$\boldsymbol{S}$-hypersimplices, pulling triangulations, and monotone paths

Sebastian Manecke Raman Sanyal Jeonghoon So

Institut für Mathematik
Goethe-Universität Frankfurt
Germany

manecke@math.uni-frankfurt.de

sanyal@math.uni-frankfurt.de

jeonghoon.so@stud.uni-frankfurt.de

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Abstract

An S-hypersimplex for $S \subseteq \{0, 1, \ldots, d\}$ is the convex hull of all 0/1-vectors of length d with coordinate sum in S. These polytopes generalize the classical hypersimplices as well as cubes, crosspolytopes, and halfcubes. In this paper we study faces and dissections of S-hypersimplices. Moreover, we show that monotone path polytopes of S-hypersimplices yield all types of multipermutahedra. In analogy to cubes, we also show that the number of simplices in a pulling triangulation of a halfcube is independent of the pulling order.

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1 Introduction

The **cube** $\square_d = [0,1]^d$ together with the **simplex** $\Delta_d = \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d)$ and the **cross-polytope** $\lozenge_d = \mathrm{conv}(\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d)$ constitute the *Big Three*, three infinite families of convex polytopes whose geometric and combinatorial features make them ubiquitous throughout mathematics. A close cousin to the cube is the **(even) halfcube**

$$H_d \;\coloneqq\; \mathrm{conv}\left(\mathbf{p} \in \{0,1\}^d \;:\; p_1 + \cdots + p_d \text{ even}\right) .$$

The halfcubes H_1 and H_2 are a point and a segment, respectively, but for $d \geqslant 3$, $H_d \subset \mathbb{R}^d$ is a full-dimensional polytope. The 5-dimensional halfcube was already described by Thomas Gosset [11] in his classification of semi-regular polytopes. In contemporary mathematics, halfcubes appear under the name of *demi(hyper)cubes* [7] or *parity polytopes* [26]. In particular the name 'parity polytope' suggests a connection to combinatorial optimization and polyhedral combinatorics; see [6, 10] for more. However, halfcubes also occur in algebraic/topological combinatorics [13, 14], convex algebraic geometry [22], and in many more areas.

In this paper, we investigate basic properties of the following class of polytopes that contains cubes, simplices, cross-polytopes, and halfcubes. For a nonempty subset S of $[0,d] := \{0,1,\dots,d\}$, we define the ***S*-hypersimplex**

$$\Delta(d,S) \ := \ \operatorname{conv}\Big(\mathbf{v}\in\{0,1\}^d \ : \ v_1+v_2+\cdots+v_d\in S\Big)\,.$$

In the context of combinatorial optimization these polytopes were studied by Grötschel [15] associated to *cardinality homogeneous set systems*. Our name and notation derive from the fact that if $S=\{k\}$ is a singleton, then $\Delta(d,S) =: \Delta(d,k)$ is the well-known **(d,k)-hypersimplex**, the convex hull of all vectors $\mathbf{v}\in\{0,1\}^d$ with exactly k entries equal to 1. This is a $(d-1)$-dimensional polytope for $0<k<d$ that makes prominent appearances in combinatorial optimization as well as in algebraic geometry [19]. We call S **proper**, if $\Delta(d,S)$ is a d-dimensional polytope, which, for $d>1$, is precisely the case if $|S|\neq 1$ and $S\neq\{0,d\}$. For appropriate choices of $S\subseteq[0,d]$, we get
– the cube $\square_d = \Delta(d,[0,d])$,
– the even halfcube $H_d = \Delta(d,[0,d]\cap 2\mathbb{Z})$,
– the simplex $\Delta_d = \Delta(d,\{0,1\})$, and
– the cross-polytope $\Delta(d,\{1,d-1\})$ (up to linear isomorphism).

In Section 2, we study the vertices, edges, and facets of S-hypersimplices.

Our study is guided by a nice decomposition of S-hypersimplices into *Cayley polytopes* of hypersimplices.

In Section 3 we return to the halfcube. A combinatorial d-cube has the interesting property that all pulling triangulations have the same number of d-dimensional simplices. The Freudenthal or staircase triangulation is a pulling triangulation and shows that the number of simplices is exactly $d!$. We show that the number of simplices in any pulling triangulation of H_d is independent of the order in which the vertices are pulled. Moreover, we relate the full-dimensional simplices in any pulling triangulation of H_d to *partial* permutations and show that their number is given by

$$t(d) \ = \ \sum_{l=3}^{d}\frac{d!}{l!}\left(2^{l-1}-l\right)\,.$$

For a polytope $P\subset\mathbb{R}^d$ and a linear function $\ell:\mathbb{R}^d\to\mathbb{R}$, Billera and Sturmfels [4] associate the **monotone path polytope** $\Sigma_\ell(P)$.This is a $(\dim P-1)$-dimensional polytope whose vertices parametrize all *coherent* ℓ-monotone paths of P. As a particularly nice example, they show in [4, Example 5.4] that the monotone path polytope $\Sigma_{\mathsf{c}}(\square_d)$, where c is the linear function $\mathsf{c}(\mathbf{x}) = x_1+x_2+\cdots+x_d$, is, up to homothety, the polytope

$$\Pi_{d-1} \ = \ \operatorname{conv}((\sigma(1),\dots,\sigma(d)) \ : \ \sigma \text{ permutation of } [d])\,.$$

For a point $\mathbf{p}\in\mathbb{R}^d$, the convex hull of all permutations of $\mathbf{p}$ is called the **permutahedron** $\Pi(\mathbf{p})$ and we refer to $\Pi_{d-1}=\Pi(1,2,\dots,d)$ as the **standard** permutahedron. If

$\mathbf{p}$ has d distinct coordinates, then $\Pi(\mathbf{p})$ is combinatorially (even normally) equivalent to Π_{d-1}. For the case that $\mathbf{p}$ has repeated entries, these polytopes were studied by Billera-Sarangarajan [3] under the name of *multipermutahedra*. In Section 4, we study maximal c-monotone paths in the vertex-edge-graph of $\Delta(d,S)$. We show that all c-monotone paths of $\Delta(d,S)$ are coherent and that essentially all multipermutahedra $\Pi(\mathbf{p})$ for $\mathbf{p} \in [0,d-1]^d$ occur as monotone path polytopes of S-hypersimplices.

We close with some questions and ideas regarding S-hypersimplices in Section 5.

