Pure $\mathcal{O}$-sequences arising from 2-dimensional PS ear-decomposable simplicial complexes

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Abstract

We show that the h-vector of a 2-dimensional PS ear-decomposable simplicial complex is a pure $\mathcal{O}$-sequence. This provides a strengthening of Stanley's conjecture for matroid h-vectors in rank 3. Our approach modifies the approach of combinatorial shifting for arbitrary simplicial complexes to the setting of 2-dimensional PS ear-decomposable complexes, which allows us to greedily construct pure multicomplex corresponding to each PS ear-decomposable complex.

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1   Introduction

In the late 1970s, Stanley [9] conjectured that the h-vector of a matroid simplicial complex is a pure $\mathcal{O}$-sequence. A significant amount of effort has been put forth to try to solve this problem, and while the conjecture still remains open, some partial progress has been made. This progress generally falls into one of two types. One approach is to prove Stanley's conjectures for special classes of matroids. For example, Merino [7] proved Stanley's conjecture for cographic matroids and Oh [8] proved it for cotransversal matroids. A second approach is to investigate matroids of low rank. The first nontrivial case of the problem comes in rank 3. Hà, Stokes, and Zanello [4] used an algebraic approach to establish Stanley's conjecture for rank at most 3, while De Loera, Kemper, and Klee [2] gave a more combinatorial proof in rank 3. Klee and Samper [6] gave an algorithmic proof in rank 4.

In the late 1990s, Chari [1] showed that any matroid simplicial complex is PS ear-decomposable. Broadly speaking, this is an inductive decomposition of a simplicial complex into subcomplexes whose contribution to the h-vector could, in principle, correspond to an interval of monomials in the divisibility lattice. The class of PS ear-decomposable

simplicial complexes is larger than the class of matroid simplicial complexes, but from a combinatorial perspective, their structure seems to more clearly reflect that of the pure $\mathcal{O}$-sequences one would like to construct. As a consequence, it is natural to extend Stanley's conjecture to the family of PS ear-decomposable simplicial complexes (see Conjecture 10).

In this paper, we use this approach to show that h-vectors of 2-dimensional PS ear-decomposable simplicial complexes (which correspond to rank-3 matroids) are pure $\mathcal{O}$-sequences (see Theorem 11). The fact that Stanley's conjecture holds for rank-3 matroids is not new, however the approach is significant for two reasons. First, we establish Stanley's conjecture for a larger class of 2-dimensional simplicial complexes. Second, our approach parallels the theory of combinatorial shifting/compression that is ubiquitous in the extremal combinatorics of f-vector theory. In the process of proving our main result, we establish a number of structural results on the extremal combinatorics of 2-dimensional PS ear-decomposable complexes. Unfortunately, the complexity of the argument presented here makes it difficult to extend the result to rank-4 matroids. However, a different perspective on the case analysis could, in principle, lead to a more general approach to Stanley's conjecture.

Before we can describe our approach in greater detail, we must establish the relevant background on simplicial complexes, PS ear-decompositions, matroids, and face enumeration. Further information can be found in Stanley's text [10]. After doing this, we give a more thorough overview of our approach in Section 2.5.

2 Background

2.1 Simplicial complexes and face numbers

A **simplicial complex** Δ on (finite) vertex set $V = V(\Delta)$ is a collection of subsets $F \subseteq V$ called **faces** with the property that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$ as well. To each abstract simplicial complex, there is an associated geometric object called its **geometric realization**, $\|\Delta\|$, which contains a geometric simplex for each face $F \in \Delta$.

The **dimension** of a face $F \in \Delta$ is $\dim(F) = |F| - 1$ and the dimension of Δ is $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$. We say that Δ is **pure** if all of its facets (maximal faces under inclusion) have the same dimension. We will typically assume that Δ is pure of dimension $d - 1$, meaning each of its facets contains d vertices.

The most fundamental combinatorial data of a $(d-1)$-dimensional simplicial complex Δ is encoded in its **f-vector**, $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_{d-1}(\Delta))$, where the **$f$-numbers** $f_i(\Delta)$ count the number of i-dimensional faces in Δ. For example, $f_0(\Delta)$, $f_1(\Delta)$, and $f_2(\Delta)$ respectively count the number of vertices, edges, and triangular faces in Δ. Unless Δ itself is the empty complex, $f_{-1}(\Delta) = 1$, corresponding to the empty face.

In many cases, it is more natural to perform a combinatorial transformation the f-vector of Δ to obtain the **h-vector**, $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, whose entries are given by

$$h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}(\Delta).$$

Concretely, when $d = 3$ (that is, $\dim(\Delta) = 2$), the h-numbers of Δ are given by

$$
\begin{aligned}
h_0(\Delta) &= 1 \\
h_1(\Delta) &= f_0(\Delta) - 3 \\
h_2(\Delta) &= f_1(\Delta) - 2f_0(\Delta) + 3 \\
h_3(\Delta) &= f_2(\Delta) - f_1(\Delta) + f_0(\Delta) - 1.
\end{aligned}
$$

2.2 PS ear-decomposable complexes

Chari [1] defined the family of $(d-1)$-dimensional PS ear-decomposable simplicial complexes. Before defining PS ear-decomposable simplicial complexes, we need to define the components that are used to build them, which are PS spheres and PS balls.

Definition 1. A **PS sphere** is a triangulated sphere that can be decomposed as a join of simplex boundaries. A **PS ball** is a triangulated ball that can be decomposed as the join of a simplex and a PS sphere.

Let us illustrate these definitions when $d = 3$. We will use σ^k to denote the k-dimensional simplex and $\partial\sigma^k$ to denote its boundary. For example, when $k = 3$, σ^3 is a solid tetrahedron and $\partial\sigma^3$ is its boundary. When $k = 0$ we set $\partial\sigma^0 = \{\emptyset\}$ by convention.

For $d = 3$, the possible PS 2-spheres and PS 2-balls are shown in Figure 1. The boundary of each PS ball is highlighted in red.

This leads us to the main object of study in this paper, PS ear-decomposable simplicial complexes.

Definition 2. Let Δ be a pure $(d-1)$-dimensional simplicial complex. A **PS ear-decomposition** of Δ is a decomposition of Δ into subcomplexes $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ such that

1. Σ_0 is a PS $(d-1)$-sphere,

2. Σ_j is a PS $(d-1)$-ball for $1 \leqslant j \leqslant t$, and

3. $\Sigma_j \cap \left(\bigcup_{i=0}^{j-1} \Sigma_i \right) = \partial\Sigma_j$ for all $1 \leqslant j \leqslant t$.

We say Δ is **PS ear-decomposable** if it admits a PS ear-decomposition.

In other words, a PS ear-decomposable complex can be constructed inductively by starting with a PS sphere, then repeatedly gluing PS balls to the existing complex such that each successive gluing takes place along the boundary of the corresponding ball.
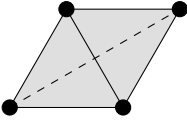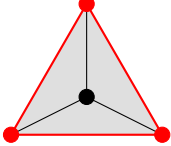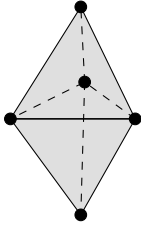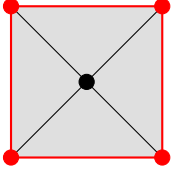
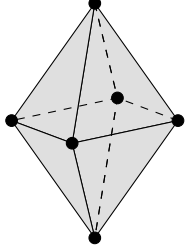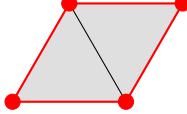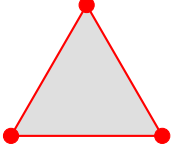| PS Spheres		PS Balls	
Decomposition	Geometry	Decomposition	Geometry
$\partial\sigma^3$		$\sigma^0 * \partial\sigma^2$	
$\partial\sigma^2 * \partial\sigma^1$		$\sigma^0 * \partial\sigma^1 * \partial\sigma^1$	
$\partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$		$\sigma^1 * \partial\sigma^1$	
		σ^2	

Figure 1: PS spheres and balls in dimension 2

2.3 Multicomplexes and $\mathcal{O}$-sequences

The second main objects of study in this paper are multicomplexes, which are multiset analogues of simplicial complexes. A **multicomplex** $\mathcal{M}$ is a (finite) collection of monomials that is closed under divisibility; i.e., if $\mu \in \mathcal{M}$ and $\nu | \mu$, then $\nu \in \mathcal{M}$. A multicomplex is **pure** if all of its maximal monomials under divisibility have the same degree (with $\deg(x_i) = 1$ for all i).

Just as the f-vector of a simplicial complex Δ counts the faces in Δ by cardinality, a multicomplex has a corresponding **F-vector** that enumerates its elements by degree. If $\mathcal{M}$ is a multicomplex and d is the maximal degree of a monomial in $\mathcal{M}$, then the F-vector is $F(\mathcal{M}) = (F_0, F_1, \ldots, F_d)$, where $F_i(\mathcal{M})$ counts the number of monomials in $\mathcal{M}$ of degree i.

A vector $F = (F_0, F_1, \ldots, F_d) \in \mathbb{Z}_{\geqslant 0}^{d+1}$ that can be realized as the F-vector of a (pure) multicomplex is called a **(pure) $\mathcal{O}$-sequence**.

Example 3. The vector $F = (1, 3, 5, 3)$ is a pure $\mathcal{O}$-sequence, with corresponding multicomplex $\mathcal{M} = \{1, x, y, z, x^2, xy, xz, y^2, yz, x^3, xyz, y^3\}$. We see that $\mathcal{M}$ is a multicomplex because the divisors of any monomial of $\mathcal{M}$ also belong to $\mathcal{M}$. For example, $xyz \in \mathcal{M}$, and its divisors — $1, x, y, z, xy, xz, yz$ — also belong to $\mathcal{M}$. Additionally, $\mathcal{M}$ is pure because every monomial in $\mathcal{M}$ divides a degree-3 monomial in $\mathcal{M}$.

Example 4. The vector $F = (1, 3, 1)$ is an $\mathcal{O}$-sequence, but not a pure $\mathcal{O}$-sequence. The monomials $\mathcal{M} = \{1, x, y, z, xy\}$ form a multicomplex with F-vector $(1, 3, 1)$. However, F is not a pure $\mathcal{O}$-sequence because any degree-2 monomial has at most two degree-1 divisors. Therefore, a pure $\mathcal{O}$-sequence with $F_2 = 1$ must have $F_1 \leqslant 2$.

