The absolute orders on the Coxeter groups A_n and B_n are Sperner

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Abstract

There are several classes of ranked posets related to reflection groups which are known to have the Sperner property, including the Bruhat orders and the generalized noncrossing partition lattices (i.e., the maximal intervals in absolute orders). In 2019, Harper–Kim proved that the absolute orders on the symmetric groups are (strongly) Sperner. In this paper, we give an alternate proof that extends to the signed symmetric groups and the dihedral groups. Our simple proof uses techniques inspired by Ford–Fulkerson's theory of networks and flows, and a product theorem.

Mathematics Subject Classifications: 05D05,05E99

1 Introduction

In 1928, Sperner [16] proved that the Boolean order $\mathbf{2}^n$ (i.e., the poset of all subsets of $\{1, 2, \ldots, n\}$) has the property that it contains no antichain (i.e., a subset of pairwise incomparable vertices) of cardinality larger than its largest rank level. In 1967, Rota [14] posed the following "Research Problem": prove or disprove that the refinement order Π_n (i.e., the poset of partitions of $\{1, \ldots, n\}$) shares this same property — today known as the *Sperner property* — for all n. Rota's Research Problem played a role in motivating the development of new theoretical tools for determining whether a class of ranked posets has the Sperner property. In particular, Harper [5] and Stanley [17] each introduced novel strengthenings of the (strong) Sperner property: the *normalized flow property* (abbr.

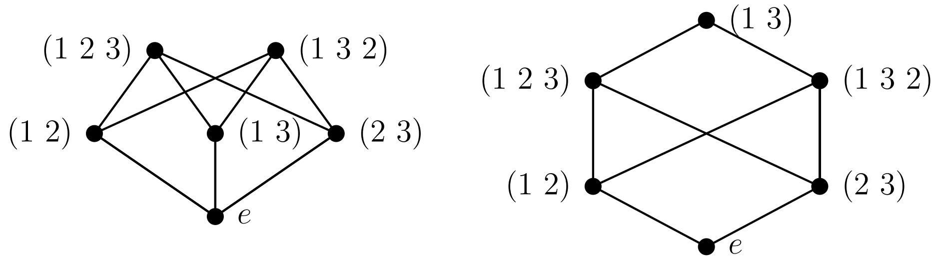

Figure 1: The absolute order (left) and a Bruhat order (right) on $A_2 \cong S_3$ (the symmetric group on 3 generators).

NFP) and the *Peck property*, respectively. The inspirations for these two properties come from seemingly disparate sources. The NFP was inspired by the Ford–Fulkerson theory of networks and flows, whereas the Peck property was inspired by the Hard Lefshetz Theorem from algebraic geometry. Despite these differences, there are striking category-theoretic parallels between the two properties (see, e.g., [7]). In particular, both NFP and Peck are "well-behaved" with respect to taking products; see [5, Product Theorem] and [11, Theorem 3.2], respectively.

There has been extensive interest in determining Spernerity for ranked posets associated to finite Coxeter groups (i.e., the finite Euclidean reflection groups). Since Coxeter groups are generated by reflections, they naturally can be associated to a ranked poset by equipping its Cayley graph — with respect to some choice of generating set of reflections — with a rank function mapping each vertex to its distance from the identity. Given a Coxeter group W with set of reflections T, a *strong Bruhat order* on W can be associated to any minimal generating subset of T, and the *absolute order* on W is associated to T itself. As noted by Kallipoliti [9, page 504], absolute orders have recently experienced a surge of interest due to their role in combinatorics, group theory, statistics, and invariant theory. Moreover, it has been observed by several separate authors (see, e.g., [13]) that the well-known "noncrossing partition lattices" appear as subposets in absolute orders (specifically, as "intervals" $[e, c]$ in W, where e is the identity and c is a Coxeter element in W).

The strong Bruhat orders were proven to be Peck (and thus strongly Sperner) by Stanley [17], and the weak Bruhat orders were proven to be Peck by Gaetz–Gao [3]. The noncrossing partition lattices were proven to be strongly Sperner in type A by Simion–Ullman [15], type B by Reiner [13], and the remaining types (as well as some complex reflection groups) by Mühle [10], providing an affirmative answer to Open Problem 3.5.12 in Armstrong's popular memoir [1].