2 S-hypersimplices

The vertices of the d-cube can be identified with sets $A \subseteq [d]$ and we write $\mathbf{e}_A \in \{0,1\}^d$ for the point with $(\mathbf{e}_A)_i = 1$ if and only if $i \in A$. Let $S \subseteq [0,d]$. Since $\Delta(d,S)$ is a vertex-induced subpolytope of the cube, it is immediate that the vertices of $\Delta(d,S)$ are in bijection to

$$\binom{[d]}{S} \coloneqq \{A \subseteq [d] : |A| \in S\}\,.$$

This gives the number of vertices as $|V(\Delta(d,S))| = \sum_{s\in S}\binom{d}{s}$.

For a polytope $P \subset \mathbb{R}^d$ and a vector $\mathbf{c} \in \mathbb{R}^d$, let

$$P^{\mathbf{c}} \coloneqq \{\mathbf{x} \in P : \langle \mathbf{c}, \mathbf{x}\rangle \geqslant \langle \mathbf{c}, \mathbf{y}\rangle \text{ for all } \mathbf{y} \in P\}$$

be the **face in direction c**. For example, unless $S = \{0\}$, $\Delta(d,S)^{\mathbf{e}_i}$ is the convex hull of all $\mathbf{e}_A$ with $A \in \binom{[d]}{S}$ with $i \in A$. Likewise, unless $S = \{d\}$, $\Delta(d,S)^{-\mathbf{e}_i} = \operatorname{conv}(\mathbf{e}_A : A \in \binom{[d]}{S}, i \notin A)$. Under the identification $\{\mathbf{x} : x_i = 1\} \cong \mathbb{R}^{d-1}$, this gives for $|S| > 1$

$$\begin{aligned} \Delta(d,S)^{\mathbf{e}_i} &\cong \Delta(d-1,S^+) &\quad& \text{where } S^+ \coloneqq \{s-1 : s \in S, s > 0\}\,, \\ \Delta(d,S)^{-\mathbf{e}_i} &\cong \Delta(d-1,S^-) && \text{where } S^- \coloneqq \{s : s \in S, s < d-1\}\,. \end{aligned} \tag{1}$$

These faces will be helpful in determining the edges of $\Delta(d,S)$. For two sets $A, B \subseteq [d]$, we denote the **symmetric difference** of A and B by $A \triangle B \coloneqq (A \cup B) \setminus (A \cap B)$. For two points $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$, we write $[\mathbf{p},\mathbf{q}]$ for the segment joining $\mathbf{p}$ to $\mathbf{q}$.

Theorem 1. *Let $S = \{0 \leqslant s_1 < \cdots < s_k \leqslant d\}$ and $A, B \in \binom{[d]}{S}$ with $|A| = s_i \leqslant s_j = |B|$. Then $[\mathbf{e}_A, \mathbf{e}_B]$ is an edge of $\Delta(d,S)$ if and only if*

(i) *$A \subset B$ and $j = i+1$, or*
(ii) *$i = j$, $|A \triangle B| = 2$, and $\{s_i - 1, s_i + 1\} \not\subset S$.*

Proof. Let $A, B \in \binom{[d]}{S}$. If $i \in A \cap B$, then $[\mathbf{e}_A, \mathbf{e}_B]$ is an edge of $\Delta(d,S)$ if and only if $[\mathbf{e}_A, \mathbf{e}_B]$ is an edge of $\Delta(d,S)^{\mathbf{e}_i}$. By (1), $\Delta(d,S)^{\mathbf{e}_i} \cong \Delta(d-1,S^+)$ and $[\mathbf{e}_A, \mathbf{e}_B] \cong [\mathbf{e}_{A\setminus i}, \mathbf{e}_{B\setminus i}]$. Hence we can assume $A \cap B = \varnothing$. For $i \in [d] \setminus (A \cup B)$, we consider $\Delta(d,S)^{-\mathbf{e}_i}$ and by the same argument we may also assume that $A \cup B = [d]$.

If $A = \varnothing$, then $B = [d]$ and $[\mathbf{e}_A, \mathbf{e}_B]$ meets every $\Delta(d,k)$ in the relative interior for $0 < k < d$. Hence $[\mathbf{e}_A, \mathbf{e}_B]$ is an edge if and only if $S = \{0, d\}$, which gives us (i).

If $0 < s_i = |A|$, then let $i \in A$ and $j \in B$. Then $[\mathbf{e}_A, \mathbf{e}_B]$ and $[\mathbf{e}_{A'}, \mathbf{e}_{B'}]$ have the same midpoint for $A' = (A \setminus i) \cup j$ and $B' = (B \setminus j) \cup i$. Thus $[\mathbf{e}_A, \mathbf{e}_B]$ is an edge of $\Delta(d, S)$ if and only if $(A', B') = (B, A)$. This is the case precisely when $|A \triangle B| = 2$ and $A \cap B, A \cup B \notin \binom{[d]}{S}$. ☐

Theorem 1 makes the number of edges readily available.

Corollary 2. *The number of edges of* $\Delta(d, S)$ *is*

$$\sum_{i=1}^{k} \binom{d - s_i}{s_{i+1} - s_i} \binom{d}{s_i} + \sum_{j} \frac{s_j (d - s_j)}{2} \binom{d}{s_j},$$

where $s_{k+1} := 0$ *and the second sum is over all* $1 \leqslant j \leqslant k$*, such that* $\{s_j - 1, s_j + 1\} \not\subset S$.

Let us illustrate Theorem 1 for the classical examples of S-hypersimplices. For the cube $\square_d = \Delta(d, [0, d])$ it states, that the edges are of the form $[\mathbf{e}_A, \mathbf{e}_B]$ for any $A \subset B \subseteq [d]$ such that $|A| + 1 = |B|$. For the halfcube $H_d = \Delta(d, [0, d] \cap 2\mathbb{Z})$ we infer that there are $d(d-1)2^{d-3}$ many edges for $d \geqslant 3$. As for the cross-polytope $\Delta(d, \{1, d-1\})$, every two vertices are connected by an edge, except for $\mathbf{e}_{\{i\}}$ and $\mathbf{e}_{[d] \setminus \{i\}}$ for all $i \in [d]$.

Theorem 1 states that there are no *long* edges of $\Delta(d, S)$. We can make use of this fact to get a canonical decomposition of $\Delta(d, S)$. For $\lambda \in \mathbb{R}$, define the hyperplane

$$H(\lambda) \; := \; \{\mathbf{x} \in \mathbb{R}^d : x_1 + \cdots + x_d = \lambda\}.$$

We note the following consequence of Theorem 1.