Example 5. The h-vector of a PS sphere is a pure multicomplex. For PS 2-spheres we exhibit their h-vectors and corresponding pure multicomplexes:

- If $\Sigma_0 = \partial\sigma^3$, then $h(\Sigma_0) = (1, 1, 1, 1)$, which corresponds to the pure multicomplex $\mathcal{M} = \{1, x, x^2, x^3\}$.

- If $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$, then $h(\Sigma_0) = (1, 2, 2, 1)$, which corresponds to the pure multicomplex $\mathcal{M} = \{1, x, y, x^2, xy, x^2y\}$.

- If $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$, then $h(\Sigma_0) = (1, 3, 3, 1)$, which corresponds to the pure multicomplex $\mathcal{M} = \{1, x, y, z, xy, xz, yz, xyz\}$.

In general, if $\Sigma_0 = \partial\sigma^{d_1} * \partial\sigma^{d_2} * \cdots * \partial\sigma^{d_k}$, its corresponding pure multicomplex contains all divisors of the monomial $x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}$.

2.4 Matroid h-vectors and Stanley's conjecture

Now that we have defined PS ear-decomposable simplicial complexes, multicomplexes, and $\mathcal{O}$-sequences, we are finally in a position to state the problem we wish to study.

In the late 1970's, Stanley [9] established a deep connection between commutative algebra and the combinatorics of f- and h-vectors of certain families of simplicial complexes in the following result.

Theorem 6. *A vector $h = (h_0, h_1, \ldots, h_d) \in \mathbb{Z}_{\geqslant 0}^d$ is the h-vector of a $(d-1)$-dimensional Cohen-Macaulay simplicial complex if and only if it is an $\mathcal{O}$-sequence.*

We will not formally define Cohen-Macaulay simplicial complexes at this point as they will not play a direct role in the rest of this paper, however it is worth noting that triangulations of spheres and balls are Cohen-Macaulay simplicial complexes. Another family of interesting simplicial complexes is the family of matroid simplicial complexes.

Definition 7. A **matroid** is a nonempty simplicial complex Δ that satisfies the following additional property: if F and G are faces in Δ with $|F| < |G|$, then there exists an element $x \in G \setminus F$ such that $F \cup \{x\}$ is also a face of Δ.

The extra structure imposed by this so-called exchange axiom adds tremendous structure to matroid simplicial complexes, as is evidenced by the following theorem of Chari.

Theorem 8. (Chari, [1, Theorem 3]) *If Δ is a coloop-free matroid simplicial complex, then Δ is PS ear-decomposable.*

Here, the condition that Δ is coloop-free means that topologically, Δ is not a cone. As such, there is no harm in considering only coloop-free matroids because cone vertices only append zeros to the end of the h-vector. Further, the following conjecture of Stanley has remained tantalizingly open for the past several decades:

Conjecture 9. (Stanley's Conjecture [9, p. 59])
The h-vector of a matroid simplicial complex is a pure $\mathcal{O}$-sequence.

It follows from Theorem 8 that matroid simplicial complexes are Cohen-Macaulay and hence their h-vectors are $\mathcal{O}$-sequences. The huge difficulty comes in establishing the purity condition. Based on Stanley's Conjecture and Chari's Theorem, the following conjecture is also quite natural.

Conjecture 10. Let Δ be a PS ear-decomposable simplicial complex. Then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.

Imani et al. [5] established this result for 1-dimensional PS ear-decomposable simplicial complexes. Our goal in this paper is to establish the same result in dimension 2. Our main theorem is the following result.

Theorem 11. *Let Δ be a 2-dimensional PS ear-decomposable simplicial complex. Then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

2.5 Overview

Before we delve into the technical results leading up to the proof, we will begin with a broad overview of our approach. Let $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ be a 2-dimensional PS ear-decomposable simplicial complex. We will transform Δ into a new PS ear-decomposable simplicial complex $\mathcal{C}(\Delta) = \Sigma_0' \cup \Sigma_1' \cup \cdots \cup \Sigma_t'$ with the same number of ears (though not

necessarily with the same initial sphere) such that $h(\Delta) = h(\mathcal{C}(\Delta))$. We call $\mathcal{C}(\Delta)$ the **compression** of Δ, which will serve as an analogue of classical compression/combinatorial shifting for PS ear-decomposable simplicial complexes.

For arbitrary simplicial complexes, compression and shifting operators play an important role in the characterization of f-vectors of simplicial complexes in the Kruskal-Katona Theorem [3, 10] and also in Stanley's characterization of h-vectors of Cohen-Macaulay simplicial complexes [9]. The difference in this setting is that $\mathcal{C}(\Delta)$ is not generally a shifted or compressed simplicial complex because these operators do not preserve PS ear-decomposability (or even purity) of the underlying simplicial complex. Instead, the ears in the PS ear-decomposition of $\mathcal{C}(\Delta)$, are added greedily with respect to revlex order (in a certain sense that will be made more precise later), and to each ear we define a corresponding set of monomials, also chosen greedily, that can be used to explicitly construct a corresponding pure multicomplex whose F-vector is $h(\mathcal{C}(\Delta))$.

The rest of the paper is structured as follows. In Section 3 we study the extremal combinatorics of the underlying graph of a 2-dimensional PS ear-decomposable simplicial complex, proving the main technical results we will use in guaranteeing the existence of the compressed complexes $\mathcal{C}(\Delta)$. In Section 4 we prove the main result. Section 4 has three main subsections corresponding to the three different PS-spheres of dimension 2. The majority of the work goes into handling the case that $\Sigma_0 = \partial\sigma^3$ is the boundary of a tetrahedron. In the case that $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ or $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ is a bipyramid or octahedron, the proof generally reduces to the case of the tetrahedron, with some exceptions for handling small boundary cases. Ultimately, the proof of the main result, Theorem 11, is given by Theorems 18, 22, and 32.

3 Shifting operators and constructible graphs

If Δ is a 2-dimensional PS ear-decomposable simplicial complex with compression $\mathcal{C}(\Delta) = \Sigma_0' \cup \Sigma_1' \cup \cdots \cup \Sigma_t'$, our goal is to assign a family of monomials $\mathcal{M}_i$ to each Σ_i' so that $\mathcal{M}_0 \cup \cdots \cup \mathcal{M}_i$ is a pure multicomplex whose F-vector is the same as the h-vector of $\Sigma_0' \cup \cdots \cup \Sigma_i'$ for each i. As we will see, it is relatively easy to describe the families $\mathcal{M}_i$ when Σ_i is one of $\sigma^0 * \partial\sigma^2$, $\sigma^0 * \partial\sigma^1 * \partial\sigma^1$, or $\sigma^1 * \partial\sigma^0$, but it is more difficult when $\Sigma_i' = \sigma^2$ fills a missing triangle. Filling a missing triangle contributes $(0, 0, 0, 1)$ to the h-vector, so the challenge in studying missing triangles is to know that there cannot be so many missing triangles to be filled that they would exceed the possible support of degree-2 monomials in the multicomplex.

Throughout this section, we will use $T(G)$ to denote the set of of triangles (3-cycles) in a simple graph G and $\#T(G)$ to denote the cardinality of that set. In order to better understand triangles in G, we begin by exploring the extremal combinatorics of the graph of a PS ear-decomposable simplicial complex.

Let $G = (V, E)$ be a simple graph with vertices $v_1, v_2, \ldots, v_n$. For any distinct $i, j \in [n]$,

define an operator $\mathcal{S}_{i,j}$ that acts on the edges of G as follows

$$\mathcal{S}_{i,j}(e) = \begin{cases} (e \setminus \{v_j\}) \cup \{v_i\} & \text{if } v_j \in e \text{ and } v_i \notin e \text{ and } (e \setminus \{v_j\}) \cup \{v_i\} \notin E \\ e & \text{otherwise.} \end{cases}$$

In other words, $\mathcal{S}_{i,j}$ shifts edges incident to vertex v_j to become incident to vertex v_i whenever possible. We slightly abuse notation and use $\mathcal{S}_{i,j}(G)$ to denote the resulting graph.

Lemma 12. *Let G be a simple graph on vertex set $\{v_1, v_2, \ldots, v_n\}$. For any distinct $i, j \in [n]$,*
$$\#T(G) \leqslant \#T(\mathcal{S}_{i,j}(G)).$$

Proof. We establish an injective map from $T(G)$ to $T(\mathcal{S}_{i,j}(G))$.

Let $\tau = \{v_k, v_\ell, v_m\}$ be a set of vertices that span a triangle in G. If $v_j \notin \tau$, then τ will remain unaffected by $\mathcal{S}_{i,j}$. Similarly, if $v_i \in \tau$ and $v_j \in \tau$, then τ also remains unaffected by $\mathcal{S}_{i,j}$. In either case, τ is also a triangle in $\mathcal{S}_{i,j}(G)$.

Thus it remains to consider the case that $\tau = \{v_j, v_k, v_\ell\}$ with $v_i \notin \tau$. If $\{v_i, v_k\}$ and $\{v_i, v_\ell\}$ are edges in G, then once again τ will remain unaffected by $\mathcal{S}_{i,j}$. Otherwise, $\{v_i, v_k, v_\ell\}$ is a triangle in $\mathcal{S}_{i,j}(G)$ but not in G. $\qquad\square$

3.1 Constructible graphs

In this section we define a family of graphs called constructible graphs, which arise as graphs of 2-dimensional PS ear-decomposable simplicial complexes. We will use the following graph theoretical notation: for a vertex v in a graph G, the **degree** of v will be denoted as $\deg(v) = \deg_G(v)$ and $\mathcal{N}(v) = \mathcal{N}_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ will denote the **neighborhood** of v. For $W \subseteq V(G)$, the **restriction** of G to W is the graph $G|_W$, with vertex set W and edge set $\{\{u, v\} \in E(G) : u, v \in W\}$.

