_\pi} = (4, 12, 10, 9, 8).$$

The straightening maps the letters in

$$[3, 10] \setminus \mathcal{A}(\kappa_\pi) = \{6, 7, 8, 10\}$$

to the letters in

$$[3, 10] \setminus \mathcal{A}(T|_{\kappa_\pi}) = \{3, 5, 6, 7\}$$

in the order-preserving manner. As a result, we obtain

$$T(\pi) = (1, 3, 6, 2, 7)(10, 9, 8, 11, 5, 4, 12) \in \mathcal{P}_{12,4}(4, 10).$$

While the core replacement affects letters only inside the cycle C_π , the straightening works on letters that are both inside and outside C_π . Similarly to the proof of Theorem 6, we can deduce the following one by one:

- (i) the width $l_\pi \leq M - m + 1$;
- (ii) the image $T(\pi)$ is a permutation of length n , and contains the cyclic factor $(i + 1)n(j + 1)$;
- (iii) the map T preserves the length of each cycle of π , and the permutation $T(\pi)$ is of odd order;
- (iv) the map T preserves the cyclic descents outside κ_π , and thus preserves the cyclic descent number of each cycle except C_π ;
- (v) the map T preserves the cyclic descent number inside the core κ_π , and the cyclic descents formed by one letter inside κ_π and the other letter outside κ_π ; thus T preserves the cyclic descent number of C_π if $C_\pi \neq (\kappa_\pi)$;
- (vi) the map T preserves the cyclic descent number of C_π if $C_\pi = (\kappa_\pi)$.

In conclusion, the map T preserves the cyclic descent number of each cycle, thus preserves the cyclic weight d of π .

We shall show (vi) in detail. Suppose that $C_\pi = (\kappa_\pi)$. We claim that

- if $i < j$, then $\text{casc}(\kappa_\pi) = 1 = \text{casc}(T|_{\kappa_\pi})$;
- if $i > j$, then $\text{cdes}(\kappa_\pi) = 1 = \text{cdes}(T|_{\kappa_\pi})$.

Consider $i < j$. Then $M \leq n - 2$ by premise. By Eq. (3.1),

$$C_\pi = (\kappa_\pi) = \overline{((m; l_\pi) nM)}.$$

If $l_\pi = 0$, then $C_\pi = (nM)$, contradicting the premise that π is of odd order. If $l_\pi = M - m + 1$, then the letters $M + 1$ and M are adjacent in the cycle C_π , which implies $l_\pi = 0$ by definition, a contradiction. Therefore, we obtain

$$1 \leq l_\pi \leq M - m. \tag{3.6}$$

As a consequence, C_π has only one cyclic ascent, that is, mn . Hence $\text{casc}(\kappa_\pi) = 1$. On the other hand, by Eq. (3.2) and Ineq. (3.6), the cycle

$$(T|_{\kappa_\pi}) = ((m + 1)n \underline{(M + 1; l_\pi)})$$

has the unique cyclic ascent $(m + 1)n$. This proves the $i < j$ part of the claim. The second part in which $i > j$ can be shown in the same fashion.

Now, the claim implies (vi) immediately, and completes the whole proof. \square

We propose the following conjecture, which turns out to be a refinement of Conjecture 2.

Conjecture 8. For all n, d , and $2 \leq j \leq n - 1$, we have $b_{n,d}(1, j) + b_{n,d}(j, 1) = 2p_{n,d}(1, j)$.

Theorem 9. *Conjecture 2 is true if Conjecture 8 is true.*

Proof. Fix d . Let n be such that $d \leq \lfloor (n-1)/2 \rfloor$ and let $i < j$. Since

$$b_{n,d}(1, j-i+1) + b_{n,d}(j-i+1, 1) = 2p_{n,d}(1, j-i+1),$$

we can infer by Theorems 6 and 7 that

$$b_{n,d}(i, j) + b_{n,d}(j, i) = 2p_{n,d}(i, j).$$

Since the number of permutations in $\mathcal{B}_{n,d}$ with the letter n appearing at the last position is $b_{n-1,d}$, we obtain

$$b_{n,d} = b_{n-1,d} + \sum_{i \neq j} b_{n,d}(i, j). \quad (3.7)$$

Since the number of permutations in $\mathcal{P}_{n,d}$ with the letter n forming a singleton is $p_{n-1,d}$, we obtain

$$p_{n,d} = p_{n-1,d} + \sum_{i \neq j} p_{n,d}(i, j). \quad (3.8)$$

Since $p_{n,d}(i, j) = p_{n,d}(j, i)$ and Conjecture 2 holds for $d = \lfloor (n-1)/2 \rfloor$, by induction on n , we derive Conjecture 2 by the recurrences Eqs. (3.7) and (3.8). \square

4 Easy cases of Conjecture 8

In this section we solve some easy cases for Conjecture 8.