Corollary 3. *Let* $S \subseteq [0, d]$ *and* $s \in S$*. Then* $\Delta(d, S) \cap H(s) = \Delta(d, s)$.

Proof. Every vertex $\mathbf{v}$ of $\Delta(d, S) \cap H(s)$ is of the form $F \cap H(s)$ for a unique inclusion-minimal face $F \subseteq \Delta(d, S)$ of dimension $\leqslant 1$. If F is an edge, then its endpoints $\mathbf{e}_A, \mathbf{e}_B$ satisfy $|A| < s < |B|$ which contradicts Theorem 1. Hence $F = \mathbf{e}_C$ for some $C \subseteq [d]$ with $|C| = s$. ☐

If $S = \{s_1 < \cdots < s_k\}$ with $k \geqslant 2$, then we can decompose

$$\Delta(d, S) \; = \; \Delta(d, s_1, s_2) \; \cup \; \Delta(d, s_2, s_3) \; \cup \; \cdots \; \cup \; \Delta(d, s_{k-1}, s_k), \tag{2}$$

where we set $\Delta(d, k, l) := \Delta(d, \{k, l\}) = \operatorname{conv}(\Delta(d, k) \cup \Delta(d, l))$ for $0 \leqslant k < l \leqslant d$. The polytope $\Delta(d, k, l)$ is the **Cayley polytope** of $\Delta(d, k)$ and $\Delta(d, l)$. Moreover, for $i < j$, we see that $\Delta(d, s_i, s_{i+1}) \cap \Delta(d, s_j, s_{j+1}) = \Delta(d, s_j)$ if $j = i + 1$ and $= \varnothing$ otherwise.

Before we determine the facets of $\Delta(d, S)$, we recall some properties of permutahedra from [3] that we will also need in Section 4. Let us say that a point $\mathbf{p} \in \mathbb{R}^d$ is **decreasing** if $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_d$. The **permutahedron** associated to $\mathbf{p}$ is the polytope

$$\Pi(\mathbf{p}) \; := \; \operatorname{conv}\big(\sigma\mathbf{p} := (p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(d)}) \; : \; \sigma \text{ permutation of } [d]\big).$$

Unless $p_i = p_j$ for all $i \neq j$, $\Pi(\mathbf{p})$ is a polytope of dimension $d - 1$ with affine hull given by $H(p_1 + \cdots + p_d)$.

Notice that $\Pi(\mathbf{p})^{\sigma\mathbf{u}} \; = \; \sigma^{-1}\Pi(\mathbf{p})^{\mathbf{u}}$. Thus, if we want to determine the face $\Pi(\mathbf{p})^{\mathbf{u}}$ up to permutation of coordinates, we can assume that $\mathbf{u}$ is decreasing. The **Minkowski sum** of two polytopes $P, Q \subset \mathbb{R}^d$ is the polytope $P + Q = \{\mathbf{p} + \mathbf{q} : p \in P, q \in Q\}$.

Proposition 4. *Let* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$ *be decreasing. Then*

$$\Pi(\mathbf{p}) + \Pi(\mathbf{q}) \ = \ \Pi(\mathbf{p}+\mathbf{q}) \,.$$

Proof. Set $P \coloneqq \Pi(\mathbf{p}) + \Pi(\mathbf{q})$. Clearly $\sigma(\mathbf{p}+\mathbf{q}) = \sigma\mathbf{p} + \sigma\mathbf{q}$ for all permutations σ and therefore every vertex of $\Pi(\mathbf{p}+\mathbf{q})$ is a vertex of P. For the converse, let $\mathbf{c}$ be such that $P^{\mathbf{c}} = \{\mathbf{v}\}$ is a vertex. Since P is invariant under coordinate permutations, we can assume that $\mathbf{c}$ is decreasing. Furthermore $(\Pi(\mathbf{p}) + \Pi(\mathbf{q}))^{\mathbf{c}} = \Pi(\mathbf{p})^{\mathbf{c}} + \Pi(\mathbf{q})^{\mathbf{c}}$ and it follows that $\mathbf{v} = \mathbf{p} + \mathbf{q}$. Hence, every vertex of P is of the form $\sigma(\mathbf{p}+\mathbf{q})$ for some permutation σ, which completes the proof. □

For $\nu_1 > \nu_2 > \cdots > \nu_r$ and $k_1, k_2, \dots, k_r \in \mathbb{Z}_{>0}$ such that $k_1 + \cdots + k_r = d$, we set

$$(\nu_1^{k_1}, \nu_2^{k_2}, \dots, \nu_r^{k_r}) \ \coloneqq \ (\underbrace{\nu_1, \dots, \nu_1}_{k_1}, \underbrace{\nu_2, \dots, \nu_2}_{k_2}, \dots, \underbrace{\nu_r, \dots, \nu_r}_{k_r}) \,.$$

For example, the (d,k)-hypersimplex is the permutahedron $\Delta(d,k) = \Pi(1^k, 0^{d-k})$.

The facets of permutahedra were described by Billera-Sarangarajan [3]. We recall their characterization. We write $I^c \coloneqq [d] \setminus I$ for the complement of $I \subseteq [d]$.

Theorem 5 ([3, Theorem 3.2])**.** *Let* $P = \Pi(\nu_1^{k_1}, \dots, \nu_r^{k_r})$ *and* $\mathbf{c} \in \mathbb{R}^d$. *Then* $P^{\mathbf{c}}$ *is a facet if and only if* $\mathbf{c} = \alpha\mathbf{e}_I + \beta\mathbf{e}_{I^c}$ *for some* $\alpha > \beta$ *and* $\varnothing \neq I \subset [d]$ *and* $h = |I|$ *satisfies*
(a) $k_1 + 1 \leqslant h \leqslant d - k_r - 1$*, or*
(b) $h = 1$ *if* $k_r < d - 1$*, or*
(c) $h = d - 1$ *if* $k_1 < d - 1$*.*

The theorem shows, for example, that $\Delta(d,k)$ for $1 < k < d-1$ has $2d$ facets with normals given by $\pm\mathbf{e}_1, \dots, \pm\mathbf{e}_d$.

