Flag-transitive non-symmetric 2-designs with $(r, \lambda) = 1$ and exceptional groups of Lie type

Yongli Zhang   Shenglin Zhou*

School of Mathematics
South China University of Technology
Guangzhou, P.R. China

slzhou@scut.edu.cn

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Abstract

This paper determines all pairs $(\mathcal{D}, G)$ where $\mathcal{D}$ is a non-symmetric 2-(v, k, λ) design with $(r, \lambda) = 1$ and G is the almost simple flag-transitive automorphism group of $\mathcal{D}$ with an exceptional socle of Lie type. We prove that if $T \trianglelefteq G \leqslant Aut(T)$ where T is an exceptional group of Lie type, then T must be the Ree group or Suzuki group, and there are five classes of designs $\mathcal{D}$.

Mathematics Subject Classifications: 05B05, 05B25, 20B25

1 Introduction

A 2-(v, k, λ) design $\mathcal{D}$ is a pair $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set of v points and $\mathcal{B}$ is a set of k-subsets of $\mathcal{P}$, called blocks, such that any two points are contained in exactly λ blocks. A flag is an incident point-block pair (α, B). An automorphism of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The design is non-trivial if $2 < k < v - 1$ and non-symmetric if $v < b$. All automorphisms of the design $\mathcal{D}$ form a group called the full automorphism group of $\mathcal{D}$, denoted by $Aut(\mathcal{D})$. Let $G \leqslant Aut(\mathcal{D})$, then $\mathcal{D}$ or G is called point (block, flag)-transitive if G acts transitively on the set of points (blocks, flags), and point-primitive if G acts primitively on $\mathcal{P}$. Note that a finite primitive group is almost simple if it is isomorphic to a group G for which $T \cong Inn(T) \leqslant G \leqslant Aut(T)$ for some non-abelian simple group T.

Let $G \leqslant Aut(\mathcal{D})$ and r be the number of blocks incident with a given point. In [4], P. Dembowski proved that if G is a flag-transitive automorphism group of a 2-design $\mathcal{D}$

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with $(r, \lambda) = 1$, then G is point-primitive and P. H. Zieschang [30] proved that G must be of almost simple or affine type. Such 2-designs have been studied in [1, 26, 27, 28, 29], where the socle of G is an elementary abelian p-group, a sporadic group or an alternating group, respectively. In this paper, we continue to study the non-symmetric case that the socle of G is an exceptional simple group of Lie type. We get the following:

Theorem 1. *Let $\mathcal{D}$ be a non-symmetric 2-(v, k, λ) design with $(r, \lambda) = 1$ and G an almost simple flag-transitive automorphism group of $\mathcal{D}$. If the socle T of G is an exceptional Lie type group in characteristic p and $q = p^e$, then for some block B of $\mathcal{D}$ one of the following holds:*

(1) *$T = {}^2G_2(q)$ with $q = 3^{2n+1} \geqslant 27$ and $\mathcal{D}$ is one of the following:*

- (i) *the Ree unital of order q with $T_B = \mathbb{Z}_2 \times L_2(q)$;*
- (ii) *a 2-$(q^3 + 1, q, q - 1)$ design with $T_B = Q_1 : K$;*
- (iii) *a 2-$(q^3 + 1, q, q - 1)$ design with $T_B = \mathbb{Z}_2 \times (Q_2 : \langle k^2 \rangle)$;*
- (iv) *a 2-$(q^3 + 1, q^2, q^2 - 1)$ design with $T_B = Q' : K$,*

where $Q \in Syl_3(T)$, $k \in K$ and Q_1, Q_2 and K are defined in Section 3.

(2) *$T = {}^2B_2(q)$ with $q = 2^{2n+1} \geqslant 8$, and $\mathcal{D}$ is a 2-$(q^2 + 1, q, q - 1)$ design with $T_B = Z(Q) : K$, where $Q \in Syl_2(T)$ and $K = \mathbb{Z}_{q-1} \cong \mathbb{F}_q^*$.*

Remark. (1) The five designs in Theorem 1 are non-symmetric. Here we just list the block stabilizer T_B for each design, and it is easily known from the proof of Propositions 24 and 26 (Section 3) that T acts 2-transitively on points set of $\mathcal{D}$ and the point stabilizer T_α is the parabolic subgroup of T.

(2) The constructions of these designs are in Section 3. Moreover, if (α, B) is any flag of the Ree unital $U_R(q) = (\mathcal{P}, \mathcal{B})$ in part (1)(i), then the design in (1)(iii) is $(\mathcal{P}, \mathcal{B}')$ with $\mathcal{B}' = (B - \{\alpha\})^T$ and $T = {}^2G_2(q)$.

(3) Two designs in part (1)(ii) and (iii) have the same parameters $(v, b, r, k, \lambda) = (q^3 + 1, q^2(q^3 + 1), q^3, q, q - 1)$, but we do not know if these two designs are isomorphic till now.

2 Preliminary results

We first give some preliminary results about designs and almost simple groups.

Lemma 2. ([27, Lemma 2.2]) *For a 2-(v, k, λ) design $\mathcal{D}$, it is well known that*

(1) *$bk = vr$;*

(2) *$\lambda(v - 1) = r(k - 1)$;*

(3) $v \leqslant \lambda v < r^2$;

(4) *if* $G \leqslant Aut(\mathcal{D})$ *is flag-transitive and* $(r, \lambda) = 1$, *then* $r \mid (|G_\alpha|, v-1)$ *and* $r \mid d$, *for any non-trivial subdegree* d *of* G.

Lemma 3. *If* G *and* $\mathcal{D}$ *satisfy the hypothesis of Theorem 1, then for every* $\alpha \in \mathcal{P}$ *and* $B \in \mathcal{B}$ *we have the following:*

(1) $G = TG_\alpha$ *and* $|G| = f|T|$ *where* f *is a divisor of* $|Out(T)|$;

(2) $|G : T| = |G_\alpha : T_\alpha| = f$;

(3) $|G_B|$ *divides* $f|T_B|$, *and* $|G_{\alpha B}|$ *divides* $f|T_{\alpha B}|$ *for any flag* (α, B).

Proof. Since G is an almost simple primitive group, (1) holds and (2) follows from (1). Note that $T \trianglelefteq G$, so $|B^T|$ divides $|B^G|$ and $|(\alpha, B)^T|$ divides $|(\alpha, B)^G|$. It follows that $|G_B : T_B|$ divides f and $|G_{\alpha B} : T_{\alpha B}|$ divides f and so (3) holds. □

Lemma 4. ([4, 2.2.5]) *Let* $\mathcal{D}$ *be a* 2-(v, k, λ) *design. If the parameters* k, r, λ *of* $\mathcal{D}$ *satisfies* $r = k + \lambda$ *and* $\lambda \leqslant 2$, *then* $\mathcal{D}$ *is embedded in a symmetric* 2-$(v + k + \lambda, k + \lambda, \lambda)$ *design.*

Lemma 5. ([4, 2.3.8]) *Let* $\mathcal{D}$ *be a* 2-(v, k, λ) *design and* $G \leqslant Aut(\mathcal{D})$. *If* G *is* 2*-transitive on points and* $(r, \lambda) = 1$, *then* G *is flag-transitive.*

Lemma 6. *Let* A, B, C *be subgroups of group* G. *If* $B \leqslant A$, *then*

$$|A : B| \geqslant |(A \cap C) : (B \cap C)|.$$

Lemma 7. ([15]) *If* T *is a simple group of Lie type in characteristic* p *acting on the set of cosets of a maximal parabolic subgroup, then* T *has a unique subdegree which is a power of* p, *except that* T *is* $L_d(q)$, $\Omega^+_{2m}(q)$ *(*m *is odd) or* $E_6(q)$.