In 2019, Harper–Kim [8] used the theory of normalized flows to prove that the absolute order on the symmetric group is strongly Sperner, a result which further emphasizes an interesting parallel between NFP and the Peck property. In this paper, we use Harper's Product Theorem to give an elegant proof of the following more general result:

Main Theorem. The absolute orders on the symmetric groups A_n, the hyperoctahedral groups B_n, and the dihedral groups $I_2(m)$, are strongly Sperner.

The proof, given in Section 5, is based on two Key Facts:

1. Any product of "claws" (a class of ranked posets defined in Example 2) is strongly Sperner.

2. Each of the absolute orders for A_n, B_n, and $I_2(m)$ contains a product of claws as a spanning subposet.

Key Fact (1) is restated in Theorem 7. Its proof is an immediate consequence of Harper's Product Theorem. The proof of Key Fact (2), given in Section 5, is primarily based on Proposition 11.

Since the initial posting of our results on arXiv, Gaetz–Gao have posted a pre-print [4] containing a proof that the absolute orders of the "generalized symmetric groups" — a class of complex Coxeter groups generalizing A_n and B_n — are strongly Sperner [4, Corollary 3.4]. Their proof was developed independently from our work in this paper, but also makes use of Harper's Product Theorem, and also is based on proving the Key Facts above. Gaetz–Gao also prove that the absolute orders of the exceptional irreducible Coxeter groups H_3, H_4, F_4, E_6, E_7, and E_8 are strongly Sperner by demonstrating explicit normalized flows on each using a computer [4, Proposition 3.7]. Finally, they prove that the Coxeter groups D_n are strongly Sperner for each n with $4 \leqslant n \leqslant 8$ using a computer. To complete the classification for finite real irreducible Coxeter groups, it remains to solve the following Open Problem; see also [4, Conjecture 1.2(1)].

Open Problem 1. Determine the values $n \geqslant 9$ for which D_n is Sperner.

2 Posets, the Sperner property, and normalized flows

This section contains a review of common terminology regarding ranked posets (except possibly our usage of the term "claw" in Example 2), as well as a review of Harper's theory of normalized flows developed in [5, 6].

A *poset* $(P, \preccurlyeq)$ is a finite set P equipped with a relation $\preccurlyeq$ that is reflexive, antisymmetric, and transitive. It is common to suppress mention of the relation $\preccurlyeq$, and refer to P as a poset. If $x, y \in P$, then y *covers* x, denoted $x \to y$, if $x \preccurlyeq z \preccurlyeq y$ implies that either $z = x$ or $z = y$. A poset P is *ranked* if there exists a function ρ (called a *rank function*) from P to $\{0, \ldots, n\}$ such that $\rho(y) = \rho(x) + 1$ whenever y covers x. A ranked poset P is partitioned by its *rank levels* $P_i = \rho^{-1}(i)$, for $i \in \{0, \ldots, n\}$.

Example 2. Let $n \geqslant 1$. Some examples of ranked posets include the:

1. n^{th} *Boolean order,* $\mathbf{2}^n = (\wp(\{1, \ldots, n\}), \subseteq)$, with rank function $r(X) = |X|$;

2. *lattice of divisors of* n, $\mathcal{D}_n = (\{1, \ldots, n\}, |)$, equipped with rank function mapping i to the number of primes in a prime factorization of i;

3. *n-chain*, $\mathcal{C}_n = (\{0, \ldots, n-1\}, \leqslant)$, with rank function $r(i) = i$; and

4. *n-antichain*, which consists of an n-set, no relations, and $r(x) = 0$ for all x.

5. *n-claw* $\mathscr{C}_n$ (for $n \geqslant 2$), the ordinal sum of a single element with an $(n-1)$-antichain; i.e., $\mathscr{C}_n$ is the ranked poset with one rank zero element covered by $n-1$ elements.

Let $k \geqslant 1$. A *k-chain* (resp. *k-antichain*) in a ranked poset $P = \bigsqcup_{i=0}^{r} P_i$ is a k-subset of P consisting of pairwise comparable (resp. incomparable) elements. A *k-family* in P is any subset of P containing no $(k+1)$-chain. For example, any union of k consecutive rank levels in P is a k-family. The poset P is *k-Sperner* if the union of the k largest rank levels is a maximal k-family; *strongly Sperner* if P is k-Sperner for all $k \in \{1, \dots, r+1\}$; and *rank unimodal* if there exists some $j \in \{0, \dots, r\}$ such that $|P_0| \leqslant |P_1| \leqslant \cdots \leqslant |P_{j-1}| \leqslant |P_j| \geqslant |P_{j+1}| \geqslant \cdots \geqslant |P_r|$. The 1-Sperner property is otherwise known simply as "the Sperner property", and a 1-family is otherwise known as an "antichain".