In order to determine the facets of $\Delta(d,S)$, we appeal to the decomposition (2). Let $S = \{0 \leqslant s_1 < s_2 < \cdots < s_k \leqslant d\}$ be proper so that $\Delta(d,S) \subset \mathbb{R}^d$ is full-dimensional. We write $\mathbf{1} \coloneqq \mathbf{e}_{[d]}$ for the all-ones vector. If $s_1 > 0$, then $\Delta(d,s_1) = \Delta(d,S)^{-\mathbf{1}}$ is a facet. Likewise, if $s_d < d$, then $\Delta(d,s_k) = \Delta(d,S)^{\mathbf{1}}$ is a facet. If $F \subset \Delta(d,S)$ is any other facet, then its vertices cannot have all the same cardinality. If $s_i \in S$ is the minimal cardinality of a vertex in F, then $F \cap \Delta(d, s_i, s_{i+1})$ is a facet of $\Delta(d, s_i, s_{i+1})$. Hence, as a first step, we determine the facets of $\Delta(d, s_i, s_{i+1})$ that are not equal to $\Delta(d, s_i)$ and $\Delta(d, s_{i+1})$.

Let $S = \{k < l\}$ be proper. An easy calculation shows that

$$\Delta(d,k,l) \cap H(\tfrac{k+l}{2}) \ = \ \tfrac{1}{2}(\Delta(d,k) + \Delta(d,l)) \,.$$

Moreover, if $F \subset \Delta(d,k,l)$ is a facet, then $F \cap H(\frac{k+l}{2})$ is a facet of the right-hand side and every facet arises that way. Hence it suffices to determine the facets of $\overline{\Delta}(d,k,l) \coloneqq \Delta(d,k) + \Delta(d,l)$. We will need the notion of a join of two polytopes: If $P, Q \subset \mathbb{R}^d$ are polytopes such that their affine hulls are *skew*, i.e., non-parallel and disjoint, then $P * Q \coloneqq \operatorname{conv}(P \cup Q)$ is called the **join** of P and Q. Every k-dimensional face of $P * Q$ is of the form $F * G$ where $F \subseteq P$ and $G \subseteq G$ are (possibly empty) faces with $\dim F + \dim G = k - 1$.

Proposition 6. *Let $1 \leqslant k < l < d$. In addition to the facets $\Delta(d,k,l)^{\mathbf{1}} = \Delta(d,l)$ and $\Delta(d,k,l)^{-\mathbf{1}} = \Delta(d,k)$, there are*

$$\Delta(d,k,l)^{\mathbf{e}_i} \;\cong\; \Delta(d-1,k-1,l-1) \quad \text{and} \quad \Delta(d,k,l)^{-\mathbf{e}_i} \;\cong\; \Delta(d-1,k,l)$$

for $i = 1,\ldots,d$. Every other facet is of the form

$$\Delta(d,k,l)^{\mathbf{c}} \;\cong\; \Delta(h,k) * \Delta(d-h,l-h)$$

where $\mathbf{c} = (l-h)\mathbf{e}_I - (h-k)\mathbf{e}_{I^c}$ for any $\varnothing \neq I \subset [d]$ with $k < h \coloneqq |I| < l$.

Proof. We first determine the facets of $\overline{\Delta}(d,k,l)$. Using Proposition 4, we see that $\overline{\Delta}(d,k,l)$ is the permutahedron $\Pi(2^k, 1^{l-k}, 0^{d-l})$. Theorem 5 yields that the facet directions of $\overline{\Delta}(d,k,l)$ are given $\mathbf{c} = \alpha\mathbf{e}_I + \beta\mathbf{e}_{I^c}$ for $\varnothing \neq I \subset [d]$ with $|I| = 1$, $|I| = d-1$, or $k < |I| < l$ and $\alpha > \beta$. In particular, for every I there is, up to scaling, a unique choice for α and β so that $\Delta(d,k,l)^{\mathbf{c}}$ is a facet.

For $I = \{i\}$ we already observed that $\mathbf{c} = \mathbf{e}_I = \mathbf{e}_i$ yields a facet linearly isomorphic to $\Delta(d-1,k-1,l-1)$. Likewise, for $[d] \setminus I = \{j\}$, we obtain for $\mathbf{c} = \mathbf{e}_I - \mathbf{1} = -\mathbf{e}_j$ a facet that is linearly isomorphic to $\Delta(d-1,k,l)$.

For $I \subseteq [d]$ with $k < |I| < l$, we observe that $\mathbf{e}_A \in \Delta(d,k)^{\mathbf{e}_I}$ if and only if $A \subset I$ and $\mathbf{e}_A \in \Delta(d,l)^{\mathbf{e}_I}$ if and only if $I \subset A$. Set $h \coloneqq |I|$ and $\mathbf{c} = (l-h)\mathbf{e}_I - (h-k)\mathbf{e}_{I^c}$. For $A \in \binom{[d]}{k}$ we compute

$$\langle \mathbf{c}, \mathbf{e}_A \rangle \;=\; (l-h)|A \cap I| - (h-k)|A \cap I^c| \;\leqslant\; (l-h)k$$

with equality if and only if $A \subset I$. For $A \in \binom{[d]}{l}$, we compute

$$\langle \mathbf{c}, \mathbf{e}_A \rangle \;=\; (l-h)|A \cap I| - (h-k)|A \cap I^c| \;\leqslant\; (l-h)h - (h-k)(l-h) \;=\; (l-h)k$$

with equality if and only if $I \subset A$. Hence the hyperplane $H = \{\mathbf{x} : \langle \mathbf{c}, \mathbf{x} \rangle = (l-h)k\}$ supports $\Delta(d,k,l)$ in a facet, since H also supports a facet of $\overline{\Delta}(d,k,l)$. In particular, $\Delta(d,k) \cap H \cong \Delta(h,k)$ under the identification $\{\mathbf{x} : x_i = 0 \text{ for } i \notin I\} \cong \mathbb{R}^h$. Likewise $\Delta(d,l) \cap H \cong \Delta(d-h,l-h)$ under the identification $\{\mathbf{x} : x_i = 1 \text{ for } i \in I\} \cong \mathbb{R}^{d-h}$. This also shows that the given subspaces are skew and, since they lie in $H(k)$ and $H(l)$ respectively, are disjoint. This shows that $\Delta(d,l,k) \cong \Delta(h,k) * \Delta(d-h,l-h)$. □

It follows from Proposition 6 that $\Delta(d,k,l)$ and $\Delta(d,l,m)$ for $0 < k < l < m < d$ never have facet normals of type (v) in common. This gives us the following description of facets of S-hypersimplices; see also [15].