Lemma 8. [24, 1.6] (Tits Lemma) *If* T *is a simple group of Lie type in characteristic* p, *then any proper subgroup of index prime to* p *is contained in a parabolic subgroup of* T.

In the following, n_p denotes the p-part of n and $n_{p'}$ denotes the p'-part of n for a positive integer n, namely, $n_p = p^t$ where $p^t \mid n$ but $p^{t+1} \nmid n$, and $n_{p'} = n/n_p$.

Lemma 9. *If* G *and* $\mathcal{D}$ *satisfy the hypothesis of Theorem 1, then* $|G| < |G_\alpha|^3$. *Moreover, if* G_α *is non-parabolic and maximal, then* $|G| < |G_\alpha||G_\alpha|^2_{p'}$ *and* $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$.

Proof. By Lemma 2(4), r divides every non-trivial subdegree of G, hence r divides $|G_\alpha|$ and $|G| < |G_\alpha|^3$ by Lemma 2(3). If G_α is not parabolic, then p divides $v = |G : G_\alpha|$ by Lemma 8. Since r divides $v - 1$, $(r, p) = 1$ and so r divides $|G_\alpha|_{p'}$. It follows that $r \leqslant |G_\alpha|_{p'}$, and hence $|G| < |G_\alpha||G_\alpha|^2_{p'}$ by Lemma 2(3) again. Now by Lemma 3(2), we have that $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$. □

Lemma 10. ([18, Theorem 2, Table III]) *If T is a finite simple exceptional group of Lie type such that $T \leqslant G \leqslant Aut(T)$ and G_α is a maximal subgroup of G such that $T_0 = Soc(G_\alpha)$ is not simple, then one of the following holds:*

(1) *G_α is parabolic;*

(2) *G_α is of maximal rank;*

(3) *$G_\alpha = N_G(E)$, where E is an elementary abelian group given in* [3, Theorem 1(II)];

(4) $T = E_8(q)$ with $p > 5$, and T_0 is either $A_5 \times A_6$ or $A_5 \times L_2(q)$;

(5) T_0 is as in Table 1.

Table 1: The cases in Lemma 10(5)

T	T_0
$F_4(q)$	$L_2(q) \times G_2(q)(p > 2, q > 3)$
$E_6^\epsilon(q)$	$L_3(q) \times G_2(q), U_3(q) \times G_2(q)(q > 2)$
$E_7(q)$	$L_2(q) \times L_2(q)(p > 3), L_2(q) \times G_2(q)(p > 2, q > 3),$ $L_2(q) \times F_4(q)(q > 3), G_2(q) \times Sp_6(q)$
$E_8(q)$	$L_2(q) \times L_3^\epsilon(q)(p > 3), L_2(q) \times G_2(q) \times G_2(q)(p > 2, q > 3),$ $G_2(q) \times F_4(q), L_2(q) \times G_2(q^2)(p > 2, q > 3)$

Lemma 11. ([17, Theorem 3]) *Let T be a finite simple exceptional group of Lie type and G a group such that $T \leqslant G \leqslant Aut(T)$. Let G_α be maximal in G and the socle $T_0(q)$ of G_α be a simple group of Lie type over $\mathbb{F}_q(q > 2)$. If $\frac{1}{2}rank(T) < rank(T_0)$, then except cases that (T, T_0) is $(E_8, {}^2A_5(5))$ or $(E_8, {}^2D_5(3))$, one of the following holds:*

(1) *G_α is a subgroup of maximal rank;*

(2) *T_0 is a subfield or twisted subgroup;*

(3) *$T = E_6(q)$ and $T_0 = C_4(q)(q$ odd$)$ or $F_4(q)$.*

Lemma 12. ([20, Theorem 1.2]) *Let T be a finite simple exceptional group of Lie type and G a group such that $T \leqslant G \leqslant Aut(T)$. Let G_α be maximal in G and the socle $T_0(q)$ of G_α be a simple group of Lie type over $\mathbb{F}_q(q > 2)$. If $rank(T_0) \leqslant \frac{1}{2}rank(T)$, we have the following bounds:*

(1) *if $T = F_4(q)$, then $|G_\alpha| < 4q^{20} \log_p q$;*

(2) *if $T = E_6^\epsilon(q)$, then $|G_\alpha| < 4q^{28} \log_p q$;*

(3) *if* $T = E_7(q)$*, then* $|G_\alpha| < 4q^{30}\log_p q$*;*

(4) *if* $T = E_8(q)$*, then* $|G_\alpha| < 12q^{56}\log_p q$*.*

In all cases, $|G_\alpha| < 12|G|^{\frac{5}{13}}\log_p q$*.*

The following lemma gives a method to check the existence of design.

Lemma 13. ([27]) *For a given* (v, b, r, k, λ) *and group* G*, the existence of design* $\mathcal{D}$ *with such values as parameters and* G *as a primitive flag-transitive automorphism group is equivalent to the following four steps hold:*

(1) G *is a primitive group on* v *points set* $\mathcal{P}$*;*

(2) G *has at least one subgroup* H *of order* $|G|/b$*;*

(3) H *has one orbit* O *of length* k *on the point-set* $\mathcal{P}$ *such that* $|O^G|$ *is* b*;*

(4) *the number of blocks which incident with any two points is the constant.*

Then $(\mathcal{P}, O^G)$ *is a* 2*-design admitting* G *as a primitive flag-transitive automorphism group.*

We now give some information about the Ree group ${}^2G_2(q)$ with $q = 3^{2n+1}$ and its subgroups, which are from [6, 9, 13] and would be used in Section 3.

Set $m = 3^{n+1}$, so $m^2 = 3q$. The Ree group ${}^2G_2(q)$ is generated by Q, K and τ, where Q is Sylow 3-subgroup of ${}^2G_2(q)$, $K = \{diag(t^m, t^{1-m}, t^{2m-1}, 1, t^{1-2m}, t^{m-1}, t^{-m}) \,|\, t \in \mathbb{F}_q^*\} \cong \mathbb{Z}_{q-1}$ and $\tau^2 = 1$ such that τ inverts K. It is well-known that $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$.