Definition 13. Let G be a simple graph on vertex set $\{v_1, v_2, \ldots, v_n\}$ with a subset of edges labeled by elements of $[n] \cup \{0\}$. We say G is **constructible** if one of the following conditions is satisfied:

1. $G = K_4$ with all edges labeled 0,

2. there exists a vertex $v_\ell \in [n]$ such that $\deg(v_\ell) = 3$ or $\deg(v_\ell) = 4$, all edges incident to vertex v_ℓ have label ℓ, and $G - v_\ell$ is constructible, or

3. there exists an unlabeled edge $v_i v_j \in G$ such that $G - v_i v_j$ is constructible.

Viewing this recursive definition as an inductive one, constructible graphs are obtained from the complete graph K_4 with edges labeled 0 through a sequence of three possible operations: adding a new vertex v_ℓ of degree three or four, all of whose edges are labeled ℓ, or inserting an unlabeled missing edge. Given a constructible graph, one can roughly see the process through which it was constructed (up to reordering) because the edges

labeled $\ell > 0$ specify which edges were created at the same time as vertex v_ℓ and all other edges were either missing edges that were inserted (unlabeled) or edges that were part of the initial K_4 (labeled 0).

Constructible graphs are relevant to us because the graph of a 2-dimensional PS ear-decomposable simplicial complex is constructible. Now we wish to bound the number of triangles in a constructible graph in terms of the types of moves that were used in its construction. For convenience, we will say that an **A-move** on a constructible graph consists of adding a new vertex of degree 3 with appropriately labeled edges, a **B-move** adds a new vertex of degree 4 with appropriately labeled edges, and an **E-move** adds an unlabeled missing edge. For a vertex $v \in V(G)$ that is not part of the initial K_4, we define its **type**, type(v), to equal 3 or 4 depending on the degree of v when it is created.

Classically, the Kruskal-Katona Theorem [10] tells us that, among all simple graphs with a given number of vertices and edges, one with the maximal number of triangles is obtained by adding edges reverse lexicographically. Such graphs are known as **compressed graphs**. Of course, this construction may lead to a graph with a large number of isolated vertices, which is not suitable to our definition of constructible graphs. Therefore, we modify this definition to better suit our needs.

Let G be a constructible graph, and let a, b, and e respectively denote the number of A-, B-, and E-moves used in constructing G. We define the **compression** $\mathcal{C}(G)$ to be the constructible graph on vertex set $\{v_1, v_2, \ldots, v_{4+a+b}\}$ that is built as follows:

1. Begin with the complete graph K_4 on vertex set $\{v_1, v_2, v_3, v_4\}$ with edges labeled 0.

2. For $5 \leqslant \ell \leqslant 4 + b$, perform a B-move to add vertex ℓ and edges $\{v_1, v_\ell\}$, $\{v_2, v_\ell\}$, $\{v_3, v_\ell\}$, $\{v_4, v_\ell\}$, all labeled ℓ.

3. For $5 + b \leqslant \ell \leqslant 4 + b + a$, perform an A-move to add vertex v_ℓ and edges $\{v_1, v_\ell\}$, $\{v_2, v_\ell\}$, $\{v_3, v_\ell\}$, all labeled ℓ.

4. Perform E-moves to insert the e smallest missing edges in reverse lexicographic order.

Given this definition, we can make the connection to the classical Kruskal-Katona theorem more precise. For graphs, the Kruskal-Katona theorem says that, among all graphs on n vertices with m edges, the graph on vertex set $\{v_1, \ldots, v_n\}$ whose edges are the first m edges in revlex order has the largest number of triangles. The structure of such a graph can be described simply – we can uniquely express $m = \binom{p}{2} + q$ with $0 \leqslant q < p$, and the compressed graph G has the following properties:

- G contains a clique on $\{v_1, \ldots, v_p\}$,

- G contains edges $\{v_i, v_{p+1}\}$ for $1 \leqslant i \leqslant q$, and

- the vertices $v_{p+2}, \ldots, v_n$ are isolated.

Similarly, the vertices of a compressed constructible graph are ordered $v_1, \ldots, v_n$ such that

- $\text{type}(v_5) \geqslant \text{type}(v_6) \geqslant \cdots \geqslant \text{type}(v_n)$,

- the maximal clique spans vertices $\{v_1, \ldots, v_p\}$ for some $p \geqslant 4$,

- if $\deg(v_{p+1}) = q$, then v_{p+1} is adjacent to the vertices $v_1, \ldots, v_q$, and

- v_i is adjacent to the first $\text{type}(v_i)$ vertices for all $i > p + 1$.

This connection to the classical Kruskal-Katona Theorem will be made precise in Theorem 16, which states that $\mathcal{C}(G)$ has at least as many triangles as G when G is constructible. The remainder of this section focuses on the proof of Theorem 16, which requires a few intermediate lemmas.

Lemma 14. *Let G be a constructible graph. Assume that G contains a clique on the vertices $\{v_1, \ldots, v_p\}$. Let u and v be vertices not among $\{v_1, \ldots, v_p\}$ such that*

- $\mathcal{N}(u) \subseteq \{v_1, \ldots, v_p\}$,

- $\mathcal{N}(v) \subseteq \{v_1, \ldots, v_p\}$,

- *u and v are not adjacent, and*

- $\deg_G(u) \leqslant \deg_G(v)$.

Let $k = \min\{\deg_G(u) - \text{type}(u), p - \deg_G(v)\}$, and let G' be the graph obtained from G by removing k unlabeled edges incident to u and adding k unlabeled edges incident to v whose neighbors lie among $\{v_1, \ldots, v_p\}$. Then G' is constructible and $\#T(G) \leqslant \#T(G')$.

Proof. The number $\deg_G(u) - \text{type}(u)$ is the number of unlabeled edges incident to u and the number $p - \deg_G(v)$ is the number of missing edges between v and vertices among $\{v_1, \ldots, v_p\}$. So k is the largest number of unlabeled edges that could be moved from u to v. The fact that G' is constructible is immediate as we are only changing the unlabeled edges in G.

Because $\{v_1, \ldots, v_p\}$ span a clique, every pair of vertices in the neighborhood of u (respectively, v) form a triangle with u (respectively v). Because of this, and because u and v are not adjacent, we see that

$$\#T(G') - \#T(G) = \binom{\deg_G(u) - k}{2} + \binom{\deg_G(v) + k}{2} - \binom{\deg_G(u)}{2} - \binom{\deg_G(v)}{2}$$
$$= (\deg_G(v) - \deg_G(u)) \cdot k + k^2 \geqslant 0. \qquad \square$$

Lemma 15. *Let G be a constructible graph, and let $W \subseteq V(G)$ be a subset of vertices. Then*
$$\#E\left(G|_W\right) \leqslant \#E\left(\mathcal{C}(G)|_{\{v_1, \ldots, v_{|W|}\}}\right).$$

Proof. We prove the claim by induction on the number of edges in $G|_W$. The claim is trivial when there are no edges.

First suppose that there exists an unlabeled edge $e \in G|_W$. By the inductive hypothesis,

$$\#E\left((G-e)|_W\right) \leqslant \#E\left(\mathcal{C}(G-e)|_{\{v_1,\dots,v_{|W|}\}}\right).$$

To obtain $\mathcal{C}(G)$ from $\mathcal{C}(G-e)$, we add the revlex smallest missing edge. If the revlex smallest missing edge is $\{v_i, v_j\}$ and $j \leqslant |W|$, then

$$
\begin{aligned}
\#E(G|_W) &= \#E\left((G-e)|_W\right) + 1 \\
&\leqslant \#E\left(\mathcal{C}(G-e)|_{\{v_1,\dots,v_{|W|}\}}\right) + 1 \\
&= \#E\left(\mathcal{C}(G)|_{\{v_1,\dots,v_{|W|}\}}\right).
\end{aligned}
$$

Otherwise, if the revlex smallest missing edge is $\{v_i, v_j\}$ and $j > |W|$, then $\mathcal{C}(G)|_{\{v_1,\dots,v_{|W|}\}}$ is complete, in which case

$$\#E\left(G|_W\right) \leqslant \#E\left(\mathcal{C}(G)|_{\{v_1,\dots,v_{|W|}\}}\right)$$

holds trivially.

Thus, we are left to handle the case that $G|_W$ contains only labeled edges. Order the vertices in W as $v_{i_1}, v_{i_2}, \dots, v_{i_{|W|}}$ in order of their creation. In G, there is at most one edge whose largest vertex is v_{i_2}, at most two edges whose largest vertex is v_{i_3}, at most three edges whose largest vertex is v_{i_4}, and at most $\text{type}(v_{i_j})$ edges whose largest vertex is v_{i_j} for $j > 4$. The same is true in $\mathcal{C}(G)$, and because the vertices of $\mathcal{C}(G)$ are ordered so that vertices of type 4 come before vertices of type 3, it must be the case that

$$\#E\left(G|_W\right) \leqslant \#E\left(\mathcal{C}(G)|_{\{v_1,\dots,v_{|W|}\}}\right). \qquad \square$$

Theorem 16. *Let G be a constructible graph. Then*

$$\#T(G) \leqslant \#T(\mathcal{C}(G)).$$

Proof. We prove the claim by induction on the number of vertices in G. When $\#V(G) = 4$, the claim is clear as G and $\mathcal{C}(G)$ are both the complete graph. Therefore, we may assume $\#V(G) = n + 1 > 4$.

If possible, pick a vertex $v \in V(G)$ with $\text{type}(v) = 4$. Otherwise, all vertices in G that are not part of the original K_4 have type 3; pick one of those vertices arbitrarily. Let $\delta = \deg_G(v)$. Note that the number of triangles in G that contain v is equal to the number of edges in $G|_{\mathcal{N}(v)}$. Therefore, by the inductive hypothesis and Lemma 15,

$$
\begin{aligned}
\#T(G) &= \#T(G - v) + \#E(G|_{\mathcal{N}(v)}) \\
&\leqslant \#T(\mathcal{C}(G - v)) + \#E\left(\mathcal{C}(G-v)|_{\{v_1,\dots,v_\delta\}}\right). \qquad (1)
\end{aligned}
$$

Let G' be the graph obtained from $\mathcal{C}(G-v)$ by adding vertex v, along with edges $\{v, v_i\}$ for $1 \leqslant i \leqslant \delta$, the first type$(v)$ of which receive label v. This label will be temporary, as we need to determine the position where v should be inserted into the given order on the vertices of $\mathcal{C}(G-v)$. By construction,

$$\#T(G') = \#T(\mathcal{C}(G-v)) + \#E\left(\mathcal{C}(G-v)|_{\{v_1,\ldots,v_d\}}\right). \tag{2}$$

What is the structure of $\mathcal{C}(G-v)$? Its vertices are ordered $v_1, v_2, \ldots, v_n$ in such a way that type$(v_5) \geqslant$ type$(v_6) \geqslant \cdots \geqslant$ type(v_n). Moreover, there exists an integer $4 \leqslant p \leqslant n$ such that

- $\{v_1, \ldots, v_p\}$ span a clique,

- $\deg(v_{p+1}) < p$ (meaning $\{v_1, \ldots, v_{p+1}\}$ do not span a clique),

- $\mathcal{N}(v_i) \subseteq \{v_1, \ldots, v_p\}$ for all $i > p$, and

- $\deg(v_i) =$ type(v_i) for all $i > p + 1$.