Theorem 7. *Let $S = \{0 \leqslant s_1 < \cdots < s_k \leqslant d\}$ be proper. Then $\Delta(d,S)$ has the following facets*

(i) $\Delta(d,S)^{\mathbf{1}} = \Delta(d,s_k)$ *provided* $s_k < d$;
(ii) $\Delta(d,S)^{-\mathbf{1}} = \Delta(d,s_1)$ *provided* $0 < s_1$;
(iii) $\Delta(d,S)^{\mathbf{e}_i} \cong \Delta(d-1,S^+)$ *for* $i = 1,\ldots,d$ *provided* S^+ *is proper;*
(iv) $\Delta(d,S)^{-\mathbf{e}_i} \cong \Delta(d-1,S^-)$ *for* $i = 1,\ldots,d$ *provided* S^- *is proper;*

(v) $\Delta(d,S)^{\mathbf{u}_I} \cong \Delta(h, h-s_i) * \Delta(d-h, s_{i+1}-h)$ *where* $I \subset [d]$ *with* $s_i < |I| =: h < s_{i+1}$ *for some* $0 < i < k$ *and* $\mathbf{u}_I := (s_{i+1}-h)\mathbf{e}_I - (h-s_i)\mathbf{e}_{I^c}$.

Proof. By decomposition (2), every facet F of $\Delta(d,S)$ determines a facet of $\Delta(d, s_i, s_{i+1})$ for some $1 \leqslant i < k$ and F is decomposed by this collection of facets. By examining the possible facet normals of $\Delta(d, s_i, s_{i+1})$, the statement readily follows. □

If $S = [0,d]$, then Theorem 7 gives us that $\square_d$ has exactly $2d$ facets in the coordinate directions $\pm\mathbf{e}_i$ for $i = 1,\ldots,d$. The facets are again cubes as $[0,d]^{\pm} = [0,d-1]$. The d-dimensional crosspolytope $\lozenge_d \cong \Delta(d,\{1,d-1\})$ has 2^d facets. The two facets of type (i), (ii), and those of type (iii) and (iv) are simplices. As for type (v) this is a join of two simplices and thus also a simplex.

The description of combinatorial type of each facet also leads to the number of k-dimensional faces for $0 \leqslant k < d$; cf. [21].

3 Pulling triangulations

A **subdivision** $\mathcal{S}$ of a d-dimensional polytope $P \subset \mathbb{R}^d$ is a collection $\mathcal{S} = \{P_1,\ldots,P_m\}$ of d-polytopes such that $P = P_1 \cup \cdots \cup P_m$ and $P_i \cap P_j$ is a face of P_i and P_j for all $1 \leqslant i < j \leqslant m$. If all polytopes P_i are simplices, then $\mathcal{S}$ is called a **triangulation**. Triangulations are the method-of-choice for various computations on polytopes including volume, lattice point counting, or, more generally, computing valuations; see [8].

A powerful method for computing a triangulation is the so-called *pulling triangulation*. Let P be a d-polytope and $\mathbf{v} \in V(P)$ a vertex. Let $F_1,\ldots,F_m$ be the facets of P not containing $\mathbf{v}$. A key insight is that the collection of polytopes

$$P_i \;:=\; \mathbf{v} * F_i \;:=\; \operatorname{conv}(F_i \cup \{\mathbf{v}\}) \qquad \text{for } i = 1,\ldots,m$$

constitutes a subdivision of P. This idea can be extended to obtain triangulations. Let $\preceq$ be a partial order on the vertex set $V(P)$ such that every nonempty face $F \subseteq P$ has a unique minimal element with respect to $\preceq$. We denote the minimal vertex of F by $\mathbf{v}_F$. The **pulling triangulation** $\mathsf{Pull}_{\preceq}(P)$ of P is recursively defined as follows. If P is a simplex, then $\mathsf{Pull}_{\preceq}(P) = \{P\}$. Otherwise, we define

$$\mathsf{Pull}_{\preceq}(P) \;=\; \bigcup_F \mathbf{v}_P * \mathsf{Pull}_{\preceq}(F)\,, \tag{3}$$

where the union is over all facets $F \subset P$ that do not contain $\mathbf{v}_P$ and where $\mathbf{v}_P * \mathsf{Pull}_{\preceq}(F) := \{\mathbf{v}_P * Q : Q \in \mathsf{Pull}_{\preceq}(F)\}$.

For the cube $\square_d$, or more generally the class of *compressed* polytopes [25], it can be shown that every simplex S in a pulling triangulation of $\square_d$ has the same volume $\frac{1}{d!}$. Thus, every pulling triangulation has exactly $d!$ many simplices, independent of the chosen order $\preceq$.

Recall that the halfcube is the S-hypersimplex $H_d = \Delta(d,[0,d] \cap 2\mathbb{Z})$. For $d \geqslant 5$ it is not true that the simplices in a pulling triangulation of H_d all have the same volume. The main result of this section is that still the number of simplices in a pulling triangulation is independent of the choice of $\preceq$.

Theorem 8. *Every pulling triangulation of H_d has the same number of simplices. The number of simplices $t(d) := |\mathsf{Pull}_{\preceq}(H_d)|$ is given by*

$$t(d) \ = \ \sum_{l=3}^{d} \frac{d!}{l!} \left(2^{l-1} - l\right) .$$

The proof of Theorem 8 is in two parts. We first show that the number of simplices of $\mathsf{Pull}_{\preceq}(H_d)$ is independent of $\preceq$. This yields a recurrence relation on $t(d)$. In the second part we review the construction of $\mathsf{Pull}_{\preceq}(H_d)$ from the perspective of choosing facets, which yields a combinatorial interpretation for $t(d)$ and which then verifies the stated expression.