Lemma 14. (1) ([13]) ${}^2G_2(q)$ *is* 2*-transitive with degree* $q^3 + 1$*.*

(2) ([5, p.252]) *The stabilizer of one point is* $Q : K$*, and* $N_{{}^2G_2(q)}(Q) = Q : K$*.*

(3) ([9, p.292]) *The stabilizer* K *of two points is cyclic of order* $q - 1$ *and the stabilizer of three points is of order* 2*.*

(4) ([9, p.292]) *The Sylow* 2*-subgroup of* ${}^2G_2(q)$ *is elementary abelian with order* 8*.*

Lemma 15. ([11],[6, Lemma 3.3]) *Let* M *be a maximal subgroup of* ${}^2G_2(q)$*. Then either* M *is conjugate to* $M_6 := {}^2G_2(3^\ell)$ *for some divisor* ℓ *of* $2n + 1$*, or* M *is conjugate to one of the subgroups* M_i *in the Table 2:*

Moreover, we see that from [6], the Sylow 3-subgroup Q can be identified with the group consisting of all triples (α, β, γ) from $\mathbb{F}_q$ with multiplication:

$$(\alpha_1, \beta_1, \gamma_1)(\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2 - \alpha_1\alpha_2^m, \gamma_1 + \gamma_2 - \alpha_1^m\alpha_2^m - \alpha_2\beta_1 + \alpha_1\alpha_2^{m+1}).$$

It is easy to check that $(0, 0, \gamma)(0, \beta, 0) = (0, \beta, \gamma)$. Set $Q_1 = \{(0, 0, \gamma)|\gamma \in \mathbb{F}_q\}$ and $Q_2 = \{(0, \beta, 0)|\beta \in \mathbb{F}_q\}$, then $Q_1 \cong Q_2 \cong \mathbb{Z}_3^{2n+1}$.

Denote the center, Frattini subgroup and the derived subgroup of Q by $Z(Q)$, $\Phi(Q)$, Q', respectively. From [6], $Q' = \Phi(Q) = Q_1 \times Q_2$, $Z(Q) = Q_1$ and Q' is an elementary abelian 3-group. For any $(\alpha, \beta, \gamma) \in Q$ and $k \in K$,

$$(\alpha, \beta, \gamma)^k = (k\alpha,\ k^{1+m}\beta,\ k^{2+m}\gamma).$$

Table 2: The maximal subgroups of $^2G_2(q)$

Group	Structure	Remarks
M_1	$Q : K$	the normalizer of Q in $^2G_2(q)$
M_2	$\mathbb{Z}_2 \times L_2(q)$	the centralizer of an involution in $^2G_2(q)$
M_3	$(\mathbb{Z}_2^2 \times D_{(q+1)/2}) : \mathbb{Z}_3$	the normalizer of a four-subgroup
M_4	$\mathbb{Z}_{q+m+1} : \mathbb{Z}_6$	the normalizer of $\mathbb{Z}_{q+m+1}$
M_5	$\mathbb{Z}_{q-m+1} : \mathbb{Z}_6$	the normalizer of $\mathbb{Z}_{q-m+1}$

Lemma 16. ([6, 13]) *If Q, M_1, Q_2, M_2 and K defined as above, then*

(1) *the normalizer of every subgroup of Q is contained in M_1;*

(2) *Q_2 is a Sylow 3-subgroup of M_2 and $N_{M_2}(Q_2) = \mathbb{Z}_2 \times (Q_2 : \langle k^2 \rangle)$ with $\langle k \rangle = K$.*

Lemma 17. ([6, Lemma 3.2]) *The following hold for the cyclic subgroup K:*

(1) *K is transitive on $Q_1 \setminus \{1\}$ acting by conjugation;*

(2) *K has two orbits $(0,1,0)^K$, $(0,-1,0)^K$ on $Q_2 \setminus \{1\}$ acting by conjugation.*

According to Lemma 17, we know that $H_1 := Q_1 : K$ is a subgroups of M_1. Moreover, $(0,0,\gamma)^K = Q_1 \setminus \{1\}$ and $(0,\beta,0)^K \cup (0,-\beta,0)^K = Q_2 \setminus \{1\}$ for any non-identity element $(0,0,\gamma)$ of Q_1 and $(0,\beta,0)$ of Q_2.

Let M_2 be a representative of the second case (as list in Table 2) of maximal subgroup of $^2G_2(q)$ and Q_2 be the Sylow 3-subgroup of M_2. If Q is the Sylow 3-subgroup of $^2G_2(q)$ such that $Q_2 \leqslant Q$, then for the normalizer $N_{M_2}(Q_2)$ of Q_2 in M_2 and the normalizer $M_1 = Q : K$ of Q in $^2G_2(q)$, we have the following conclusions.

Lemma 18. *Let H be a subgroup of M_2 such that $|H| = q(q-1)$. Then*

(1) *H is conjugate to $N_{M_2}(Q_2)$;*

(2) *$N_{M_2}(Q_2) \leqslant M_1$ and $N_{M_2}(Q_2)$ is not conjugate to H_1;*

(3) *H is contained in a conjugacy of M_1.*

Proof. Let $H \leqslant M_2$ such that $|H| = q(q-1)$. Note that $M_2 \cong \mathbb{Z}_2 \times L_2(q)$ and $|N_{M_2}(Q_2)| = q(q-1)$. Then by the list of maximal subgroups of $L_2(q)$ we know that $H \cong N_{M_2}(Q_2)$. Let σ be an automorphism from $N_{M_2}(Q_2)$ to H. Then $Q_2^\sigma \trianglelefteq H$ since $Q_2 \trianglelefteq N_{M_2}(Q_2)$. Moreover, since $q \mid |H|$, the Sylow 3-subgroup of H is conjugate to Q_2 in M_2 and so $Q_2^\sigma = Q_2^c \trianglelefteq H$ for some $c \in M_2$. It follows that

$$H \leqslant N_{M_2}(Q_2^c) = N_{M_2}(Q_2)^c.$$

Therefore, $H = N_{M_2}(Q_2)^c$ and (1) holds.

By Lemma 16(1), $N_{M_2}(Q_2) \leqslant M_1$. Suppose that $N_{M_2}(Q_2)$ is conjugate to H_1 in M_1. Then $N_{M_2}(Q_2) = H_1^u = Q_1 : K^u$ for some $u \in M_1$, which implies that $Q_1 \trianglelefteq N_{M_2}(Q_2)$ and $Q_1 \times Q_2 \leqslant N_{M_2}(Q_2)$, a contradiction. So (2) holds and (3) follows from (1) and (2). $\square$

Lemma 19. *Let H be a subgroup of M_1 such that $|H| = q(q-1)$. Then $H = A : K^u$ for the Sylow 3-subgroup A of H and some $u \in M_1$.*

Proof. Obviously, M_1 is solvable, so H is solvable. Let K_1 be a subgroup of order $q-1$ of H. Since $|M_1| = q^3(q-1)$, K_1 and K are two Hall subgroups of M_1, which implies that $K_1 = K^u$ for some $u \in M_1$ by [8, Chapter 6, Theorem 4.1]. Let $A = H \cap Q$. We have $A \trianglelefteq H$ by $Q \trianglelefteq M_1$ and $A \cap K_1 = 1$. Hence $H = A : K^u$ for some $u \in M_1$. □

Lemma 20. *Let H be a subgroup of M_1 such that $|H| = q(q-1)$. Then H is conjugate to H_1 or $N_{M_2}(Q_2)$ in M_1.*

Proof. By Lemma 19, we have $H = A : K^u$ where A is a Sylow 3-subgroup of H and so $H^{u^{-1}} = A^{u^{-1}} : K$. Clearly, $A^{u^{-1}} \leqslant Q$ since $A \leqslant Q$ and $u \in M_1$. Let F be a maximal subgroup of Q such that $A^{u^{-1}} \leqslant F$. If $A^{u^{-1}} \cap Q' = 1$, then by Lemma 6 and the fact $Q' \leqslant F$, $|F : A^{u^{-1}}| \geqslant |F \cap Q' : A^{u^{-1}} \cap Q'| = q^2$, and so $|F| \geqslant q^3$, a contradiction. Therefore, $A^{u^{-1}} \cap Q' \neq 1$.