Moreover, for all $i > 4$, the labeled edges incident to v_i are the those incident to the first type(v_i) vertices — either $\{v_1, v_2, v_3\}$ or $\{v_1, v_2, v_3, v_4\}$.

By Eqs (1) and (2), it follows that $\#T(G) \leqslant \#T(G')$, so we need only show that $\#T(G') \leqslant \#T(\mathcal{C}(G))$. We examine several cases.

Case 1: $\deg(v) = p$

In this case, $\mathcal{C}(G)$ can be obtained from G' in two steps:

First, insert vertex v into the given vertex order so that $v_4 < v < v_5$. Second, if necessary, replace the revlex largest unlabeled edge $\{v_k, v_{p+1}\}$ with the unlabeled edge $\{v, v_{p+1}\}$. Whether type$(v) = 4$ or all vertices have type 3, this order respects the rule that all vertices of type 4 come before the vertices of type 3. Moreover, $\mathcal{C}(G)$ will contain a clique of size $p + 1$ on $\{v, v_5, \ldots, v_p\}$, with unlabeled edges $\{v, v_{p+1}\}, \{v_5, v_{p+1}\}, \ldots, \{v_{k-1}, v_{p+1}\}$ being the remaining revlex smallest edges under this new vertex order. Neither of these operations changes the number of triangles in the graph, and hence $\#T(G') = \#T(\mathcal{C}(G))$.

Case 2: $\deg(v) > p$

As above, let $\delta = \deg(v)$ and insert v into the given vertex order so that $v_4 < v < v_5$. Having done this, the graph G' fails to be compressed because there are unlabeled edges of the form $\{v, v_i\}$ for $p + 2 \leqslant i \leqslant \delta$ that are not the revlex smallest edges that could have been added. However, it is still the case that $\deg_{G'}(v_{p+1}) \geqslant \deg_{G'}(v_{p+2}) \geqslant \cdots \geqslant \deg_{G'}(v_\delta)$. Therefore, we may repeatedly apply Lemma 14, first to the pair of vertices $\{v_{p+1}, v_\delta\}$, then $\{v_{p+1}, v_{\delta-1}\}$, and so on, each time moving the unlabeled edge $\{v, v_k\}$ for $k > p + 1$ to the revlex smallest missing edge incident to v_{p+1}. Each of these moves weakly increases the number of triangles in the graph, and hence $\#T(G') \leqslant \#T(\mathcal{C}(G))$ as desired. We note that at some point in this process, it could be the case that v_{p+1} becomes adjacent to all vertices among $\{v, v_5, \ldots, v_p\}$. In this case, we simply start moving unlabeled edges so that they form the revlex smallest missing edge incident to v_{p+2} instead (and, if necessary, then to v_{p+3}, v_{p+4}, etc.).

Case 3: $\deg(v) < p$

This is the most complicated of the cases and requires a few sub-cases.

If $\text{type}(v) = \text{type}(v_{p+1})$, we insert v into the vertex order on $\mathcal{C}(G - v)$ as follows: if $\deg(v) \geqslant \deg(v_{p+1})$, then $v_p < v < v_{p+1}$; otherwise, if $\deg(v_{p+1}) > \deg(v)$, then $v_{p+1} < v < v_{p+2}$. Then we apply Lemma 14 to the pair of vertices $\{v, v_{p+1}\}$ to obtain $\mathcal{C}(G)$ and see that $\#T(G') \leqslant \#T(\mathcal{C}(G))$.

If $\text{type}(v) \neq \text{type}(v_{p+1})$, then by our choice of v it must be the case that $\text{type}(v) = 4$ and $\text{type}(v_{p+1}) = 3$. If there exists an index $5 \leqslant i \leqslant p$ such that $\text{type}(v_i) = 3$, we can pick the smallest such i and perform an operation that declares $\text{type}(v_i) = 4$ and $\text{type}(v) = 3$. Since $4 < i \leqslant p$, we know that $\{v_4, v_i\} \in G'$, so we can give that edge the label i and remove the label v from the edge $\{v_4, v\}$. This does not change the number of trees in G', but now we have arranged for $\text{type}(v) = 3 = \text{type}(v_{p+1})$ and we can apply the argument used in the previous case.

Thus we need only examine the case that $\text{type}(v_p) = \text{type}(v) = 4$, but $\text{type}(v_{p+1}) = 3$. If $\deg(v) \geqslant \deg(v_{p+1})$, we insert v into the vertex order on $\mathcal{C}(G-v)$ so that $v_p < v < v_{p+1}$. This preserves the condition that all vertices of type 4 come before the vertices of type 3 in the vertex order, and then we can apply Lemma 14 to the pair of vertices $\{v, v_{p+1}\}$ to obtain $\mathcal{C}(G)$ from G'. On the other hand, if $\deg(v) < \deg(v_{p+1})$, we first perform the above operation to swap edge labels so that $\text{type}(v) = 3$ and $\text{type}(v_{p+1}) = 4$. Then we insert v into the vertex order on $\mathcal{C}(G-v)$ so that $v_{p+1} < v < v_{p+2}$ and then apply Lemma 14. $\qquad\square$

4 Proofs of the main result

Let $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ be a 2-dimensional PS ear-decomposable simplicial complex. By a slight abuse of notation from the previous section, we will say the PS balls that are added by $\Sigma_i = \sigma^0 * \partial\sigma^2$, $\Sigma_i = \sigma^0 * \partial\sigma^1 * \partial\sigma^1$, $\Sigma_i = \sigma^1 * \partial\sigma^1$, or $\Sigma_i = \sigma^2$, are called ears of type A, B, E, and F respectively. Similarly, we let η_A, η_B, η_E, and η_F respectively denote the number of ears of each type used in constructing Δ. This implies that

$$
\begin{aligned}
h_0(\Delta) &= 1 \\
h_1(\Delta) &= h_1(\Sigma_0) + \eta_A + \eta_B \\
h_2(\Delta) &= h_2(\Sigma_0) + \eta_A + 2\eta_B + \eta_E \\
h_3(\Delta) &= 1 + \eta_A + \eta_B + \eta_E + \eta_F.
\end{aligned}
$$

In other words, a move of type A contributes $(0, 1, 1, 1)$ to the h-vector, a move of type B contributes $(0, 1, 2, 1)$ to the h-vector, a move of type E contributes $(0, 0, 1, 1)$, and a move of type F contributes $(0, 0, 0, 1)$ to the h-vector.

Our goal in this section is to prove that the h-vector of a PS ear-decomposable simplicial complex is a pure $\mathcal{O}$-sequence. Throughout the remainder of the paper, we will label the vertices of the tetrahedron, bipyramid, and octahedron as shown in Figure 2.

The proof breaks into three main cases for the three possible base spheres Σ_0, along with several subcases. We begin with a broad overview of the main cases and subcases in

$$\Sigma_0 = \partial\sigma^3 \qquad\qquad \Sigma_0 = \partial\sigma^2 * \partial\sigma^1 \qquad\qquad \Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$$
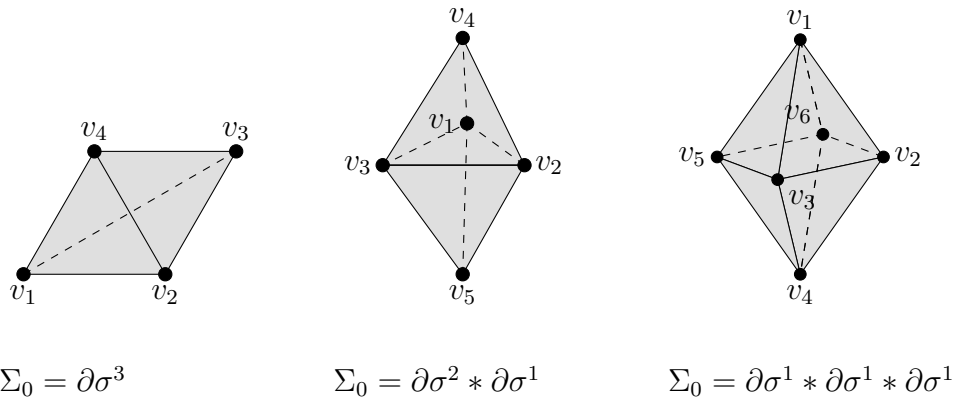
Figure 2: Vertex labelings for 2-dimensional PS spheres.

the the remainder of the paper.

1. The case that $\Sigma_0 = \partial\sigma^3$ is the boundary of a tetrahedron is the simplest case given Theorem 16. Algorithm 17 defines the compressed complex $\mathcal{C}(\Delta)$ and corresponding pure multicomplex $\mathcal{M}(\Delta)$. The fact that this algorithm terminates and produces a pure multicomplex is proved in Theorem 18.

2. The case that $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ is the boundary of a bipyramid requires two subcases.

   (a) When $\eta_F > 0$, we can reduce the problem to the case that $\Sigma_0 = \partial\sigma^3$ in Theorem 19.

   (b) When $\eta_F = 0$, Algorithm 20 defines the compressed complex $\mathcal{C}(\Delta)$ and corresponding pure multicomplex $\mathcal{M}(\Delta)$. The fact that this algorithm terminates and produces a pure multicomplex is proved in Theorem 21.

3. The case that $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ is the boundary of an octahedron is the most complicated case.

   (a) When $\eta_F = 0$, Algorithm 23 defines the compressed complex $\mathcal{C}(\Delta)$ and corresponding pure multicomplex $\mathcal{M}(\Delta)$. The fact that this algorithm terminates and produces a pure multicomplex is proved in Theorem 24.