Proposition 10. *Conjecture 8 is true for $d = 0$ and $d = 1$.*

Proof. When $d = 0$, both sides of the desired equality are equal to 1. Let $d = 1$. By Lemma 3, we can infer that

$$b_{n,1}(1, 2) = b_{n,1}(2, 1) = b_{n-2,0} = 1 \quad \text{and} \quad p_{n,1}(1, 2) = p_{n-2,0} = 1.$$

For $j \geq 3$, it is easy to show that

$$b_{n,1}(j, 1) = 2^{j-2}, \quad b_{n,1}(1, j) = 0, \quad \text{and} \quad p_{n,1}(1, j) = 2^{j-3}.$$

This verifies Conjecture 8 for $d = 1$ by direct calculation. \square

Lemma 11. $p_{n,d}(1, 2) = p_{n,d}(1, 3)$.

Proof. Define $X_2 \subseteq \mathcal{P}_{n,d}(1, 2)$ to be the set of permutations which do not have a cycle of the form $(1n23\alpha)$, where α is a word. Similarly, define $X_3 \subseteq \mathcal{P}_{n,d}(1, 3)$ to be the set of permutations which do not have a cycle of the form $(1n32\alpha)$. By considering the exchange of the letters 2 and 3, we know that $|X_2| = |X_3|$.

It remains to define a map $\phi: \mathcal{P}_{n,d}(1, 2) \setminus X_2 \rightarrow \mathcal{P}_{n,d}(1, 3) \setminus X_3$. Let $\pi \in \mathcal{P}_{n,d}(1, 2) \setminus X_2$. Suppose that π contains a cycle $c = (1n23\alpha)$. Define $\phi(\pi)$ to be the permutation obtained

from π by replacing the cycle c by the cycle $u = (1n32\alpha')$. It is clear that ϕ is an involution. Let $d = \text{des}(\alpha)$. Then

$$\text{cdes}(c) = d + 2 = \text{casc}(u) \quad \text{and} \quad \text{casc}(c) = \ell(c) - d - 2 = \text{cdes}(u).$$

It follows that c and u have the same weight, and so do the permutations π and $\phi(\pi)$. This completes the proof. \square

Proposition 12. *If Conjecture 8 is true for $j = 2$, then it is true for $j = 3$.*

Proof. Suppose that Conjecture 8 is true for $j = 2$, that is,

$$b_{n,d}(1, 2) + b_{n,d}(2, 1) = 2p_{n,d}(1, 2).$$

For the sake of showing Conjecture 8 for $j = 3$, i.e.,

$$b_{n,d}(1, 3) + b_{n,d}(3, 1) = 2p_{n,d}(1, 3),$$

by Lemma 11, it suffices to show that

$$b_{n,d}(1, 2) - b_{n,d}(1, 3) = b_{n,d}(3, 1) - b_{n,d}(2, 1).$$

For words u and v , let $b_{n,d}(u, v)$ be the number of permutations in $\mathcal{B}_{n,d}$ that contain the factor unv . By exchanging the letters 2 and 3, we obtain

$$b_{n,d}(1, 2) - b_{n,d}(1, 3) = b_{n,d}(1, 23) - b_{n,d}(1, 32), \quad \text{and} \quad (4.1)$$

$$b_{n,d}(3, 1) - b_{n,d}(2, 1) = b_{n,d}(23, 1) - b_{n,d}(32, 1). \quad (4.2)$$

Using the proof of Lemma 3, one may show that

$$b_{n,d}(1, 23) = b_{n-3, d-1} = b_{n,d}(23, 1) \quad \text{and} \quad (4.3)$$

$$b_{n,d}(1, 32) = b_{n-3, d-2} = b_{n,d}(32, 1). \quad (4.4)$$

Combining Eqs. (4.1) to (4.4), we can deduce

$$\begin{aligned} b_{n,d}(1, 2) - b_{n,d}(1, 3) &= b_{n,d}(1, 23) - b_{n,d}(1, 32) \\ &= b_{n,d}(23, 1) - b_{n,d}(32, 1) = b_{n,d}(3, 1) - b_{n,d}(2, 1). \end{aligned}$$

This completes the proof. \square

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