If $A^{u^{-1}} \cap Q'$ has an element $(0, 0, \gamma)$ such that $\gamma \neq 0$, then $(0, 0, \gamma)^K = Q_1 \setminus \{1\} \subseteq A^{u^{-1}} \setminus \{1\}$ and so $A^{u^{-1}} = Q_1$ which implies that $H^{u^{-1}} = H_1$. Similarly, if $A^{u^{-1}} \cap Q'$ has an element $(0, \beta, 0)$ such that $\beta \neq 0$, then $A^{u^{-1}} = Q_2$. Hence $H^{u^{-1}} = Q_2 : K$. In particular, $N_{M_2}(Q_2)^c = Q_2 : K$ for some $c \in M_1$ by Lemma 19. Then $H^{u^{-1}} = N_{M_2}(Q_2)^c$.

Suppose that $A^{u^{-1}} \cap Q'$ has an element $(0, \beta, \gamma)$ such that $\beta \neq 0$ and $\gamma \neq 0$. Note that $(0, \beta, \gamma)^{-1} = (0, -\beta, -\gamma) \in A^{u^{-1}} \cap Q'$. Since $|A^{u^{-1}}| = q$, $(0, \beta, \gamma)^K = (0, -\beta, -\gamma)^K$ and so $(0, \beta, \gamma)^k = (0, -\beta, -\gamma)$ for some $k \in K$, which implies that $(0, \beta, 0)^k = (0, -\beta, 0)$, contradicts Lemma 17(2). □

Corollary 21. *Let $H \leqslant {}^2G_2(q)$ and $|H| = q(q-1)$, then H is conjugate in ${}^2G_2(q)$ to H_1 or $N_{M_2}(Q_2)$.*

Proof. By Lemma 15, H is contained in a conjugacy of M_1 or M_2. The result follows immediately from Lemmas 18 and 20. □

Lemma 22. *Let $H \leqslant {}^2G_2(q)$ and $|H| = q^2(q-1)$, then H is conjugate in ${}^2G_2(q)$ to $Q' : K$.*

Proof. Since Q' *char* $Q \trianglelefteq M_1$, so $Q' : K$ is a subgroup of M_1 with order $q^2(q-1)$. Suppose that $H \leqslant {}^2G_2(q)$ and $|H| = q^2(q-1)$. By Lemma 15, we have $H^{g^{-1}} \leqslant M_1$. Similar as the proof of Lemma 19, we get that $H^{g^{-1}}$ has the structure $A : K$ where A is the Sylow 3-subgroup of $H^{g^{-1}}$. Let F be a maximal subgroup of Q satisfying $A \leqslant F$. Since $|F : A| \geqslant |F \cap Q_i : A \cap Q_i|$, we have $|A \cap Q_i| > 1$, which implies $Q_i = (A \cap Q_i)^K \leqslant A^K = A$ for $i = 1, 2$. So $Q' \leqslant A$, and it follows that $Q' = A$ and $H^{g^{-1}} = Q' : K$ in M_1. □

Similarly, we have the following result on the Suzuki group ${}^2B_2(q)$ by [7] and [5, p.250].

Lemma 23. *Suppose that Q is the Sylow 2-subgroup of ${}^2B_2(q)$ and $M_1 = Q : K$ is the normalizer of Q. Let $H \leqslant {}^2B_2(q)$ and $|H| = q(q-1)$. Then H is conjugate in ${}^2B_2(q)$ to $Z(Q) : K$.*

3 Proof of Theorem 1

3.1 T is the Ree group

Proposition 24. *Let G and $\mathcal{D}$ satisfy hypothesis of Theorem 1 and B be a block. If $T = {}^2G_2(q)$ with $q = 3^{2n+1}$, then $\mathcal{D}$ is the Ree unital or one of the following:*

(1) *$\mathcal{D}$ is a 2-$(q^3+1, q, q-1)$ design with $T_B = Q_1 : K$ or $\mathbb{Z}_2 \times (Q_2 : \langle k^2 \rangle)$;*

(2) *$\mathcal{D}$ is a 2-(q^3+1, q^2, q^2-1) design with $T_B = Q' : K$.*

This proposition will be proved into two steps. We first assume that there exists a design satisfying the assumptions and obtain the possible parameters (v, b, r, k, λ) in Lemma 25, then prove the existence of the designs using Lemma 13.

Lemma 25. *Let G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T = {}^2G_2(q)$ with $q = 3^{2n+1}$, then $(v, b, r, k, \lambda) = (q^3+1, q^2(q^3+1), q^3, q, q-1)$ or $(q^3+1, q(q^3+1), q^3, q^2, q^2-1)$ or $\mathcal{D}$ is the Ree unital.*

Proof. Let $T_\alpha := G_\alpha \cap T$. Since G is primitive on $\mathcal{P}$, then T_α is one of the cases in Lemma 15 by [11]. By Lemma 9, we know that the cases that $T_\alpha = \mathbb{Z}_2^2 \times D_{(q+1)/2}$ and $\mathbb{Z}_{q\pm m+1} : \mathbb{Z}_6$ are impossible. If $T_\alpha = \mathbb{Z}_2 \times L_2(q)$, then $v = q^2(q^2-q+1)$ and $(|T_\alpha|, v-1) = (q(q^2-1), q^4-q^3+q^2-1) = q-1$. But since r divides $f(|T_\alpha|, v-1)$, r is too small to satisfy $v < r^2$. Similarly, T_α cannot be ${}^2G_2(3^\ell)$.

Therefore $T_\alpha = Q : K$ and $v = q^3+1$. Moreover, from [5, p.252], T is 2-transitive on $\mathcal{P}$, so T is flag-transitive by Lemma 5. Hence we may assume that $G = T = {}^2G_2(q)$. The equations in Lemma 2 show

$$b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda q^3(q^3+1)}{k(k-1)},$$

then by the flag-transitivity of T, we have

$$|T_B| = \frac{|T|}{b} = \frac{(q-1)k(k-1)}{\lambda}.$$

Let M be a maximal subgroup of T such that $T_B \leqslant M$. Then since $|T_B| \mid |M|$ and $q \geqslant 27$, M is either M_1 or M_2 as shown in Lemma 15.