   (b) When $\eta_E = 0$, Algorithm 26 defines the compressed complex $\mathcal{C}(\Delta)$ and corresponding pure multicomplex $\mathcal{M}(\Delta)$. The fact that this algorithm terminates and produces a pure multicomplex is proved in Theorem 27.

   (c) When $\eta_F > 0$ and $\eta_E > 0$, we must consider two further subcases based on whether $\{v_1, v_4\} \in \Delta$.

      i. When $\{v_1, v_4\} \in \Delta$, we reduce the problem to the case that $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ is a bipyramid in Proposition 28.

ii. When $\{v_1, v_4\} \notin \Delta$, we use shifting operators $\mathcal{S}_{i,j}$ to reduce to the case that $\Sigma_0 = \partial\sigma^3$ is a tetrahedron, but with two extra monomials of degree-3 in Theorem 30. We then show that two monomials of degree-3 can be removed in Theorem 31, which completes the proof.

4.1 The case that $\Sigma_0 = \partial\sigma^3$

We now present the main algorithm that will be used to prove that h-vectors of 2-dimensional PS ear-decomposable complexes are pure $\mathcal{O}$-sequences when $\Sigma_0 = \partial\sigma^3$.
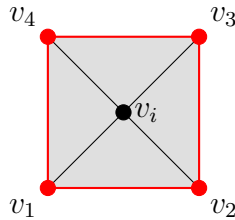
Algorithm 17. Let Δ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^3$. We define a new PS ear-decomposable complex $\mathcal{C}(\Delta)$ and a set of monomials $\mathcal{M}(\Delta)$ in the variables $\{x_4, \ldots, x_{\eta_A+\eta_B+4}\}$ inductively as follows.

Step 0:

- Set $\Sigma_0 = \partial\sigma^3$ on vertex set $\{v_1, v_2, v_3, v_4\}$.
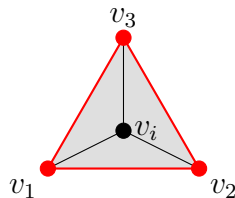
- Set $\mathcal{M}(\Delta) = \{1, x_4, x_4^2, x_4^3\}$.

Step 1:

- For $5 \leqslant i \leqslant \eta_B + 4$:

  - Introduce vertex v_i to $\mathcal{C}(\Delta)$ through a B-move whose boundary is glued along the cycle $v_1 v_2 v_3 v_4$.

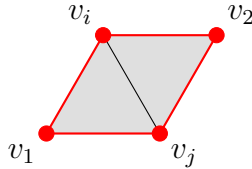 - Add $\{x_i, x_4 x_i, x_i^2, x_4 x_i^2\}$ to $\mathcal{M}(\Delta)$.

- For $\eta_B + 5 \leqslant i \leqslant \eta_A + \eta_B + 4$:

  - Introduce vertex v_i to $\mathcal{C}(\Delta)$ through an A-move whose boundary is glued along the cycle $v_1 v_2 v_3$,

– Add $\{x_i, x_i^2, x_i^3\}$ to $\mathcal{M}(\Delta)$.

Step 2:

- For $1 \leqslant \ell \leqslant \eta_E$:

    – Let $\{v_i, v_j\}$ be the revlex smallest missing edge in $\mathcal{C}(\Delta)$.

    – Add the edge $\{v_i, v_j\}$ to $\mathcal{C}(\Delta)$ through an E-move whose boundary is the cycle $v_1 v_i v_2 v_j$.

    – Add $\{x_i x_j, x_i^2 x_j\}$ to $\mathcal{M}(\Delta)$.

Step 3:

- For $1 \leqslant \ell \leqslant \eta_F$:

    – Let G be the revlex smallest missing 2-face in $\mathcal{C}(\Delta)$. Add G to $\mathcal{C}(\Delta)$.

    – Let μ be the revlex smallest degree-3 monomial not belonging to $\mathcal{M}(\Delta)$ whose proper divisors all belong to $\mathcal{M}(\Delta)$. Add μ to $\mathcal{M}(\Delta)$.

Note first that upon the completion of Step 1, $\mathcal{M}(\Delta)$ is a pure multicomplex containing x_j^2 for all $4 \leqslant j \leqslant \eta_A + \eta_B + 4$. The degree-2 monomials that do not belong to $\mathcal{M}(\Delta)$ at this point are of two types: those of the form $x_4 x_j$ for which vertex v_j was introduced through an A-move, and those of the form $x_i x_j$ with $5 \leqslant i < j \leqslant \eta_A + \eta_B + 4$. The number of such monomials is $\eta_A + \binom{\eta_A + \eta_B}{2}$. But this is exactly the same as the number of missing edges in $\mathcal{C}(\Delta)$ after the completion of Step 1, meaning that Step 2 of the Algorithm will terminate and the resulting multicomplex $\mathcal{M}(\Delta)$ is still a pure multicomplex.

Now we turn our attention to Step 3, which is slightly more complicated. Again we count the number of missing 2-faces in $\mathcal{C}(\Delta)$ and the number of degree-3 monomials not belonging to $\mathcal{M}(\Delta)$ upon the completion of Step 2.

Let G be the underlying graph of Δ and G' the underlying graph of $\mathcal{C}(\Delta)$. By the construction of $\mathcal{C}(\Delta)$ in Steps 0, 1, and 2, $G' = \mathcal{C}(G)$ is a compressed constructible graph. Let p be the size of the largest clique in G', and let q denote the degree of v_{p+1}. Furthermore, let a' be the number of vertices among $\{v_5, \ldots, v_{p+1}\}$ that were introduced through a type-A move and let a'' be the number of vertices among $\{v_{p+2}, \ldots, v_n\}$ that were introduced through a type-A move. Define b' and b'' similarly for vertices introduced through a type-B move.

Observe that $p + 1 = 4 + a' + b'$, so

$$p = a' + b' + 3. \tag{3}$$

Further,

$$\eta_E = \binom{p}{2} + q - 3a' - 4b' - 6, \tag{4}$$

because there are $\binom{p}{2} + q$ edges among the first $p + 1$ vertices of G', but $3a' + 4b'$ were introduced as part of the A- or B-moves used to create vertices and 6 were part of the initial PS sphere Σ_0.

Next, we can directly count that

$$\#T(G') = \binom{p}{3} + \binom{q}{2} + 6b'' + 3a''. \tag{5}$$

However, $3\eta_A + 4\eta_B + 2\eta_E + 4$ of those triangles span triangular faces in $\mathcal{C}(\Delta)$ after the completion of Step 2 because A-, B-, and E-moves respectively introduce 3, 4, and 2 triangular faces, and there are 4 triangular faces in Σ_0. Finally, by Theorem 16, the graph of Δ cannot have more triangles than G', and Δ also has $3\eta_A + 4\eta_B + 2\eta_E$ triangular faces that are introduced through A-, B-, and E-moves, and 4 additional triangular faces in Σ_0. Therefore,

$$
\begin{aligned}
\eta_F &\leqslant \#T(G) - 3\eta_A - 4\eta_B - 2\eta_E - 4 \\
&\leqslant \#T(G') - 3\eta_A - 4\eta_B - 2\eta_E - 4 \\
&\overset{(5)}{=} \binom{p}{3} + \binom{q}{2} + 6b'' + 3a'' - 3(a' + a'') - 4(b' + b'') - 2\eta_E - 4 \\
&\overset{(4)}{=} \left[\binom{p}{3} - 2\binom{p}{2}\right] + \left[\binom{q}{2} - 2q\right] + 3a' + 4b' + 2b'' + 8 \\
&= \left[\binom{p-2}{3} - 3p + 4\right] + \left[\binom{q-2}{2} - 3\right] + 3a' + 4b' + 2b'' + 8 \\
&\overset{(3)}{=} \binom{p-2}{3} + \binom{q-2}{2} + b' + 2b''.
\end{aligned}
$$

On the other hand, upon the completion of Step 2, the degree-2 monomials in $\mathcal{M}(\Delta)$ support the following degree-3 monomials:

- the $\binom{p-1}{3}$ monomials of degree 3 in the variables $x_4, \ldots, x_p$,

- the $\binom{q-1}{2}$ monomials of degree 3 in the variables $x_4, \ldots, x_q, x_{p+1}$ that are divisible by x_{p+1},

- the a'' monomials x_j^3 corresponding to vertices v_j that were introduced through an A-move for $j > p + 1$, and

- the $3b''$ monomials of the form $\{x_4^2 x_j, x_4 x_j^2, x_j^3\}$ corresponding to vertices v_j that were introduced through a B-move for $j > p + 1$.

Use $m = \binom{p-1}{3} + \binom{q-1}{2} + a'' + 3b''$ denote the number of available monomials. Upon the completion of Step 2, $1 + \eta_A + \eta_B + \eta_E$ of these monomials have been added to $\mathcal{M}(\Delta)$, and hence the number of available degree-3 monomials that can be added to $\mathcal{M}(\Delta)$ is

$$
\begin{aligned}
m - \eta_A - \eta_B - \eta_E - 1 \;=\;& \binom{p-1}{3} + \binom{q-1}{2} + a'' + 3b'' - \eta_A - \eta_B - \eta_E - 1 \\
\overset{(4)}{=}\;& \left[\binom{p-1}{3} - \binom{p}{2}\right] + \left[\binom{q-1}{2} - q\right] \\
& + a'' + 3b'' + 3a' + 4b' - \eta_A - \eta_B + 5 \\
=\;& \left[\binom{p-2}{3} - 2p + 3\right] + \left[\binom{q-2}{2} - 2\right] \\
& + a'' + 3b'' + 3a' + 4b' - \eta_A - \eta_B + 5 \\
\overset{(3)}{=}\;& \binom{p-2}{2} + \binom{q-2}{2} + b' + 2b''.
\end{aligned}
$$

This tells us that η_F is bounded by the number of degree-3 monomials that can be added to $\mathcal{M}(\Delta)$ upon the completion of Step 2 while still preserving the property that $\mathcal{M}(\Delta)$ is a multicomplex. Consequently, Step 3 will terminate and upon its completion, $\mathcal{M}(\Delta)$ will be a pure multicomplex. This proves the following theorem.