Lemma 3. *Suppose that P is a spanning subposet of P'; i.e., suppose P has the same vertex set and rank function as P'. If P is rank unimodal and strongly Sperner, then so is P'.*

Proof. It is clear that P' is rank unimodal. We proceed to prove that P' is strongly Sperner. Since P is rank unimodal, its largest k rank levels can be chosen so that their ranks are consecutive. Their union is a k-family in both P and P'. Since P is k-Sperner, this union is a k-family in P of maximal size, and therefore a k-family in P' of maximal size. □

It is not the case that a product of Sperner (or even strongly Sperner) posets is necessarily Sperner. However, there is a strengthening of strong Spernerity called the *normalized flow property* (abbr. NFP) which is "well-behaved" under taking products. In the following paragraph, we recall [5, 6] what it means for a ranked poset P to have a normalized flow with respect to the counting measure. This condition is stronger than NFP, and is both easier to define and sufficient for our purposes. A general definition of NFP may be found in [6, page 173].

Let $P = \sqcup_{i=0}^{r} P_i$ be a ranked poset, with vertex set V and edge set E. A *weight function* ν on P — i.e., a function $\nu : V \to \mathbb{R}_{>0}$ — has a unique extension to a measure via $\nu(X) = \Sigma_{x \in X} \nu(x)$ for each subset X of V. For example, the weight function on P defined by $x \mapsto 1$ for all x in V extends to the *counting measure on* P, defined by $X \mapsto |X|$ for all subsets X of P. A *normalized flow on P with respect to the measure* ν is a function $f : E \to \mathbb{R}_{\geqslant 0}$ that satisfies, for each $i \in \{0, \dots, r-1\}$,

1. for each $a \in P_i$, $\sum_{a \to b} f(a, b) = \nu(a)/\nu(P_i)$, and

2. for each $b \in P_{i+1}$, $\sum_{a \to b} f(a, b) = \nu(b)/\nu(P_{i+1})$.

Theorem 4 (Corollary to Theorem III in [5])**.** *If a ranked poset P has a normalized flow with respect to the counting measure, then P is strongly Sperner.*

Example 5. Let $n \geqslant 2$. It is easily verified that the function mapping each edge of the n-chain to 1 defines a normalized flow with respect to the counting measure. Likewise, the function mapping each edge of the n-claw to $1/(n-1)$ is also a normalized flow with respect to the corresponding counting measure. Thus all chains and claws are strongly Sperner by Theorem 4.

Let $(P, \preccurlyeq_P)$ and $(Q, \preccurlyeq_Q)$ be ranked posets, with rank functions r_P and r_Q. The *(Cartesian) product $P \times Q$ of P and Q* is the ranked poset with vertex set $\{(p,q) : p \in P, q \in Q\}$, relation $\preccurlyeq_{P\times Q}$ defined by $(p,q) \preccurlyeq_{P\times Q} (p',q')$ whenever $p \preccurlyeq_P p'$ and $q \preccurlyeq_Q q'$, and rank function $r_{P\times Q}$ defined by $r_{P\times Q}((p,q)) = r_P(p) r_Q(q)$. If P and Q are equipped with weight functions ν_P and ν_Q, then the weight function $\nu_{P\times Q}$ on the product $P \times Q$ is defined by $\nu_{P\times Q}((p,q)) = \nu_P(p)\nu_Q(q)$.

Theorem 6 (Harper's Product Theorem/Theorem I.C in [5])**.** *Let $P = \sqcup_i P_i$ and $Q = \sqcup_j Q_j$ be ranked posets which have normalized flows with respect to the weights ν_P and ν_Q, respectively. Suppose that the sequences $(\nu_P(P_i))_i$ and $(\nu_Q(Q_j))_j$ are log-concave. Then $(\nu_{P\times Q}((P\times Q)_k))_k$ is log-concave, and $P \times Q$ has a normalized flow with respect to $\nu_{P\times Q}$.*

It follows immediately (see the proof of Corollary 7 below) from Harper's Product Theorem that any finite product of nontrivial chains or claws is strongly Sperner. To demonstrate the power of product theorems (such as Harper's or [11, Theorem 3.2]), note that $\mathbf{2}^n = (\mathcal{C}_2)^n$ and $\mathcal{D}_n = \Pi_{\text{prime } p|n} \mathcal{C}_p$. It follows that the Boolean orders $\mathbf{2}^n$ and the lattices of divisors $\mathcal{D}_n$ are each strongly Sperner.