If $T_B \leqslant M_1$, then $k(k-1) \mid \lambda q^3$. Furthermore, since $(r, \lambda) = 1$ and so $\lambda \mid (k-1)$ by Lemma 2(2). Therefore $\lambda = k-1$, and it follows that $r = v-1 = q^3$ and $k \mid q^3$. Note that M_1 is point stabilizer of T in this action. So there exists η such that $M_1 = T_\eta$ and $T_B \leqslant T_\eta$. However, the flag-transitivity of T implies $\eta \notin B$. For any point $\gamma \in B$, $T_{\gamma B} \leqslant T_{\eta\gamma}$. By Lemma 14, $|T_{\eta\gamma}| = q-1$, and so $|T_{\gamma B}| \mid (q-1)$. On the other hand, from

$$|B^{T_\gamma}| = |T_\gamma : T_{\gamma B}| \leqslant |B^{G_\gamma}| = |G_\gamma : G_{\gamma B}| = r = q^3,$$

we have $T_{\gamma B} = T_{\eta\gamma} \leqslant T_B$. Since the stabilizer of three points is of order 2 by Lemma 14, the size of $T_{\eta\gamma}$-orbits acting on $\mathcal{P} \setminus \{\eta, \gamma\}$ is $q-1$ or $\frac{1}{2}(q-1)$. This, together with

$T_{\eta\gamma} \leqslant T_B$ and $\eta \notin B$, implies that $k-1 = a(\frac{q-1}{2})$ for an integer a. Recall that $k \mid q^3$ and $k < r$, we get $k = q$ or $k = q^2$. If $k = q$, then

$$b = q^2(q^3+1), r = q^3, \lambda = q-1.$$

If $k = q^2$, we have

$$b = q(q^3+1), r = q^3, \lambda = q^2-1.$$

Now we deal with the case that $T_B \leqslant M_2$ by the similar method in [10, Theorem 3.2].

If T_B is a solvable subgroup of $M_2 \cong \mathbb{Z}_2 \times L_2(q)$, then T_B must map into either $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_2 \times D_{q\pm1}$ or $\mathbb{Z}_2 \times ([q] : \mathbb{Z}_{\frac{q-1}{2}})$. Obviously, the former two cases are impossible. For the last case, $T_B \lesssim \mathbb{Z}_2 \times ([q] : \mathbb{Z}_{\frac{q-1}{2}})$ and so T_B is a subgroup of $H \leqslant M_2$, where the order of H is $q(q-1)$. Hence by Lemma 18(3), this case can be reduced to the case that $T_B \leqslant M_1$.

If T_B is non-solvable, then it is embedded in $\mathbb{Z}_2 \times L_2(q_0)$ with $q_0^\ell = q = 3^{2n+1}$. The condition that $|T_B|$ divides $|\mathbb{Z}_2 \times L_2(q_0)|$ forces $q_0 = q$ and so T_B is isomorphic to $\mathbb{Z}_2 \times L_2(q)$ or $L_2(q)$.

If $T_B \cong \mathbb{Z}_2 \times L_2(q)$, then $T_B = M_2$ and so $b = q^2(q^2-q+1)$. Hence, from Lemma 2, we have $k \mid q(q+1)$, $q^2 \mid r$ and $r \mid q^3$. Since $k \geqslant 3$, T_B cannot acting trivially on the block B by the fact that the stabilizer of three points is of order 2. Moreover, since $q+1$ is the smallest degree of any non-trivial action of $L_2(q)$, we have $k = \frac{\lambda(v-1)}{r} + 1 \geqslant q+1$. If the design $\mathcal{D}$ is a linear space, then $\mathcal{D}$ is the Ree unital (see [10]) with parameters

$$(v, b, r, k, \lambda) = (q^3+1, q^2(q^2-q+1), q^2, q+1, 1)$$

and T is flag-transitive with the block stabilizer M_2.

If $\lambda > 1$, then we claim that $\lambda = k-1$. Clearly, $\lambda \mid (k-1)$ as $(r, \lambda) = 1$ by Lemma 2(2). If $3 \mid (k-1)$ and $(k,3) = 1$, then since $k \mid q(q+1)$ and $k \geqslant q+1$, we have $k = q+1$ and so $\lambda \mid q$, which contradicts $(r, \lambda) = 1$ as $q^2 \mid r$. Hence $(k-1, 3) = 1$. Moreover, $(k-1) \mid \lambda q^3$ implies that $(k-1) \mid \lambda$. So we have $\lambda = k-1$.

Let Δ_1, Δ_2,..., Δ_t be the orbits of M_2 with the action that T is 2-transitive on q^3+1 points. Since M_2 is the block stabilizer of the Ree unital, it has an orbit of size $q+1$. Without loss of generality, let $|\Delta_1| = q+1$. On one hand, recall that $k \mid q(q+1)$ and T is flag transitive, $T_B = M_2$ has at least one orbit with size less than $q(q+1)$. On the other hand, we show that $|\Delta_i| > q(q+1)$ for all i such that $i \neq 1$ in the following and obtain the desired contradiction. For any point δ such that $\delta \in \mathcal{P} \setminus \Delta_1$, we claim that $(M_2)_\delta$ is a 2-group. Let p be a prime divisor of $|(M_2)_\delta|$ and P be a Sylow p-subgroup of $(M_2)_\delta$. If $p \neq 2$ and $p \neq 3$, then since $(M_2)_\delta \leqslant T_\delta$, we have $p \mid (q-1)$. Obviously, since Δ_1 is an orbit of M_2 and $P \leqslant (M_2)_\delta$, and so P acts invariantly on Δ_1 and $\mathcal{P} \setminus \Delta_1$. Note that the length of any nontrivial P-orbit divided by p, so P fixes at least two points in Δ_1. Moreover, P also fixes δ. Therefore P fixes at least three points of $\mathcal{P}$, which is impossible as the order of the stabilizer of three points is 2 by Lemma 14(3). If $p = 3$, since P fixes the point $\delta \in \mathcal{P} \setminus \Delta_1$ and $|\mathcal{P} \setminus \Delta_1| = q^3 - q$, then P fixes at least three points in $\mathcal{P} \setminus \Delta_1$, which is also impossible. As a result, $(M_2)_\delta$ is a 2-group. The fact that the Sylow 2-subgroup of T is of order 8 implies that the sizes of the M_2-orbits Δ_i $(i \neq 1)$ are at least

$\frac{q(q^2-1)}{8}$ and hence larger than $q(q+1)$, which contradicts the fact $k \mid q(q+1)$. Therefore, $T_B \ncong \mathbb{Z}_2 \times L_2(q)$. Similarly, $T_B \ncong L_2(q)$. Thus when T_B is a non-solvable subgroup in M_2, $\mathcal{D}$ is a Ree unital. $\square$

Proof of Proposition 24 We use Lemma 13 to prove the existence of the design with parameters listed in Lemma 25.

Assume that $(v, b, r, k, \lambda) = (q^3+1, q^2(q^3+1), q^3, q, q-1)$. Then from Corollary 21 we known that there are only two conjugacy classes of subgroups of order $q(q-1)$ in T and $H_1 = Q_1 : K \leqslant T_\alpha$ and $N_{M_2}(Q_2) = \mathbb{Z}_2 \times (Q_2 : \langle k^2 \rangle)$ be representatives, respectively.

First, we consider the orbits of H_1. Let $\gamma \neq \alpha$ be the point fixed by K. Since $K \leqslant H_1$, then $K_\gamma = K \leqslant (H_1)_\gamma \leqslant T_{\alpha\gamma} = K$, which implies $(H_1)_\gamma = T_{\alpha\gamma}$ and so $|H_1 : (H_1)_\gamma| = |\gamma^{H_1}| = q$. It is easy to see that $|\delta^{H_1}| \neq q$ for any point $\delta \neq \alpha$ or γ. Therefore, H_1 has only one orbit of size q. Let $B_1 = \gamma^{H_1}$.