Theorem 18. *The set of monomials $\mathcal{M}(\Delta)$ output by Algorithm 17 is a pure multicomplex. Moreover $F(\mathcal{M}(\Delta)) = h(\mathcal{C}(\Delta)) = h(\Delta)$. Consequently, if Δ is a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^3$, then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

4.2 The case that $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$

The case that Σ_0 is a bipyramid is very similar to the case that Σ_0 is the boundary of a tetrahedron. We can start by labeling the vertices of the bipyramid as in Figure 2.

Now we examine two cases. First suppose $\eta_F > 0$. If at some point the missing face $\{v_1, v_2, v_3\}$ is filled through an F-move, then we can view Δ as a PS ear-decomposable complex with $\Sigma_0 = \partial\sigma^3$ by starting with the boundary of the tetrahedron on $\{v_1, v_2, v_3, v_4\}$, then adding vertex v_5 through an A-move, and then attaching the remaining ears in order to construct Δ. On the other hand, if $\{v_1, v_2, v_3\}$ is not filled through an F-move, pick any 2-face filled through an F-move, remove it from Δ, and fill $\{v_1, v_2, v_3\}$ instead. This creates a new PS ear-decomposable complex Δ' with $h(\Delta') = h(\Delta)$. In either of these cases, we can then apply Theorem 18 to see that $h(\Delta)$ is a pure $\mathcal{O}$-sequence. This proves the following theorem.

Theorem 19. *Let $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ and $\eta_F > 0$. There exists a PS ear-decomposable simplicial complex $\Delta' = \Sigma'_0 \cup \Sigma'_1 \cup \cdots \cup \Sigma'_t$ with $\Sigma'_0 = \partial\sigma^3$ and $h(\Delta) = h(\Delta')$. Consequently, $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

On the other hand, if $\eta_F = 0$, we can use the following algorithm to construct a pure multicomplex whose F-vector is the same as $h(\Delta)$.

Algorithm 20. Let Δ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ and $\eta_F = 0$. We define a new PS ear-decomposable complex $\mathcal{C}(\Delta)$ and a set of monomials $\mathcal{M}(\Delta)$ in the variables $\{x_4, x_5, \ldots, x_{\eta_A+\eta_B+5}\}$ inductively as follows.

Step 0:

- Set $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ on vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ as shown in Figure 2.

- Set $\mathcal{M}(\Delta) = \{1, x_4, x_5, x_4^2, x_4 x_5, x_4^2 x_5\}$.

Step 1:

- For $6 \leqslant i \leqslant \eta_B + 5$:

  - Introduce vertex v_i to $\mathcal{C}(\Delta)$ through a B-move whose boundary is glued along the cycle $v_1 v_2 v_3 v_4$.
  - Add $\{x_i, x_4 x_i, x_i^2, x_4 x_i^2\}$ to $\mathcal{M}(\Delta)$.

- For $\eta_B + 6 \leqslant i \leqslant \eta_A + \eta_B + 5$:

  - Introduce vertex v_i to $\mathcal{C}(\Delta)$ through an A-move whose boundary is glued along the cycle $v_1 v_2 v_3$,
  - Add $\{x_i, x_i^2, x_i^3\}$ to $\mathcal{M}(\Delta)$.

Step 2:

- For $1 \leqslant \ell \leqslant \eta_E$:

  - Let $\{v_i, v_j\}$ be the revlex smallest missing edge in $\mathcal{C}(\Delta)$.
  - Add the edge $\{v_i, v_j\}$ to $\mathcal{C}(\Delta)$ through an E-move whose boundary is the cycle $v_1 v_i v_2 v_j$.
  - If $\{v_i, v_j\} = \{v_4, v_5\}$, add $\{x_5^2, x_5^3\}$ to $\mathcal{M}(\Delta)$.
  - Else, add $\{x_i x_j, x_i^2 x_j\}$ to $\mathcal{M}(\Delta)$.

The proof that this algorithm terminates and produces a pure multicomplex is identical to the proof that Steps 0, 1, and 2 in Algorithm 20 terminate and produce a pure multicomplex. Upon the completion of Step 1, the number of missing edges in $\mathcal{C}(\Delta)$ is $\binom{\eta_A+\eta_B}{2} + 2\eta_A + \eta_B + 1$. Indeed, none of the newly introduced vertices are adjacent, giving $\binom{\eta_A+\eta_B}{2}$ missing edges; each vertex v_j with $j > 5$ is part of a missing edge with v_5, and it is part of a missing edge with v_4 if it was introduced through an A-move; finally, the edge $\{v_4, v_5\}$ is missing. On the other hand, the number of missing degree-2 monomials in $\mathcal{M}(\Delta)$ upon the completion of Step 1 is

$$
\binom{2+\eta_A+\eta_B+1}{2} - \eta_A - 2\eta_B - 2 \;=\; \binom{\eta_A+\eta_B}{2} + (3\eta_A + 3\eta_B + 3) - \eta_A - 2\eta_B - 2
$$
$$
=\; \binom{\eta_A+\eta_B}{2} + 2\eta_A + \eta_B + 1
$$

Once again, this guarantees that Step 2 will terminate. Moreover, $\{v_4, v_5\}$ is the revlex smallest missing edge in $\mathcal{C}(\Delta)$ after the completion of Step 1. This means that if $\eta_E > 0$, then the first missing edge that is inserted also adds x_5^2 to $\mathcal{M}(\Delta)$, at which point $x_j^2 \in \mathcal{M}(\Delta)$ for all j. This guarantees that the monomials added to $\mathcal{M}(\Delta)$ in subsequent iterations of Step 2 preserve the property that $\mathcal{M}(\Delta)$ is a pure multicomplex. Thus, we have proved the following theorem.

Theorem 21. *The set of monomials $\mathcal{M}(\Delta)$ output by Algorithm 20 is a pure multicomplex. Moreover $F(\mathcal{M}(\Delta)) = h(\mathcal{C}(\Delta)) = h(\Delta)$. Consequently, if Δ is a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$ and $\eta_F = 0$, then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

Theorems 19 and 21 together handle the case that $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$.

Theorem 22. *Let $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^2 * \partial\sigma^1$. Then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

4.3 The case that $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$

The case that $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ is the boundary complex of the octahedron is more complicated than the previous two cases and cannot immediately be reduced to the case that $\Sigma_0 = \partial\sigma^3$. This case requires special handling of base cases for small values of η_A, η_B, η_E, and η_F. We can start by labeling the vertices of the octahedron as in Figure 2.

4.3.1 The case that $\eta_F = 0$

Algorithm 23. Let Δ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ and $\eta_F = 0$. We define a new PS ear-decomposable complex $\mathcal{C}(\Delta)$ and a set of monomials $\mathcal{M}(\Delta)$ in the variables $\{x_4, x_5, x_6, \ldots, x_{\eta_A + \eta_B + 6}\}$ inductively as follows.

Step 0:

- Set $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ on vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown in Figure 2.

- Set $\mathcal{M}(\Delta) = \{1, x_4, x_5, x_6, x_4x_5, x_4x_6, x_5x_6, x_4x_5x_6\}$.

Step 1:

- For $7 \leqslant i \leqslant \eta_B + 6$:

  - Introduce vertex v_i to $\mathcal{C}(\Delta)$ through a B-move whose boundary is glued along the cycle $v_1 v_2 v_4 v_3$.

  - Add $\{x_i, x_4x_i, x_i^2, x_4x_i^2\}$ to $\mathcal{M}(\Delta)$.

- For $\eta_B + 7 \leqslant i \leqslant \eta_A + \eta_B + 6$:

- Introduce vertex v_i to $\mathcal{C}(\Delta)$ through an A-move whose boundary is glued along the cycle $v_1 v_2 v_3$.
- Add $\{x_i, x_i^2, x_i^3\}$ to $\mathcal{M}(\Delta)$.

Step 2:

- For $1 \leqslant \ell \leqslant \eta_E$:

    - Let $\{v_i, v_j\}$ be the revlex smallest missing edge in $\mathcal{C}(\Delta)$.
    - If $\{v_i, v_j\} = \{v_1, v_4\}$:
        * Add edge $\{v_1, v_4\}$ to $\mathcal{C}(\Delta)$ through an E-move whose boundary is $v_1 v_2 v_4 v_5$.
        * Add $\{x_4^2, x_4^3\}$ to $\mathcal{M}(\Delta)$.
    - Else if $\{v_i, v_j\} = \{v_2, v_5\}$:
        * Add edge $\{v_2, v_5\}$ to $\mathcal{C}(\Delta)$ through an E-move whose boundary is $v_1 v_2 v_4 v_5$.
        * Add $\{x_5^2, x_5^3\}$ to $\mathcal{M}(\Delta)$.
    - Else if $\{v_i, v_j\} = \{v_3, v_6\}$:
        * Add edge $\{v_3, v_3\}$ to $\mathcal{C}(\Delta)$ through an E-move whose boundary is $v_1 v_3 v_4 v_6$.
        * Add $\{x_6^2, x_6^3\}$ to $\mathcal{M}(\Delta)$.
    - Else:
        * Add edge $\{v_i, v_j\}$ to $\mathcal{C}(\Delta)$ through an E-move whose boundary is $v_1 v_i v_2 v_j$.
        * Add $\{x_i x_j, x_i^2 x_j\}$ to $\mathcal{M}(\Delta)$.

Step 0 and Step 1 in Algorithm 23 proceed as they have in all other cases, and it is clear that $\mathcal{M}(\Delta)$ is a pure multicomplex upon the completion of Step 1. In Step 2, the revlex smallest missing edges $\{v_1, v_4\}$, $\{v_2, v_5\}$, and $\{v_3, v_6\}$ correspond to the values $\ell = 1$, $\ell = 2$, and $\ell = 3$ respectively. When $\ell \geqslant 4$, we see that $x_i^2 \in \mathcal{M}(\Delta)$ for all $4 \leqslant i \leqslant \eta_A + \eta_B + 6$ and moreover that the revlex smallest missing edge $\{v_i, v_j\}$ satisfies $j \geqslant 7$ and $i \geqslant 4$. Therefore the contribution of the monomials $\{x_i x_j, x_i^2 x_j\}$ preserves the property that $\mathcal{M}(\Delta)$ is a pure multicomplex. This proves the following theorem.