Corollary 7. *Any finite product of claws is strongly Sperner.*

Proof. As discussed in Example 5, any claw $\mathscr{C}_n$ $(n \geqslant 2)$ has a normalized flow with respect to the counting measure, and the sequence $(|(\mathscr{C}_n)_j|)_j$ is trivially log-concave. As an inductive hypothesis, let $k_1, \ldots, k_n \geqslant 2$, and suppose that $\Pi_{i=1}^{n-1} \mathscr{C}_{k_i}$ has a normalized flow with respect to the counting measure, and that $(|(\Pi_{i=1}^{n-1} \mathscr{C}_{k_i})_j|)_j$ is log-concave. Then the product $\Pi_{i=1}^{n} \mathscr{C}_{k_i}$ has a normalized flow with respect to the counting measure by Harper's Product Theorem, and thus is strongly Sperner by Theorem 4. □

It can similarly be shown that any product of chains is strongly Sperner. For example, the Boolean order $\mathbf{2}^n = (\mathcal{C}_2)^n$ and the lattice of divisors $\mathcal{D}_n = \Pi_{\text{prime } p|n} \mathcal{C}_p$ are each strongly Sperner.

3 Regular simplexes and cubes

The ranked posets of interest in this paper are the *absolute orders* associated to finite (real) Coxeter groups (i.e., finite Euclidean reflection groups). The finite irreducible Coxeter groups fall into two overlapping classes: the symmetry groups of regular polytopes, and the Weyl groups (i.e., the symmetry groups of irreducible "root systems"). The classification (see, e.g., [1, Figure 1.1]) of the finite irreducible Coxeter groups is one of the crowning achievements of group theory and Euclidean geometry. This classification has three infinite classes which are symmetry groups of regular polytopes: A_n, B_n, and $I_2(m)$. These groups are recalled below.

The regular n-simplex Δ_n is the convex hull of the standard basis $\{e_i\}_i^{n+1}$ for $\mathbb{R}^{n+1}$. There is an "obvious" bijective correspondence between faces of Δ_n and subsets of $\{i\}_{i=1}^{n+1}$, with i-dimensional faces (or *i-faces*) of Δ_n corresponding with subsets of size $i+1$. Via this correspondence, vertices are identified with singletons and facets (i.e., $(n-1)$-faces) are identified with n-sets. The symmetry group A_n of Δ_n is the group of permutations of $\{i\}_{i=1}^{n+1}$ (i.e., the symmetric group S_{n+1}). The set of reflections in A_n consists of all transpositions $(i\ j)$, $i \neq j$.

The n-cube $\square_n$ is the convex hull in $\mathbb{R}^n$ of the Cartesian product $\{-1,1\}^n \subset \mathbb{R}^n$. The dual polytope to the n-cube is the n-cross-polytope $\diamondsuit_n$, which is the convex hull of $\{\pm e_1, \pm e_2, \ldots, \pm e_n\} \subset \mathbb{R}^n$. Each i-face of $\diamondsuit_n$ corresponds to a subset $S \subset \{\pm j\}_{j=1}^n$ of size $i+1$ with the property that $k \in S$ implies $-k \notin S$. The symmetry group B_n for each of the dual polytopes $\square_n$ and $\diamondsuit_n$ is the group of *signed permutations*; i.e., the permutations w of the set $\{\pm j\}_{j=1}^n$ with the property that $w(-i) = -w(i)$ for all i. The group B_n is commonly known as the *signed symmetric group* or *hyperoctahedral group*. Following [9], we denote the signed permutation with cycle form $(a_1\ a_2\ \cdots\ a_k)(-a_1\ -a_2\ \cdots\ -a_k)$ by $(\!(a_1, a_2, \ldots, a_k)\!)$, and $(a_1\ a_2\ \cdots\ a_k\ -a_1\ -a_2\ \cdots\ -a_k)$ by $[a_1, a_2, \ldots, a_k]$. The set of reflections in B_n corresponds to the union of $\{[i]\}_{i=1}^n$ and $\{(\!(i,j)\!), (\!(i,-j)\!)\}_{1 \leqslant i < j \leqslant n}$.