From Theorem 7 we infer the following description of facets of H_d for $d \geqslant 3$: For every $i = 1, \ldots, d$ we have

$$\begin{aligned} H_d^{-\mathbf{e}_i} \ &= \ H_d \cap \{\mathbf{x} : x_i = 0\} \ \cong \ H_{d-1} \,, \\ H_d^{\mathbf{e}_i} \ &= \ H_d \cap \{\mathbf{x} : x_i = 1\} \ \cong \ H_{d-1} \,, \end{aligned}$$

where the last isomorphism is realized by reflection in a hyperplane $\{\mathbf{x} : x_j = \frac{1}{2}\}$ for $j \neq i$. The remaining facets of H_d are provided by Theorem 7(v) and, in case d is odd, by (i): For $B \subseteq [d]$ with $|B|$ odd and $\mathbf{u}_B = \mathbf{e}_B - \mathbf{e}_{B^c}$, we have

$$H_d^{\mathbf{u}_B} \ = \ H_d \cap \{\mathbf{x} : \langle \mathbf{e}_B, \mathbf{x} \rangle - \langle \mathbf{e}_{B^c}, \mathbf{x} \rangle = |B| - 1\} \ \cong \ \Delta_{d-1} \,.$$

Proposition 9. *The number $t(d)$ of simplices in a pulling triangulation of H_d satisfies*

$$t(d) \ = \ d \cdot t(d-1) + 2^{d-1} - d$$

for $d \geqslant 4$ and $t(d) = 1$ for $d \leqslant 3$.

Proof. We prove the result by induction on d. For $d = 1, 2, 3$, we note that H_d is itself a simplex and thus there is nothing to prove.

For $d \geqslant 4$, let $A \subseteq [d]$ be an even subset such that $\mathbf{e}_A \in \{0,1\}^d$ is the minimal vertex of P with respect to $\preceq$. By the discussion preceding the proposition, the facets not containing $\mathbf{e}_A$ are $H_d^{\mathbf{e}_i} \cong H_{d-1}$ for $i \notin A$, $H_d^{-\mathbf{e}_i} \cong H_{d-1}$ for $i \in A$, and $H_d^{\mathbf{u}_B} \cong \Delta_{d-1}$ for

$$B \in \mathcal{B} \ := \ \{B \subseteq [d] : |B| \text{ odd}, \, |A \triangle B| > 1\} \,.$$

Note that $|\mathcal{B}| = 2^{d-1} - d$. Thus it follows from (3) that

$$\begin{aligned} t(d) \ &= \ |\mathsf{Pull}_{\preceq}(H_d)| \ = \ \sum_{i \in A} |\mathsf{Pull}_{\preceq}(H_d^{-\mathbf{e}_i})| + \sum_{i \notin A} |\mathsf{Pull}_{\preceq}(H_d^{\mathbf{e}_i})| + \sum_{B \in \mathcal{B}} |\mathsf{Pull}_{\preceq}(H_d^{\mathbf{u}_B})| \\ &= \ d \cdot t(d-1) + 2^{d-1} - d \,, \end{aligned}$$

where the last equality follows by induction. □

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with suitable partial order $\preceq$ on $V(P)$. Every simplex in $\mathsf{Pull}_{\preceq}(P)$ corresponds to a chain of faces

$$P \;=\; G_0 \;\supset\; G_1 \;\supset\; G_2 \;\supset\; \cdots \;\supset\; G_k \tag{4}$$

such that $\dim G_i = d-i$ and G_k is a simplex of dimension $d-k$. The corresponding simplex is then given by $\mathbf{v}_{G_0} * \mathbf{v}_{G_1} * \cdots * G_k$. If P is a simple polytope with facets $F_1, \ldots, F_m$, then any such chain of faces is given by an ordered sequence of distinct indices $h_1, h_2, \ldots, h_k$ such that

$$G_i \;=\; F_{h_1} \cap F_{h_2} \cap \cdots \cap F_{h_i}$$

for all $i = 0, \ldots, k$.

For the d-dimensional cube $\square_d$, the facets can be described by $(i, \delta) \in [d] \times \{0, 1\}$ so that

$$K_i^\delta \;:=\; \square_d \cap \{x_i = \delta\} \;\cong\; \square_{d-1} \,.$$

The only faces of $\square_d$ that are simplices have dimensions $\leqslant 1$ and thus simplices in $\mathsf{Pull}_{\preceq}(\square_d)$ correspond to sequences $(i_1, \delta_1), \ldots, (i_{d-1}, \delta_{d-1}) \in [d] \times \{0, 1\}$ with $i_s \neq i_t$ for $s \neq t$. Thus, if we choose i_d such that $\{i_1, \ldots, i_{d-1}, i_d\} = [d]$, then every simplex of $\mathsf{Pull}_{\preceq}(\square_d)$ determines a permutation $\sigma = i_1 i_2 \cdots i_d$ of $[d]$.

Observe that for any vertex $\mathbf{v} \in \square_d$ and $i \in [d]$, we have that $\mathbf{v} \in K_i^0$ or $\mathbf{v} \in K_i^1$. This means that for any permutation $\sigma = i_1 i_2 \cdots i_d$ of $[d]$ there are $\delta_1, \delta_2, \ldots, \delta_{d-1} \in \{0, 1\}$ such that $(i_1, \delta_1), \ldots, (i_{d-1}, \delta_{d-1})$ come from a simplex in $\mathsf{Pull}_{\preceq}(\square_d)$. This shows that $|\mathsf{Pull}_{\preceq}(\square_d)| = d!$ independent of the order $\preceq$.

We call a sequence $\tau = i_1 i_2 \ldots i_k$ with $i_1, \ldots, i_k \in [d]$ a **partial permutation** if $i_s \neq i_t$ for $s \neq t$. We simply write $[d] \setminus \tau$ for $[d] \setminus \{i_1, \ldots, i_k\}$. The following Proposition completes the proof of Theorem 8.

Proposition 10. *For any suitable partial order $\preceq$, the simplices of $\mathsf{Pull}_{\preceq}(H_d)$ for $d \geqslant 3$ are in bijection to pairs (τ, B) where τ is a partial permutation of $[d]$ and $B \subseteq [d] \setminus \tau$ is a non-singleton subset of odd cardinality.*