Theorem 24. *The set of monomials $\mathcal{M}(\Delta)$ output by Algorithm 23 is a pure multicomplex. Moreover $F(\mathcal{M}(\Delta)) = h(\mathcal{C}(\Delta)) = h(\Delta)$. Consequently, if Δ is a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ and $\eta_F = 0$, then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*
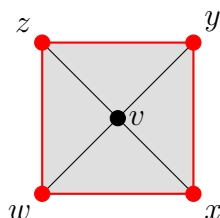
4.3.2 The case that $\eta_E = 0$

Lemma 25. *Let Δ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ and $\eta_E = 0$. If $\eta_B = 0$, then $\eta_F = 0$. Otherwise, if $\eta_B > 0$, then*

$$\eta_F \leqslant \begin{cases} 2\eta_B - 1 & \text{if } \eta_A = 0, \\ 2\eta_B & \text{if } \eta_A > 0. \end{cases}$$

Proof. Attaching a PS ball by an A-move does not change the number of missing triangles in a PS ear-decomposable simplicial complex. Therefore, since Σ_0 does not contain any missing triangles, neither will Δ if $\eta_B = 0$.

Now suppose $\eta_B > 0$. When a new vertex is introduced through a type-B move, there is a potential for two missing triangles to be introduced to Δ. Specifically, if vertex v is introduced through a B-move whose boundary vertices are labeled as shown below, then the new missing triangles that are created are $\{v, x, z\}$ (as long as $\{x, z\}$ was already an edge) and $\{v, w, y\}$ (as long as $\{w, y\}$ was already an edge).

Therefore, $\eta_F \leqslant 2\eta_B$ whenever $\eta_B > 0$. However, if $\eta_A = 0$, then the first ear attached through a B-move cannot contribute two missing triangles because the graph of the octahedron does not contain an induced K_4 subgraph (meaning either $\{x, z\}$ or $\{w, y\}$ will be missing). This means that when $\eta_A = 0$ and $\eta_B > 0$, it must be the case that $\eta_F \leqslant 2\eta_B - 1$. $\qquad\square$

Algorithm 26. Let Δ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ and $\eta_E = 0$. We define a new PS ear-decomposable complex $\mathcal{C}(\Delta)$ and a set of monomials $\mathcal{M}(\Delta)$ in the variables $\{x_4, x_5, x_6, \ldots, x_{\eta_A + \eta_B + 6}\}$ inductively as follows.

Step 0:

- Set $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ on vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown above.

- Set $\mathcal{M}(\Delta) = \{1, x_4, x_5, x_6, x_4x_5, x_4x_6, x_5x_6, x_4x_5x_6\}$.

Step 1:

- If $\eta_A = 0$ and $\eta_B > 0$:

  - Introduce vertex v_7 to $\mathcal{C}(\Delta)$ through a B-move whose boundary is glued along the cycle $v_1v_2v_4v_3$.

  - Add $\{x_7, x_4x_7, x_7^2, x_4x_7^2\}$ to $\mathcal{M}(\Delta)$

  - For $8 \leqslant i \leqslant 7 + (\eta_B - 1)$:

    * Introduce vertex v_i to $\mathcal{C}(\Delta)$ through a B-move whose boundary is glued along the cycle $v_1v_2v_3v_7$.

    * Add $\{x_i, x_i^2, x_7x_i, x_7x_i^2\}$ to $\mathcal{M}(\Delta)$.

- If $\eta_A > 0$:

  – For $7 \leqslant i \leqslant \eta_A + 6$:

    * Introduce vertex v_i to $\mathcal{C}(\Delta)$ through an A-move whose boundary is glued along the cycle $v_1 v_2 v_3$.
    * Add $\{x_i, x_i, x_i^3\}$ to $\mathcal{M}(\Delta)$.

  – For $\eta_A + 7 \leqslant i \leqslant \eta_A + \eta_B + 6$:

    * Introduce vertex v_i to $\mathcal{C}(\Delta)$ through a B-move whose boundary is glued along the cycle $v_1 v_2 v_3 v_7$.
    * Add $\{x_i, x_i^2, x_7 x_i, x_7 x_i^2\}$ to $\mathcal{M}(\Delta)$.

Step 2:

- If $\eta_A = 0$:

  – Let $\mathcal{S} = \{x_7^3\} \cup \{x_7^2 x_i, x_i^3 \ : \ 8 \leqslant i \leqslant \eta_B + 6\}$.
  – Let $\mathcal{F} = \{\{v_1, v_4, v_7\}\} \cup \{\{v_1, v_3, v_i\}, \{v_2, v_7, v_i\} \ : \ 8 \leqslant i \leqslant \eta_B + 6\}$.
  – Add the first η_F monomials in $\mathcal{S}$ (under revlex order) to $\mathcal{M}(\Delta)$.
  – Add the first η_F faces in $\mathcal{F}$ (under revlex order) to $\mathcal{C}(\Delta)$.

- If $\eta_A > 0$:

  – Let $\mathcal{S} = \{x_7^2 x_i, x_i^3 \ : \ \eta_A + 7 \leqslant i \leqslant \eta_A + \eta_B + 6\}$.
  – Let $\mathcal{F} = \{\{v_1, v_3, v_i\}, \{v_2, v_7, v_i\} \ : \ \eta_A + 7 \leqslant i \leqslant \eta_A + \eta_B + 6\}$.
  – Add the first η_F monomials in $\mathcal{S}$ (under revlex order) to $\mathcal{M}(\Delta)$.
  – Add the first η_F faces in $\mathcal{F}$ (under revlex order) to $\mathcal{C}(\Delta)$.

This algorithm requires more case analysis because of the bound on η_F when $\eta_A = 0$ and $\eta_B > 0$ in Lemma 25. In Step 1, we introduce all vertices through A- and B-moves. In the case that $\eta_A > 0$, we break with our previous strategy and introduce all type-A vertices first out of convenience. As noted in the proof of Lemma 25, we do this so that there will be a K_4 subgraph on the vertices $\{v_1, v_2, v_3, v_7\}$ where the boundaries of the type-B balls can be glued.

In Step 2, when $\eta_A = 0$, there are $2\eta_B - 1$ faces in set $\mathcal{F}$, which correspond to the maximum number of missing triangles in a PS ear-decomposable complex with $\eta_E = 0$ and $\eta_A = 0$ by Lemma 25. There are $2\eta_B - 1$ corresponding monomials in the set $\mathcal{S}$ that can be added to $\mathcal{M}(\Delta)$. Similarly, when $\eta_A > 0$, there are $2\eta_B$ faces in set $\mathcal{F}$ and $2\eta_B$ monomials in set $\mathcal{S}$. Therefore, Algorithm 26 terminates and produces a pure $\mathcal{O}$-sequence. This proves the following theorem.

Theorem 27. *The set of monomials $\mathcal{M}(\Delta)$ output by Algorithm 26 is a pure multicomplex. Moreover $F(\mathcal{M}(\Delta)) = h(\mathcal{C}(\Delta)) = h(\Delta)$. Consequently, if Δ is a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial \sigma^1 * \partial \sigma^1 * \partial \sigma^1$ and $\eta_E = 0$, then $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

4.3.3 The case that $\eta_E > 0$ and $\eta_F > 0$

Our goal in the case that $\eta_E > 0$ and $\eta_F > 0$ is to reduce to the cases already established in which Σ_0 is the boundary of a tetrahedron or a bipyramid.

Proposition 28. *Let $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ be a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$. Assume the vertices of Σ_0 are labeled as in Figure 2. If $\{v_1, v_4\} \in \Delta$, then there exists a PS ear-decomposable simplicial complex $\Delta' = \Sigma_0' \cup \Sigma_1' \cup \cdots \cup \Sigma_t'$ such that $\Sigma_0' = \partial\sigma^2 * \partial\sigma^1$ and $h(\Delta) = h(\Delta')$. Consequently, $h(\Delta)$ is a pure $\mathcal{O}$-sequence.*

Proof. Let Σ_i be the ear in which edge $\{v_1, v_4\}$ is introduced. The ear Σ_i must be of type-E, so there are vertices u and w such that Σ_i has triangular faces $\{v_1, v_4, u\}$ and $\{v_1, v_4, w\}$.

The proof proceeds in two steps. First we handle the case that $\{u, w\} = \{v_2, v_5\}$. Next we reduce to that case.

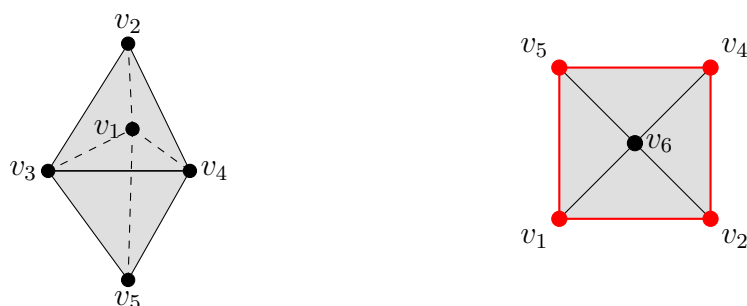If $\{u, w\} = \{v_2, v_5\}$, let Σ_0' be the bipyramid and let Σ_1' be the PS ear shown in Figure 3.

Figure 3: The new PS sphere Σ_0' (left) and PS ball Σ_1' (right).

Note that $\Sigma_0' \cup \Sigma_1' = \Sigma_0 \cup \Sigma_i$. Therefore, we can decompose Δ as $\Delta = \Sigma_0' \cup \Sigma_1' \cup \Sigma_2 \cup \cdots \cup \Sigma_{i-1} \cup \Sigma_{i+1} \cup \cdots \cup \Sigma_t$, which satisfies the claim.

Now suppose instead that $\{u, w\} \neq \{v_2, v_5\}$. First, assume $\{u, w\}$ and $\{v_2, v_5\}$ share one element. Without loss of generality, $u = v_2$ but $w \neq v_5$. Consider the set $\tau = \{v_1, v_4, v_5\}$. The three edges $\{v_1, v_4\}$, $\{v_1, v_5\}$, and $\{v_4, v_5\}$ all belong to Δ.