Lemma 8. *For any pair (C, C') of distinct facets in Δ_n (resp. $\square_n$), there is a unique reflection in A_n (resp. B_n) mapping C to C'.*

Proof. Let $C \neq C'$ be facets in Δ_n. Since $C \neq C'$ correspond to subsets of $\{i\}_{i=1}^{n+1}$ of size n, it follows that $C - C' = \{i\}$ and $C' - C = \{j\}$ for some $i \neq j$. The unique reflection mapping C to C' is $(i\ j)$.

Now let $C \neq C'$ be facets in $\square_n$. The facets of $\square_n$ correspond to the vertices of $\diamondsuit_n$, which in turn correspond to elements of $\{\pm j\}_{j=1}^n$. Suppose without loss of generality that C corresponds to 1. Either C' corresponds to -1, j for some $j \neq 1$, or $-j$ for some $j \neq 1$. In any case, there is a unique reflection in B_n mapping C to C' (specifically, the reflections $[1]$, $(\!(1,j)\!)$, and $(\!(1,-j)\!)$, respectively). $\square$

Define a *(complete) flag* $\mathscr{F} = (\mathcal{P}_i)_{i=0}^n$ in an n-dimensional regular polytope $\mathcal{P}$ to be a sequence of faces in $\mathcal{P}$, ordered by containment, with $\dim(\mathcal{P}_i) = i$. The action of A_n (resp. B_n) on Δ_n (resp. $\square_n$) induces a simply transitive action on the associated set of flags. Hence if we designate some flag in Δ_n or $\square_n$ — call it the *standard flag* $\mathscr{F}^{\text{std}} = (\mathcal{P}_i^{\text{std}})_{i=0}^n$ — then a correspondence between elements in the polytope's symmetry group and its set of flags can be defined via $w \mapsto w \cdot \mathscr{F}^{\text{std}}$. Note that, for all $i \in \{0, \ldots, n\}$, the i-faces for the n-simplex (resp. the n-cube) are i-simplexes (resp. i-cubes).

We briefly recall some generalities about absolute orders; see, e.g., [1] for details. Let W be a finite Coxeter group with set of reflections T. The *absolute length* l_T on W is the word length with respect to T. The *absolute order* on W is defined by

$$\pi \leqslant \mu \text{ if and only if } l_T(\mu) = l_T(\pi) + l_T(\pi^{-1}\mu)$$

for all $\pi, \mu \in W$. Equivalently, the absolute order is the poset on W generated by the covering relations $w \to tw$, where $w \in W$, $t \in T$, and $l_T(w) < l_T(tw)$. This order is graded with rank function l_T. The absolute length generating function $\mathrm{P}_W(q) = \sum_{w\in W} q^{l_T(w)}$ satifies $\mathrm{P}_W(q) = \prod_{i=1}^{n}(1 + (d_i - 1)q)$, where $n = \mathrm{rank}(W)$ and $(d_i)_{i=1}^n$ is a sequence of positive integer invariants for W called the *degree sequence*; see Remark 9 below. It follows that $|T| = |l_T^{-1}(1)| = \sum_{i=1}^{n}(d_i - 1)$. Moreover, the rank sequence $\left(|l_T^{-1}(i)|\right)_{i=0}^{n}$ for any absolute order is strictly log-concave by [18, Theorem 4.5.2], and thus all of the absolute orders are rank unimodal.

Remark 9. The degree sequence $(d_i)_{i=1}^n$ for A_n is defined by $d_i = i + 1$, and for B_n by $d_i = 2i$. Additional details regarding degrees for Coxeter groups are not necessary for our purposes in this paper, but may be found in, e.g., [1, Section 2.7].

4 A factorization result

In Proposition 11 below, we prove that any element of A_n or B_n can be written uniquely as a product of reflections "with respect to a fixed flag" in the associated regular polytope, a fundamental result in the proof (given in Section 5) that each of the groups A_n and B_n contain a product of claws. For all that follows, $\mathcal{P}$ denotes the regular n-simplex Δ_n or n-cube $\square_n$, and W denotes the corresponding symmetry group. Note that if $\mathcal{P}$ is Δ_n or $\square_n$, each reflective symmetry of an i-face $\mathcal{P}_i$ of $\mathcal{P}$ uniquely extends to a reflective symmetry of $\mathcal{P}$. Define $T_{\mathcal{P}_i}$ to be the embedding of the set of reflections of $\mathcal{P}_i$ into W.