Now we show that $H_1 = T_{B_1}$, which implies $|B_1^T| = b$. Since $H_1 \leqslant T_{B_1}$ and $B_1 = \gamma^{H_1} = \gamma^{T_{B_1}}$, then $|H_1 : (H_1)_\gamma| = |T_B : T_{\gamma B_1}| = q$. If $K = (H_1)_\gamma < T_{\gamma B_1}$, then 3 divides $|T_{\gamma B_1} : T_{\delta\gamma B_1}|$ for any $\delta \in B_1 \setminus \{\gamma\}$ by Lemma 14(3). It follows that $3 \mid (q-1)$, a contradiction. As a result, $K = (H_1)_\gamma = T_{\gamma B_1}$ and so $H_1 = T_{B_1}$. Let $\mathcal{B}_1 := B_1^T$. Therefore $|\mathcal{B}_1| = |T : H_1| = b$. Let $\mathcal{B}_1$ be the set of blocks.

Finally, since T is 2-transitive on $\mathcal{P}$, the number of blocks which incident with two points is a constant. Hence $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}_1)$ is a 2-$(q^3+1, q, q-1)$ design admitting T as a flag transitive automorphism group by Lemma 13.

In a similar way, we can construct the design $\mathcal{D}_2$ satisfying all hypothesis when the subgroup is $N_{M_2}(Q_2) = \mathbb{Z}_2 \times (Q_2 : \langle k^2 \rangle)$. However, at this stage we do not know if $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic.

3.2 *T* is the Suzuki group

Proposition 26. *Let G and $\mathcal{D}$ satisfy hypothesis of Theorem 1. If $T = {}^2B_2(q)$ with $q = 2^{2n+1}$, then $\mathcal{D}$ is a 2-$(q^2+1, q, q-1)$ design with $T_B = Z(Q) : K$ where $Q \in Syl_2(T)$ and $K = \mathbb{Z}_{q-1}$.*

Proof. Let $T = {}^2B_2(q)$ with order $(q^2+1)q^2(q-1)$. Then $|G| = f(q^2+1)q^2(q-1)$ where f divides $|Out(T)|$. By [7] or [25], the order of G_α is one of the following:

(1) $fq^2(q-1)$;

(2) $2f(q-1)$;

(3) $4f(q \pm \sqrt{2q} + 1)$;

(4) $f(q_0^2+1)q_0^2(q_0-1)$ with $q_0^\ell = q$.

Since $|G| < |G_\alpha|^3$, we first have that $|G_\alpha| \neq 2f(q-1)$. If $|G_\alpha| = 4f(q \pm \sqrt{2q}+1)$, then from the inequality $|G| < |G_\alpha|^3$, we get $f(q^2+1)q^2(q-1) < (4f)^3(2q)^3$, and so $q^2+q+1 \leqslant 4^3 f^2 2^3$. It follows that $q+1 < 4^3 2^3$ and $q = 2^7$, 2^5 or 2^3 by $f \leqslant |Out(T)| = 2n+1$ and $q = p^{2n+1}$. If $q = 2^7$, then $|G| = f2^{14}(2^{14}-1)(2^7-1) > f^3 4^3 (2^7+2^4+1)^3 = |G_\alpha|^3$ where

$f = 7$ or 1, a contradiction. If $q = 2^5$, then $v = 198400$ or 325376 for $|G_\alpha| = 4f(q+\sqrt{2q}+1)$ or $4f(q-\sqrt{2q}+1)$ respectively. By calculating $(|G_\alpha|, v-1)$, since r divides $(|G_\alpha|, v-1)$, we know that r is too small. Similarly, we get $q \neq 2^3$.

If $|G_\alpha| = f(q_0^2+1)q_0^2(q_0-1)$ with $q_0^\ell = q$, then the inequality $|G| < |G_\alpha||G_\alpha|_{p'}^2$ forces $m = 3$. So $v = (q_0^4 - q_0^2 + 1)q_0^4(q_0^2 + q_0 + 1)$. Since r divides $(|G_\alpha|_{p'}, v-1)$, then $r \leqslant |G_\alpha|_{p'} \leqslant fq_0^3 < q_0^{9/2}$. From $v < r^2$, we get $(q_0^4 - q_0^2 + 1)q_0^4(q_0^2 + q_0 + 1) < r^2 < q_0^9$, which is impossible.

Now assume that $|G_\alpha| = fq^2(q-1)$. Then $v = q^2+1$ and T is 2-transitive by [5, p.250]. Hence, T is flag-transitive by Lemma 5. Similarly, we have $|T_B| = \frac{|T|}{b} = \frac{k(k-1)(q-1)}{\lambda}$. Let M be the maximal subgroup of T such that $T_B \leqslant M$. The fact that $|T_B|$ divides $|M|$ implies that $|M| = q^2(q-1)$ and $k(k-1)$ divides λq^2. Similar to the proof of Lemma 25, we have $T_{\gamma B} = T_{\alpha\gamma}$ with the order $q-1$. Furthermore, we get

$$(v, b, r, k, \lambda) = (q^2+1, q(q^2+1), q^2, q, q-1).$$

Next we prove the existence of the design with above parameters by Lemma 13. Firstly, from Lemma 23 we know that the Suzuki group has a unique conjugacy class of subgroups of order $q(q-1)$, let $H := Z(Q) : K \leqslant T_\alpha$ be the representative.

Note that K is the stabilizers of two points in ${}^2B_2(q)$ by [9, p.187]. Let $\gamma \neq \alpha$ be the point fixed by K and $B = \gamma^H$. A similar argument to that of Proposition 24 implies that B is the only H-orbit of length q and $H = T_B$. Let $\mathcal{B} = B^T$ be the set of blocks. Finally, since T is 2-transitive on $\mathcal{P}$, the number of blocks which incident with two points is a constant. Hence $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a 2-$(q^2+1, q, q-1)$ design admitting T be a flag transitive automorphism group by Lemma 13. □

3.3 T is one of the remaining families

In this subsection, let

$$\mathcal{T} = \{{}^2F_4(q), {}^3D_4(q), G_2(q), F_4(q), E_6^\epsilon(q), E_7(q), E_8(q)\}.$$

We will prove that there are no new design arise when $T \in \mathcal{T}$.

Firstly, we show that G_α cannot be a parabolic subgroup of G for any $T \in \mathcal{T}$.

Lemma 27. *Let G and $\mathcal{D}$ satisfy hypothesis of Theorem 1. If $T \in \mathcal{T}$, then G_α cannot be a parabolic subgroup of G.*

Proof. By Lemma 7, for all cases that $T \in \mathcal{T} \setminus E_6(q)$, there is a unique subdegree which is a power of p, so r is a power of p by Lemma 2(4). We can easily check that r is too small and the condition $r^2 > v$ cannot be satisfied. If $T = E_6(q)$, for the cases where G contains a graph automorphism and $G_\alpha \cap T$ is P_2 or P_4 Lemma 7 still applies (see [23, p.345]) and can also be ruled out similarly. If $G_\alpha \cap T$ is P_3 with type A_1A_4, then

$$v = \frac{(q^3+1)(q^4+1)(q^9-1)(q^6+1)(q^4+q^2+1)}{(q-1)}.$$

Since r divides $(|G_\alpha|, v-1)$, we have $r \mid eq(q-1)^5(q^5-1)$ and so r is too small to satisfy $r^2 > v$. If $G_\alpha \cap T$ is P_1 with type D_5, then

$$v = \frac{(q^8+q^4+1)(q^9-1)}{q-1}.$$

From [14], we know that there exists two non-trivial subdegrees:

$$d = \frac{q(q^3+1)(q^8-1)}{(q-1)} \quad \text{and} \quad d' = \frac{q^8(q^4+1)(q^5-1)}{(q-1)}.$$

Since $(d, d') = q(q^4+1)$, we have $r \mid q(q^4+1)$ by Lemma 2(4), which contradicts with $r^2 > v$. □

Let $\mathcal{T}_1 = \{F_4(q), E_6^\epsilon(q), E_7(q), E_8(q)\}$.