Proof. Since H_3 is a simplex and the only admissible pair (τ, B) is given by the empty partial permutation and $B = [3]$, we assume $d \geqslant 4$. For $i = 1, \ldots, d$ and $\delta \in \{0, 1\}$, let

$$F_i^\delta \;:=\; H_d \cap \{x_i = \delta\} \;\cong\; H_{d-1}$$

be the halfcube facets of H_d. The halfcube H_d for $d \geqslant 4$ is not a simple polytope. However, it follows from Theorem 7 that the faces of H_d are halfcubes or simplices. If $G \subset H_d$ is a face linearly isomorphic to a halfcube of dimension $d-k \geqslant 4$, then G is a simple face in the sense that G is precisely the intersection of k halfcube facets. Every chain of faces (4) corresponds to some $(i_1, \delta_1), \ldots, (i_{k-1}, \delta_{k-1}) \in [d] \times \{0, 1\}$ such that $G_{k-1} = F_{i_1}^{\delta_1} \cap \cdots \cap F_{i_{k-1}}^{\delta_{k-1}}$ is isomorphic to H_{d-k+1} and G_k is a simplex facet of G_{k-1} not containing $\mathbf{v}_{G_{k-1}}$. This gives rise to a unique partial permutation $\tau = i_1 i_2 \ldots i_{k-1}$. To see that any such partial permutation can arise, we observe that again $V(H_d) \subset F_i^0 \cup F_i^1$ for all $i = 1, \ldots, d$. We can identify G_{k-1} with H_{d-k+1} embedded in $\{\mathbf{x} : x_{i_1} = \cdots = x_{i_{k-1}} = 0\}$ and $v_{G_{k-1}} = \mathbf{0}$. Now any simplex facet of H_{d-k+1} corresponds to an odd-cardinality subset $B \subset [d] \setminus \tau$ with $|B| \neq 1$. $\square$

4 Monotone paths

Let $P \subset \mathbb{R}^d$ be a polytope and $\ell : \mathbb{R}^d \to \mathbb{R}$ a linear function. An **ℓ-monotone path** of P is a sequence of vertices $W = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ such that $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ is an edge of P for $i = 1, \ldots, k-1$ and

$$\min \ell(P) \ = \ \ell(\mathbf{v}_1) \ < \ \ell(\mathbf{v}_2) \ < \ \cdots \ < \ \ell(\mathbf{v}_k) \ = \ \max \ell(P)\,.$$

More generally, a collection of faces $F_1, F_2, \ldots, F_k$ of P is an **induced subdivision** of the segment $\ell(P)$ if $F_1^{-\ell}$ and F_k^{ℓ} is a face of $P^{-\ell}$ and P^{ℓ}, respectively, and

$$F_i^{\ell} \ = \ F_{i+1}^{-\ell}$$

for $i = 1, \ldots, k-1$. If ℓ is **generic**, that is, if ℓ is not constant on edges of P, then the minimum/maximum of ℓ on every nonempty face F is attained at a unique vertex. In this case $F_i^{\pm\ell}$ is a vertex for all i and an induced subdivision is called a **cellular string**. An induced subdivision $F'_1, \ldots, F'_h$ is a refinement if for every $1 \leqslant i \leqslant k$, there are $1 \leqslant s < t \leqslant h$ such that $F'_s, \ldots, F'_t$ is an induced subdivision of $\ell(F_i)$. The collection of all induced subdivisions of $\ell(P)$ is partially ordered by refinement and is called the **Baues poset** of (P, ℓ). The minimal elements in the Baues poset are exactly the ℓ-monotone paths. Monotone paths are quintessential in the study of simplex-type algorithms in linear programming but they are also studied in topology in connection with iterated loop spaces; see [2, 20]. For the linear function $\mathsf{c}(x) = x_1 + \cdots + x_d$, Corollary 2 readily yields the c-monotone paths of $\Delta(d, S)$.

Corollary 11. *Let $S = \{s_1 < s_2 < \cdots < s_k\}$ be proper. The c-monotone paths correspond to sequences $A_1 \subset A_2 \subset \cdots \subset A_k$ with $|A_i| = s_i$ for all $i = 1, \ldots, k$.*

A ℓ-monotone path W is **coherent** if W is a monotone path with respect to the *shadow-vertex algorithm*; see [5, 17]. That is, if there is linear function $h_W : \mathbb{R}^d \to \mathbb{R}$ such that under the projection $\pi : \mathbb{R}^d \to \mathbb{R}^2$ given by $\pi(\mathbf{x}) = (\ell(\mathbf{x}), h_W(\mathbf{x}))$, the path W is mapped to one of the two paths in the boundary of the polygon $\pi(P)$. Figure 1 shows that in general coherent paths constitute a proper subset of all ℓ-monotone paths and it is interesting to determine for which pairs (P, ℓ) all ℓ-monotone paths are coherent; see, for example, the recent paper [9]. The S-hypersimplices with the linear function $\mathsf{c}(x)$ are examples of this.

Proposition 12. *Let $S \subseteq [0, d]$ be proper. Then all c-monotone path of $\Delta(d, S)$ are coherent.*

Proof. Let $A_1 \subset A_2 \subset \cdots \subset A_k$ be a c-monotone path. For the linear function

$$h(\mathbf{x}) \ \coloneqq \ \langle \mathbf{1}_{A_1} + \cdots + \mathbf{1}_{A_k}, \mathbf{x} \rangle$$

it is easy to see that $h(\mathbf{1}_B)$ with $B \in \binom{[d]}{S}$ is maximal if and only if $B \in \{A_1, \ldots, A_k\}$. ☐

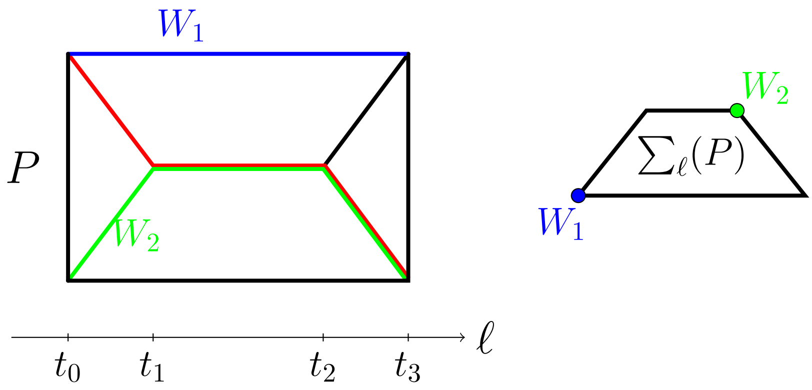

Figure 1: Left: Top view of triangular prism P and linear function ℓ. Three ℓ-monotone paths (in red, green, and blue) but the red path is not coherent. Right: Monotone path polytope $\Sigma_\ell(P)$.