Lemma 10. *Let $\mathcal{P}$ be the n-simplex or n-cube, and let W be the corresponding group of symmetries with degree sequence $(d_i)_{i=1}^n$. Fix a standard flag $(\mathcal{P}_i^{std})_{i=0}^n$ in $\mathcal{P}$, and set $T_i = T_{\mathcal{P}_i^{std}}$. It follows that, for all $i \in \{1, \ldots, n\}$, $|T_i - T_{i-1}| = d_i - 1$.*

Proof. The n-simplex (resp. the n-cube) has the property that, for each i, each of its i-faces is an i-simplex (resp. i-cube). Hence the symmetry group for any of its i-faces is A_i (resp. B_i). If the degree sequence for the n-simplex (resp. n-cube) is $(d_j)_{j=1}^n$, then the degree sequence associated to an i-face is $(d_j)_{j=1}^i$. It follows that $|T_i - T_{i-1}| = |T_i| - |T_{i-1}| = \sum_{j=1}^{i}(d_j - 1) - \sum_{j=1}^{i-1}(d_j - 1) = d_i - 1$. $\square$

Proposition 11. *Let $\mathcal{P}$ be the n-simplex or n-cube, and let W be the associated symmetry group. Fix a standard flag $\mathscr{F}^{std} = (\mathcal{P}_i^{std})_{i=0}^n$ in $\mathcal{P}$, and set $T_i = T_{\mathcal{P}_i^{std}}$.*

1. *Any element $w \in W$ has a unique factorization of the form*

$$w = r_n r_{n-1} \cdots r_2 r_1$$

with $r_i \in (T_i - T_{i-1}) \sqcup \{e\}$ for each i, where e is the identity in W.

2. *Given such a factorization, the length can be computed via*

$$l_T\left(\prod_{i=0}^{n-1} r_{n-i}\right) = |\{i : r_i \neq e\}|.$$

3. Finally, $\prod_{i=0}^{n-1} r_{n-i}$ covers $\prod_{i=0}^{n-1} r'_{n-i}$ if there exists k such that $r_k \neq r'_k = e$ and $r_j = r'_j$ for all $j \neq k$.

Proof. We begin by proving (1). The claim is clearly true for $n = 1$. Now let $n > 1$ be arbitrary, and suppose the claim is true for $n-1$. Let $w \in W$, with corresponding flag $\mathscr{F} = (\mathcal{P}_i)_{i=0}^{n}$. If $(\mathcal{P}_i)_{i=0}^{n-1}$ is a flag in the "standard facet" $\mathcal{P}_{n-1}^{\text{std}}$, then the claim follows by the inductive hypothesis. Suppose instead that $(\mathcal{P}_i)_{i=0}^{n-1}$ is a flag in some other facet C. Lemma 8 implies that there is a unique reflection $r_n \in (T_n - T_{n-1}) - \{e\}$ mapping C to $\mathcal{P}_{n-1}^{\text{std}}$. By the inductive hypothesis, it follows that $r_n \cdot \mathscr{F} = (r_{n-1} \cdots r_2 r_1) \cdot \mathscr{F}^{\text{std}}$ with $r_i \in (T_i - T_{i-1}) \sqcup \{e\}$ for all $i \in \{1, \ldots, n-1\}$. Therefore $w \cdot \mathscr{F}^{\text{std}} = \mathscr{F} = r_n(r_{n-1} \cdots r_2 r_1) \cdot \mathscr{F}^{\text{std}}$, and the claim follows.