Lemma 28. *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T \in \mathcal{T}_1$ and G_α is non-parabolic, then G_α cannot be a maximal subgroup of maximal rank.*

Proof. If G_α is non-parabolic and of maximal rank, then for any $T \in \mathcal{T}_1$, we have a complete list of $T_\alpha := G_\alpha \cap T$ in [16, Tables 5.1-5.2]. All subgroups in [16, Table 5.2] and some cases in [16, Table 5.1] can be ruled out by the inequality $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$ in Lemma 9. Since r divides $(|G_\alpha|, v-1)$, for the remaining cases we have that $r^2 < v$, a contradiction.

For example, if $T = F_4(q)$ with order $q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$. Then T_α is one of the following: (1) $2.(L_2(q) \times PSp_6(q)).2$ (q odd); (2) $d.\Omega_9(q)$; (3) $d^2.P\Omega_8^+(q).S_3$; (4) ${}^3D_4(q).3$; (5) $Sp_4(q^2).2$ (q even); (6) $(Sp_4(q) \times Sp_4(q)).2$ (q even); (7) $h.(L_3^\epsilon(q) \times L_3^\epsilon(q)).h.2$, with $d = (2, q-1)$ and $h = (3, q-\epsilon)$.

If $T_\alpha = 2.(L_2(q) \times PSp_6(q)).2$ with q odd, then

$$|T_\alpha| = q^{10}(q^2-1)^2(q^4-1)(q^6-1) \ \text{ and } \ v = q^{14}(q^4+1)(q^4+q^2+1)(q^6+1).$$

Since $(q^2+1) \mid v$ and $(q^4+q^2+1) \mid v$, $(|G_\alpha|, v-1) \mid |Out(T)|(q^2-1)^4$ and so $r^2 < q^9 < v$, a contradiction.

If $T_\alpha = 2.P\Omega_9(q)$ with q odd, then

$$|T_\alpha| = q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1) \ \text{ and } \ v = q^8(q^8+q^4+1).$$

Since $q \mid v$, $(q^4+q^2+1) \mid v$, $v-1 \equiv 2 \pmod{q^4-1}$, we get r divides $2^4|Out(T)|(q^4+1)$ and so $r^2 < v$, a contradiction.

Cases (3)-(6) can be ruled out similarly, and Case (7) cannot occur because of $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$. □

Lemma 29. *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T \in \mathcal{T}_1$ and G_α is non-parabolic, then $T_0 = Soc(G_\alpha \cap T)$ is simple and $T_0 = T_0(q_0) \in Lie(p)$.*

Proof. Assume that $T_0 = Soc(G_\alpha \cap T)$ is not simple. Then by Lemma 10 and Lemma 28, one of the following holds:

(1) $G_\alpha = N_G(E)$, where E is an elementary abelian group given in [3, Theorem 1(II)];

(2) $T = E_8(q)$ with $p > 5$, and T_0 is either $A_5 \times A_6$ or $A_5 \times L_2(q)$;

(3) T_0 is as in Table 1.

From [3, Theorem 1(II)], we check that all subgroups in Case (1) are local and too small to satisfy $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$.

The order of subgroup in Case (2) is too small.

For Case (3), since G_α is not simple and not local by [3, Theorem 1], G_α is of maximal rank by [23, p.346], which has already been ruled out in Lemma 28. Therefore, T_0 is simple.

Now assume that $T_0 = T_0(q_0) \notin Lie(p)$. Then for all T, we find the possibilities of T_0 in [19, Table 1]. Some cases can be ruled out by the inequality $|T| < |Out(T)|^2|T_\alpha||T_\alpha|^2_{p'}$. In each of the remaining cases, since r must divides $(|G_\alpha|, v-1)$, r is too small to satisfy $v < r^2$. For example, assume that $T = F_4(q)$. If $T_0 \notin Lie(p)$, then according to [19, Table 1], it is one of the following: A_{5-10}, $L_2(7)$, $L_2(8)$, $L_2(13)$, $L_2(17)$, $L_2(25)$, $L_2(27)$, $L_3(3)$, $U_3(3)$, $U_4(2)$, $Sp_6(2)$, $\Omega_8^+(2)$, ${}^3D_4(2)$, J_2, J_2, $A_{11}(p = 11)$, $L_3(4)(p = 3)$, $L_4(3)(p = 2)$, ${}^2B_2(8)(p = 5)$, $M_{11}(p = 11)$. Since $|G| < |G_\alpha|^3$, T_0 is $A_9(q = 2)$, $A_{10}(q = 2)$, $Sp_6(2)(q = 2)$, $\Omega_8^+(2)(q = 2, 3)$, ${}^3D_4(2)(q = 2, 3)$, $J_2(q = 2)$ or $L_4(3)(q = 2)$. But, since $r \mid (|G_\alpha|, v-1)$, we have $r^2 < v$ for all these cases, which is a contradiction. □

Lemma 30. *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If $T_0 = T_0(q_0)$ is a simple group of Lie type and G_α is non-parabolic, then $T \notin \mathcal{T}_1$.*

Proof. First assume that $T = F_4(q)$. If $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$, then by Lemma 11 and Lemma 28, the only possible cases of $G_\alpha \cap T$ satisfying $|G| < |G_\alpha|^3$ are $F_4(q^{\frac{1}{2}})$ and $F_4(q^{\frac{1}{3}})$ when $q_0 > 2$. If $G_\alpha \cap T = F_4(q^{\frac{1}{2}})$, then $v = q^{12}(q^6+1)(q^4+1)(q^3+1)(q+1) > q^{26}$. Since q, $q+1$, q^2+1 and q^3+1 are factors of v, then $r \mid 2e(q-1)^2(q^3-1)^2$ by $r \mid (|G_\alpha|, v-1)$, which implies that $r^2 < v$, a contradiction. If $G_\alpha \cap T = F_4(q^{\frac{1}{3}})$, then since $p \mid vr$ divides $|G_\alpha|_{p'}$, which also implies $r^2 < v$. When $q_0 = 2$, the subgroups $T_0(2)$ with $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$ that satisfy $|G| < |G_\alpha|^3$ are $A_4^\epsilon(2)$, $B_3(2)$, $B_4(2)$, $C_3(2)$, $C_4(2)$ or $D_4^\epsilon(2)$. But in each case, $r \mid (|G_\alpha|, v-1)$ forces $r^2 < v$, a contradiction. If $\mathrm{rank}(T_0) \leqslant \frac{1}{2}\mathrm{rank}(T)$, then from Lemma 12, we have $|G_\alpha| < 4q^{20}\log_p q$. By further checking the orders of groups of Lie type, we find that if $|G_\alpha| < 4q^{20}\log_p q$, then $|G_\alpha|_{p'} < q^{12}$, and so $|G_\alpha||G_\alpha|^2_{p'} < |G|$, contrary to Lemma 9.