The **monotone path polytope** $\Sigma_\ell(P)$ is a convex polytope of dimension $\dim P - 1$ whose face lattice is isomorphic to the poset of coherent subdivisions. The construction is a special case of fiber polytopes of Billera and Sturmfels [4]. Let $\ell(P) = [a, b] \subset \mathbb{R}$. A **section** of (P, ℓ) is a continuous function $\gamma : [a, b] \to P$ such that $\ell(\gamma(t)) = t$ for all $a \leqslant t \leqslant b$. Following [4], the monotone path polytope is defined as

$$\Sigma_\ell(P) \;=\; \operatorname{conv}\left\{ \frac{1}{b-a} \int_P \gamma \, d\mathbf{x} : \gamma \text{ section} \right\} .$$

We now determine the monotone path polytopes of $\Delta(d, S)$ with respect to the natural linear function $\mathsf{c}(x) = x_1 + \cdots + x_d$. Let us first observe that for $S \subset [d-1]$ the c-monotone paths of $\Delta(d, S)$ and $\Delta(d, S \cup \{0, d\})$ are in bijection. Clearly every c-monotone path of $\Delta(d, S \cup \{0, d\})$ restricts to a c-monotone path of $\Delta(d, S)$. Conversely, if $A_1 \subset \cdots \subset A_k$ corresponds to a c-monotone path, then $\varnothing =: A_0 \subset A_1 \subset \cdots \subset A_k \subset A_{k+1} = [d]$ is the unique extension to a c-monotone path of $\Delta(d, S \cup \{0, d\})$.

Theorem 13. *Let $S = \{0 = s_0 \leqslant s_1 < s_2 < \cdots < s_{k-1} < s_k = d\}$ be proper. Then*

$$\tfrac{1}{2}\mathbf{1} + d \cdot \Sigma_{\mathsf{c}}(\Delta(d, S)) \;=\; \Pi(k^{s_1 - s_0}, (k-1)^{s_2 - s_1}, \ldots, 1^{s_k - s_{k-1}}) .$$

Proof. Let $P \subset \mathbb{R}^d$ be a polytope with vertex set V and let ℓ be a linear function. Let $\ell(V) = \{a = t_0 < t_1 < \cdots < t_k = b\}$. We write $P_i := P \cap \ell^{-1}(t_i)$ for $0 \leqslant i \leqslant k$. Theorem 1.5 of [4] together with the fact that

$$P \cap \ell^{-1}\left(\frac{t_i + t_{i+1}}{2}\right) \;=\; \frac{1}{2}(P_i + P_{i+1})$$

for $0 \leqslant i < m$ yields that

$$(b-a)\Sigma_\ell(P) \;=\; \tfrac{1}{2}P_0 + \sum_{i=1}^{k-1} P_i + \tfrac{1}{2}P_k .$$

If $P = \Delta(d, S)$ and $\ell(\mathbf{x}) = \mathsf{c}(\mathbf{x})$, then $P_i = \Delta(d, s_i)$ for $0 \leqslant i \leqslant k$. In particular, $P_0 = \{\mathbf{0}\}$ and $P_k = \{\mathbf{1}\}$. Therefore

$$\tfrac{1}{2}\mathbf{1} + d \cdot \Sigma_{\mathsf{c}}(\Delta(d, S)) \;=\; \sum_{i=0}^{k} \Delta(d, s_i)\,.$$

Since $\Delta(d, s_i) = \Pi(1^{s_i}, 0^{d-s_i})$ we conclude from Proposition 4 that the above sum is the permutahedron $\Pi(\mathbf{p})$ for

$$\mathbf{p} = (1^{s_0}, 0^{d-s_0}) + \cdots + (1^{s_k}, 0^{d-s_k})\,.$$

This finishes the argument. □

5 Further questions

Volumes and Gröbner bases

Laplace and later Stanley [24] showed that the volume of $\Delta(d, i, i + 1)$ is $\frac{A(d,i)}{d!}$ where $A(d, i)$ counts the number of permutations σ of $[d]$ with i **descents**, that is, the number of $1 \leqslant i < d$ such that $\sigma(i) > \sigma(i+1)$; see also [18, 23]. This implies that $d!\ \mathrm{vol}\,\Delta(d, [k, l])$ is the number of permutations of $[d]$ with descent number in $[k, l] = \{k, k + 1, \ldots, l\}$ for any $k < l$. It would be very interesting to know if $\mathrm{vol}\,\Delta(d, S)$ has a combinatorial interpretation for all S. In light of (2) it would be sufficient to determine $\mathrm{vol}\,\Delta(d, k, l)$ for $l - k > 1$.

For $0 \leqslant k < d$, the hypersimplices $\Delta(d, k, k + 1) \cong \Delta(d, k + 1)$ are *alcoved polytopes* in the sense of Lam–Postnikov [18] and hence come with a canonical square-free and unimodular triangulation. This is reflected by the fact that the associated toric ideals have quadratic and square-free Gröbner bases with respect to the reverse-lexicographic term order.

For general $k < l$, the polytopes $\Delta(d, k, l)$ are not alcoved anymore. It would be interesting if $\Delta(d, k, l)$ has a unimodular triangulation or square-free Gröbner basis.

5.1 Extension complexity

An **extension** of a polytope P is a polytope Q together with a surjective linear projection $Q \to P$. The **extension complexity** $\mathrm{ext}(P)$ of P is the minimal number of facets of an extension of P. This is a parameter that is of interest in combinatorial optimization [16]. It was shown in [12] that $\mathrm{ext}(\Delta(d, k, k + 1)) = 2d$ for $1 \leqslant k \leqslant d - 2$.

A realization of the join of two convex polytopes $P, Q \subset \mathbb{R}^d$ is given by $P * Q = \mathrm{conv}((P \times \mathbf{0} \times 0) \cup (\mathbf{0} \times Q \times 1))$. If P and Q has m and n facets, respectively, then $P * Q$ has $m+n$ facets. Balas' union bound [1] is the observation that $P * Q \to P \cup Q$ and hence $\mathrm{ext}(P \cup Q) \leqslant \mathrm{ext}(P) + \mathrm{ext}(Q)$. Iterating the join over the pieces of the decomposition 2 shows the following.

Proposition 14. *If $S \subseteq [0, d]$ is proper, then*

$$\mathrm{ext}(\Delta(d, S)) \leqslant 2d\,(|S| - 1)\,.$$

This is a nontrivial bound as the number of facets of $\Delta(d, S)$ is at least $2+2d+\sum_{r \notin S}\binom{d}{r}$. To illustrate, note that the number of facets of the halfcube H_d for $d \geqslant 5$ is $2d + 2^{d-1}$ whereas the bounded afforded by Proposition 14 is $\leqslant d^2$. Carr and Konjevod [6] gave an extension of H_d of size linear in d. It would be interesting to know lower bounds on the extension complexity of $\Delta(d, S)$, maybe using the approach via rectangular covering; c.f. [12].

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