To prove (2), we first let $\mathcal{P}$ be the n-simplex and let W be its symmetry group. Assume without loss of generality that $\mathscr{F}^{\text{std}} = (\{1, \ldots, i+1\})_{i=0}^{n}$. Then $T_i - T_{i-1}$ consists of all transpositions $(j\ (i+1))$ with $j \in \{1, \ldots, i\}$. If w is a product of elements in $T_{i-1} \sqcup \{e\}$, then w is a permutation of $\{1, \ldots, i\}$. Hence $l_T(r_i w) > l_T(w)$, which implies that $l_T(r_i w) = l_T(w) + 1$. The claim follows from a straight-forward induction on n. Now let $\mathcal{P}$ be the n-cube and W its symmetry group. Assume without loss of generality $\mathscr{F}^{\text{std}} = (\mathcal{P}_i^{\text{std}})_{i=0}^{n}$ is chosen so that the symmetries of $\mathcal{P}_i^{\text{std}}$ correspond to symmetries of $\{\pm 1, \pm 2, \ldots, \pm i\}$. Then T_i consists of all reflections of the form $[j]$, $(\!(j,k)\!)$, and $(\!(-j,k)\!)$ with $j,k \in \{1, \ldots, i\}$ and $j \neq k$, and $T_i - T_{i-1}$ consists of all reflections of the form $[i]$, $(\!(j,i)\!)$, and $(\!(-j,i)\!)$ with $j \in \{1, \ldots, i-1\}$. Similar to the case above, the product $r_i w$ of r_i in $T_i - T_{i-1}$ with a product w of reflections in T_{i-1} has $l_T(r_i w) > l_T(w)$. Hence $l_T(r_i w) = l_T(w) + 1$, and the claim follows by induction.

Finally, to prove (3), let w and w' be elements of W with the property that their expansions $w = \prod_{i=0}^{n-1} r_i$ and $w' = \prod_{i=0}^{n-1} r'_i$ satisfy $r_k \neq r'_k = e$ for some k and $r_j = r'_j$ for all $j \neq k$. By Proposition 11.2, it follows that $l_T(w') + 1 = l_T(w)$. Set $\sigma = \prod_{i=0}^{k-1} r_i$ and $\tau = \prod_{i=k+1}^{n-1} r_i$, so that $w = \sigma r_k \tau$ and $w' = \sigma\tau$. Then $l_T((w')^{-1} w) = l_T(\tau^{-1}\sigma^{-1}\sigma r_k \tau) = l_T(\tau^{-1} r_k \tau) = 1$. Since $l_T(w') + l_T((w')^{-1} w) = l_T(w)$, it follows that w covers w'. □

5 Proof of Main Theorem

The Main Theorem is recalled below, followed by its proof:

Main Theorem. The absolute orders on the symmetric groups A_n, the hyperoctahedral groups B_n, and the dihedral groups $I_2(m)$, are strongly Sperner.

Proof of Main Theorem. Let $\mathcal{P}$ be the n-simplex or n-cube, and let W be the associated symmetry group. Fix a standard flag $\mathscr{F}^{\text{std}} = (\mathcal{P}_i^{\text{std}})_{i=0}^{n}$ in $\mathcal{P}$, and set $T_i = T_{\mathcal{P}_i^{\text{std}}}$. Let $(d_i)_{i=1}^{n}$ be the degree sequence for W. Consider the product poset

$$\prod_{i=0}^{n-1} \mathscr{C}_{d_{n-i}} = \mathscr{C}_{d_n} \times \cdots \times \mathscr{C}_{d_2} \times \mathscr{C}_{d_1}$$

of claws $\mathscr{C}_{d_i}$. For each i, define a bijective correspondence between the vertices of the claw $\mathscr{C}_{d_i}$ and the elements of $(T_i - T_{i-1}) \sqcup \{e\}$ by mapping the $d_i - 1$ rank one vertices in $\mathscr{C}_{d_i}$

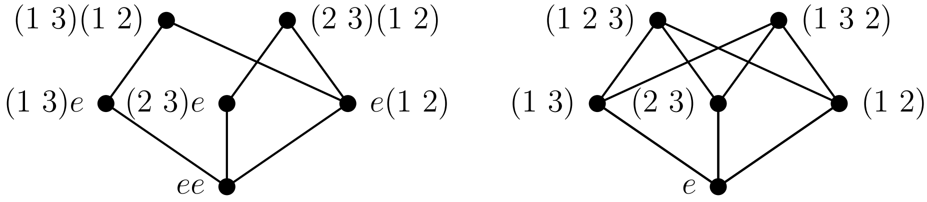

Figure 2: The product of claws $\mathscr{C}_3 \times \mathscr{C}_2$ (left) can be viewed as a spanning subposet of the absolute order on A_2 (right).

bijectively onto $T_i - T_{i-1}$ (such a bijection exists by Lemma 10) and the rank zero vertex in $\mathscr{C}_{d_i}$ to e. These bijective correspondences between claws and sets of reflections induce a bijective correspondence $\phi(r_n, \ldots, r_2, r_1) = r_n \cdots r_2 r_1$ between the vertices of the product poset $\prod_{i=0}^{n-1} \mathscr{C}_{d_{n-i}}$ and the vertices of the absolute order W by Proposition 11(1).