For $T = E_6^\epsilon(q)$, if $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$, then when $q_0 > 2$, by Lemma 11 the only possibilities are $E_6^\epsilon(q^{\frac{1}{2}})$, $E_6^\epsilon(q^{\frac{1}{3}})$, $C_4(q)$ and $F_4(q)$. But in all these cases, simple calculation shows that r are too small to satisfy $v < r^2$. When $q_0 = 2$, since $|G| < |G_\alpha|^3$, the possible subgroups $T_0(2)$ of $E_6^\epsilon(2)$ are $A_5^\epsilon(2)$, $B_4(2)$, $C_4(2)$, $D_4^\epsilon(2)$ and $D_5^\epsilon(2)$. However, the facts that $r \mid (|G_\alpha|, v-1)$ and $v < r^2$ implies that all these cases are impossible. If $\mathrm{rank}(T_0) \leqslant \frac{1}{2}\mathrm{rank}(T)$, then from Lemma 12, we have $|G_\alpha| < 4q^{28}\log_p q$. Considering the orders of groups of Lie type, we see that $|G_\alpha|_{p'} < q^{17}$, and so $|G_\alpha||G_\alpha|^2_{p'} < |G|$, a contradiction.

Assume that $T = E_7(q)$. If $rank(T_0) \leqslant \frac{1}{2}rank(T)$, then by Lemma 12 $|G_\alpha|^3 \leqslant |G|$, a contradiction. If $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$, then when $q_0 > 2$, B by Lemma 11, the only cases $G_\alpha \cap T$ satisfying $|G| < |G_\alpha|^3$ are $G_\alpha \cap T = E_7(q^{\frac{1}{s}})$, where $s = 2$ or 3. But in all cases we have $r^2 < v$. If $q_0 = 2$, then the possible subgroups $T_0(2)$ of $E_7(2)$ such that $|G| < |G_\alpha|^3$ are $A_6^\epsilon(2)$, $A_7^\epsilon(2)$, $B_5(2)$, $C_5(2)$, $D_5^\epsilon(2)$ and $D_6^\epsilon(2)$. However, the facts that $r \mid (|G_\alpha|, v-1)$ and $v < r^2$ implies that all these cases are impossible.

Assume that $T = E_8(q)$. If $\mathrm{rank}(T_0) \leqslant \frac{1}{2}\mathrm{rank}(T)$, then by Lemma 12 we get $|G_\alpha|^3 < |G|$, a contradiction. Therefore, $\mathrm{rank}(T_0) > \frac{1}{2}\mathrm{rank}(T)$. If $q_0 > 2$, then Lemma 11 implies $G_\alpha \cap T = E_8(q^{\frac{1}{s}})$, with $s = 2$ or 3. However in both cases we get a small r with $r^2 < v$, a contradiction. If $q_0 = 2$, all subgroups satisfying $|G_\alpha|^3 > |G|$ are $A_8^\epsilon(2)$, $B_7(2)$, $B_8(2)$, $C_7(2)$, $C_8(2)$, $D_8^\epsilon(2)$ and $D_7^\epsilon(2)$. But for all these cases we have $r^2 < v$. □

Lemma 31. *If $T = G_2(q)$ with $q = p^e > 2$, then G_α cannot be a non-parabolic maximal subgroup of G.*

Proof. Suppose that $T = G_2(q)$ with $q > 2$ since $G_2(2)' = PSU_3(3)$. All maximal subgroups of G can be found in [11] for odd q and in [2] for even q.

Assume that G_α be a non-parabolic maximal subgroup of G. First we deal with the case where $G_\alpha \cap T = SL_3^\epsilon(q).2$ with $\epsilon = \pm$. Then we have $v = \frac{1}{2}q^3(q^3 + \epsilon 1)$. By Lemma 2 and [23, Section 8] we conclude that r divides $\frac{(q^3-\epsilon 1)}{2}$ for odd q (cf. [23, Section 4, Case 1, $i = 1$]) and r divides $(q^3 - \epsilon 1)$ for even q (cf. [23, Section 3, Case 8]). The case that q odd is ruled out by $v < r^2$. If q is even, then $r = q^3 - \epsilon 1$. This, together with $k < r$, implies $k - 1 = \lambda\frac{q^3+\epsilon 2}{2}$, and so $\lambda = 1$ or $\lambda = 2$. From the result of [23] we known that $\lambda \neq 1$. If $\lambda = 2$, then since $k < r$, we have $\epsilon = -$. It follows that $k = q^3 - 1$ and $r = q^3 + 1$. This is impossible by Lemma 4 and [22, Theorem 1].

Now, if $G_\alpha \cap T = {}^2G_2(q)$ with $q = 3^{2n+1} \geqslant 27$, then $v = q^3(q+1)(q^3-1)$. Note that $q \mid v$ and $(q^2-1, v-1) = 1$, we have $(|G_\alpha|, v-1) \mid e(q^2-q+1)$, and it follows that $r^2 < v$, a contradiction.

The cases that $G_\alpha \cap T$ is $G_2(q_0)$ or $(SL_2(q) \circ SL_2(q)) \cdot 2$ can be ruled out similarly.

Using the inequality $|G| < |G_\alpha|^3$ and the fact that r divides $(|G_\alpha|, v-1)$, we find r too small to satisfy $r^2 > v$ for every other maximal subgroup. □

Lemma 32. *If $T = {}^2F_4(q)$, then G_α cannot be a non-parabolic maximal subgroup.*

Proof. Let $T = {}^2F_4(q)$ and G_α be a non-parabolic maximal subgroup of G. Then from the list of the maximal subgroups of G in [21], there are no subgroups G_α satisfying $|G| < |G_\alpha||G_\alpha|^2_{p'}$, except for the case $q = 2$. For the case $q = 2$, $G_\alpha \cap T$ is $L_3(3).2$ or $L_2(25)$. However in each case, since r divides $(|G_\alpha|, v-1)$, and so r is too small. □

Lemma 33. *If $T = {}^3D_4(q)$, then G_α cannot be a non-parabolic maximal subgroup.*

Proof. If $T = {}^3D_4(q)$ and G_α is a non-parabolic maximal subgroup of G, then all possibilities of $G_\alpha \cap T$ are listed in [12]. However, for all cases, the fact that r divide $(|G_\alpha|, v-1)$ give a small r which cannot satisfy the condition $v < r^2$. For example, if $G_\alpha \cap T$ is $G_2(q)$ of order $q^6(q^2-1)(q^6-1)$, then $v = q^6(q^8+q^4+1)$. Since $q \mid v$ and $(q^4+q^2+1) \mid v$, then $r \mid 3e(q^2-1)^2$, which contradicts with $v < r^2$. □

Lemma 34. *Suppose that G and $\mathcal{D}$ satisfy the hypothesis of Theorem 1. If the socle $T \in \mathcal{T}$, then G_α cannot be a non-parabolic maximal subgroup.*

Proof. It is follows from Lemmas 28−33. □

Now Theorem 1 is an immediate consequence of Propositions 24-26 and of Lemmas 27 and 34.

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