We claim that $\prod_{i=0}^{n-1} \mathscr{C}_{d_{n-i}}$ can be viewed as a spanning subposet of W via the above bijection between of the vertex sets. It suffices to prove that if y covers x in $\prod_{i=0}^{n-1} \mathscr{C}_{d_{n-i}}$, then $\phi(y)$ covers $\phi(x)$ in W. Suppose that $(r_n, \ldots, r_2, r_1)$ covers $(r'_n, \ldots, r'_2, r'_1)$ in the product of claws. Then there exists k for which $r_k \neq r'_k = e$ and $r_j = r'_j$ for all $j \neq k$. By Proposition 11(3), the claim immediately follows. By Lemma 7, $\prod_{i=0}^{n-1} \mathscr{C}_{d_{n-i}}$ is strongly Sperner. Since $\prod_{i=0}^{n-1} \mathscr{C}_{d_{n-i}}$ is a spanning subposet of W, it follows by Lemma 3 that W is strongly Sperner.

Finally, let $\mathcal{P}$ be the regular m-gon. The associated symmetry group is $I_2(m)$, and its degree sequence is $(2, m)$. It is clear that $\mathcal{P}$ and its symmetry group satisfy Lemmas 8, Lemma 10, and Proposition 11. The same arguments used above show that $I_2(m)$ contains $\mathscr{C}_m \times \mathscr{C}_2$ as a spanning subposet, and so $I_2(m)$ is strongly Sperner. $\square$

References

[1] D. Armstrong. Generalized noncrossing partitions and combinatorics of Coxeter groups. *Mem. Amer. Math. Soc.*, 202, 2009.

[2] E. R. Canfield. On a problem of Rota. *Bull. Amer. Math. Soc.*, 84(1): 164, 1978.

[3] C. Gaetz and Y. Gao. The weak Bruhat order on the symmetric group is Sperner. *Séminaire Lotharingien de Combinatoire*, 82B, 2019.

[4] C. Gaetz and Y. Gao. On the Sperner property for the absolute order on complex reflection groups. arXiv:1903.02033, 2019.

[5] L. H. Harper. The morphology of partially ordered sets. *J. Combin. Theory, Ser. A*, 17:44–58, 1974.

[6] L. H. Harper. The global theory of flows in networks. *Advances in Appl. Math.*, 1:158–181, 1980.

[7] L. H. Harper. Morphisms for the Strong Sperner Property of Stanley and Griggs. *Linear and Multilinear Algebra*, 16:323–337, 1984.

[8] L. H. Harper and G. B. Kim. Is the Symmetric Group Sperner? arXiv:1901.00197, 2019.

[9] M. Kallipoliti. The absolute order on the hyperoctahedral group. *J. Algebr. Comb.*, 34:183–211, 2011.

[10] H. Mühle. Symmetric decompositions and the strong Sperner property for noncrossing partition lattices. *J. Algebr. Comb.*, 45:745–775, 2017.

[11] R. A. Proctor, M. E. Saks, and D. G. Sturtevant. Product partial orders with the Sperner property. *Discrete Math.*, 30:173–180, 1980.

[12] R. A. Proctor. Representations of $\mathfrak{sl}_2$ on posets and the Sperner property. *SIAM J. Alg. Disc. Meth.*, 30(2):275–280, 1982.

[13] V. Reiner. Non-Crossing Partitions for Classical Reflection Groups. *Discrete Math.*, 177:195–222, 1997.

[14] G. C. Rota. Research problem: A generalization of Sperner's theorem. *J. Combin. Theory*, 2:104, 1967.

[15] R. Simion and D. Ullmann. On the Structure of the Lattice of Noncrossing Partitions. *Discrete Math.*, 98:193–206, 1991.

[16] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27:544–548, 1928.

[17] R. P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Alg. Disc. Meth.*, 1:168–184, 1980.

[18] H. S. Wilf, *Generatingfunctionology,* 2nd ed., Academic Press, 1994.