If $\tau \in \Delta$, then τ is added to Δ through an F-move because $\{v_1, v_5\}$ and $\{v_4, v_5\}$ are edges in Σ_0 and $\{v_1, v_4\}$ is introduced through Σ_i, and neither Σ_0 nor Σ_i contains τ. Therefore, there is some $j > i$ such that Σ_j is an F-move introducing face τ. We can replace Σ_i with the PS ear Σ_i', which is an E-move introducing edge $\{v_1, v_4\}$ using the triangles $\{v_1, v_2, v_4\}$ and $\{v_1, v_4, v_5\}$, and replace Σ_j with the PS ear Σ_j', which is an F-move introducing the face $\{v_1, v_4, w\}$. This is an alternate PS ear decomposition of Δ that satisfies the claim.

On the other hand, if $\tau \notin \Delta$, let Δ' be the complex obtained from Δ by removing the face $\{v_1, v_4, w\}$ and adding the face τ. This complex is PS ear-decomposable because we can replace Σ_i with the ear Σ_i', which is an E-move with triangles $\{v_1, v_2, v_4\}$ and $\{v_1, v_4, v_5\}$. Clearly $h(\Delta') = h(\Delta)$, so Δ' satisfies the claim.

Finally, if $\{u, w\} \cap \{v_2, v_5\} = \emptyset$, we apply the previous argument twice. $\qquad\square$

Now we turn our attention to the case that $\{v_1, v_4\} \notin \Delta$. We begin with a lemma whose proof is essentially Algorithm 17.

Lemma 29. *Let G be a constructible graph and $\mathcal{C}(G)$ its compression. There exists a 2-dimensional PS ear-decomposable simplicial complex Δ whose underlying graph is $\mathcal{C}(G)$.*

Proof. We begin with the boundary of a tetrahedron on vertices $\{v_1, v_2, v_3, v_4\}$. For each vertex v_i of type-B in $\mathcal{C}(G)$, we attach an ear of type-B whose boundary is the cycle $v_1 v_2 v_3 v_4$, and for each vertex of type-A, we attach an ear of type-A whose boundary is the cycle $v_1 v_2 v_3$. If $\{v_i, v_j\}$ is an edge added to $\mathcal{C}(G)$, then v_1 and v_2 are common neighbors of v_i and v_j by construction, so we attach an ear of type-E whose boundary is the cycle $v_1 v_i v_2 v_j$. The underlying graph of the resulting complex is $\mathcal{C}(G)$. $\qquad\square$

For the remainder of the paper, we assume Δ is a 2-dimensional PS ear-decomposable simplicial complex with $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ such that $\{v_1, v_4\} \notin \Delta$. Let G be the underlying graph of Δ. Let G' be the graph obtained from G by applying the shifting operator $\mathcal{S}_{4,5}$. We know G is constructible and claim that G' is also constructible. First, $\mathcal{S}_{4,5}$ acts on the graph of Σ_0 by removing edge $\{v_1, v_5\}$ and adding edge $\{v_1, v_4\}$.
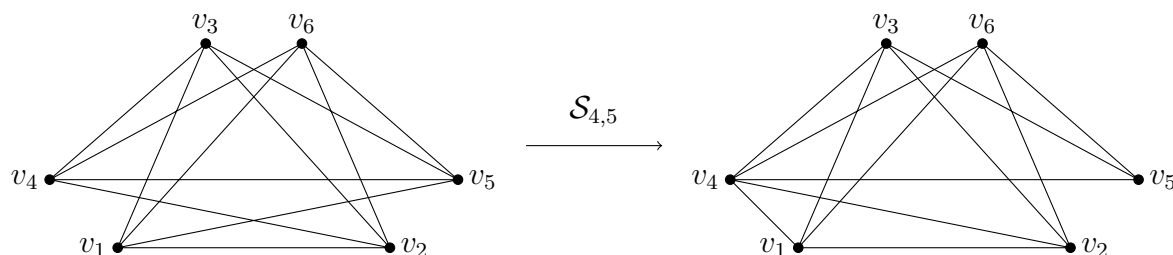
The resulting graph is constructible, starting with the complete graph on vertices $\{v_1, v_2, v_3, v_4\}$, then adding edges $\{v_1, v_6\}$, $\{v_3, v_6\}$, and $\{v_4, v_6\}$ through an A-move that introduces vertex v_6, and adding edges $\{v_2, v_5\}$, $\{v_4, v_5\}$, and $\{v_5, v_6\}$ through an A-move that introduces vertex v_5. We now proceed inductively. For each vertex v_ℓ with $\ell > 6$, consider the ℓ-labeled edges incident to v_ℓ in G. Let N denote the set of vertices incident to v_ℓ by such edges. If $v_5 \notin N$ or if $\{v_4, v_5\} \subset N$, add ℓ-labeled edges $\{v_i, v_\ell\}$ to G' for all $i \in G$. Otherwise, if $v_5 \in N$ but $v_4 \notin N$ add ℓ-labeled edges $\{v_4, v_\ell\}$, and $\{v_i, v_n\}$ for $i \in N \setminus \{v_5\}$ to G'.

Having done this, we move on to consider the unlabeled edges in G. Suppose $\{v_i, v_j\}$ is such an edge and $i < j$. If $i > 5$, add the unlabeled edge $\{v_i, v_j\}$ to G' as well. If $i = 4$, it is possible that $\{v_4, v_j\}$ already belongs to G' as a j-labeled edge. If this happens, then $\{v_4, v_j\}$ is an unlabeled edge in G and $\{v_5, v_j\}$ is a j-labeled edge in G. This means

$\{v_5, v_j\} \notin G'$, so we can add the edge $\{v_5, v_j\}$ to G'. Finally, if $i = 5$ and $\{v_4, v_j\}$ is not an edge in G', add the unlabeled edge $\{v_4, v_j\}$ to G'. Otherwise, add the unlabeled edge $\{v_5, v_j\}$ to G'. This proves that G' is also constructible.

Next, by Lemma 12 we know $\#T(G) \leqslant \#T(G')$, and because G' is constructible, Theorem 16 tells us $\#T(G') \leqslant \#T(\mathcal{C}(G'))$. Finally, by Lemma 29, there is a PS ear-decomposable simplicial complex Δ' whose underlying graph is $\mathcal{C}(G')$. Use η'_A, η'_B, η'_E, and η'_F to denote the number of ears of type A, B, E, and F respectively in the construction of Δ'. It follows from the proof of Lemma 29 that the PS sphere used in the construction of Δ' is $\partial\sigma^3$ and that $\eta'_A = \eta_A + 2$, $\eta'_B = \eta_B$, $\eta'_E = \eta_E$, and $\eta'_F = 0$. Finally, since $\#T(G) \leqslant \#T(\mathcal{C}(G'))$, there are at least η_F missing triangles in Δ' that can be filled. This proves the following theorem.

Theorem 30. *Let* $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ *be a 2-dimensional PS ear-decomposable simplicial complex with* $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ *such that* $\{v_1, v_4\} \notin \Delta$.

There exists a PS ear-decomposable simplicial complex $\Delta' = \Sigma'_0 \cup \Sigma'_1 \cup \cdots \cup \Sigma'_{t+2}$ *such that*

- $\Sigma'_0 = \partial\sigma^3$,

- $h(\Delta') = h(\Delta) + (0, 0, 0, 2)$,

- $\eta'_A = \eta_A + 2$,

- $\eta'_B = \eta_B$,

- $\eta'_E = \eta_E$, *and*

- $\eta'_F = \eta_F$.

This brings us to the final subcase.

Theorem 31. *Let* $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ *be a 2-dimensional PS ear-decomposable simplicial complex with* $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$ *such that* $\{v_1, v_4\} \notin \Delta$. *Assume further that* $\eta_E > 0$ *and* $\eta_F > 0$. *Then* $h(\Delta)$ *is a pure* $\mathcal{O}$-*sequence.*

Proof. Let Δ' be the complex whose existence is guaranteed by Theorem 30, and let $\mathcal{C}(\Delta')$ and $\mathcal{M} = \mathcal{M}(\Delta')$ be the PS ear-decomposable simplicial complex and corresponding pure multicomplex output by Algorithm 17. It is important to recall that the vertex labels in $\mathcal{C}(\Delta')$ may be permuted from their initial labels in Δ and Δ'. Our goal is to show that there are two degree-3 monomials in $\mathcal{M}$ that can be removed without destroying the purity of the multicomplex. We examine three cases.

Case 1: Vertices v_5 and v_6 have type A in $\mathcal{C}(\Delta')$.

In Step 1 of Algorithm 17, the monomials $\{x_5, x_5^2, x_5^3\}$ and $\{x_6, x_6^2, x_6^3\}$ are added to the initial set of monomials $\{1, x_4, x_4^2, x_4^3\}$ in $\mathcal{M}$. Because $\eta_E > 0$, and $x_4 x_5$ is the revlex smallest monomial of degree-2 that does not belong to $\mathcal{M}$ upon the completion of Step 1, the monomials $\{x_4 x_5, x_4^2 x_5\}$ will be added to $\mathcal{M}$ in Step 2 of Algorithm 17. Finally, because $\eta_F > 0$, the monomial $x_4 x_5^2$ will be added in Step 3 as it is the revlex smallest monomial

of degree 3 that does not belong to $\mathcal{M}$ upon the completion of Step 2. This means that the following monomials will belong to $\mathcal{M}$ upon the completion of Algorithm 17

$$
\begin{array}{ccccc}
x_4^3 & x_5^3 & x_6^3 & x_4^2 x_5 & x_4 x_5^2 \\
x_4^2 & x_5^2 & x_6^2 & x_4 x_5 \\
x_4 & x_5 & x_6 \\
1
\end{array}
$$

Now we can observe that $\mathcal{M} \setminus \{x_4^3, x_5^3\}$ is a pure multicomplex whose F-vector is $h(\Delta)$.

Case 2: Vertex v_5 has type B and vertex v_6 has type A in $\mathcal{C}(\Delta')$.

In Step 1 of Algorithm 17, the monomials $\{x_5, x_4 x_5 x_5^2, x_4 x_5^2\}$ and $\{x_6, x_6^2, x_6^3\}$ are added to the initial set of monomials $\{1, x_4, x_4^2, x_4^3\}$ in $\mathcal{M}$. Because $\eta_E > 0$, the monomials $\{x_4 x_6, x_4^2 x_6\}$ will be added to $\mathcal{M}$ in Step 2 of Algorithm 17. Finally, because $\eta_F > 0$, the monomial $x_4^2 x_5$ will be added in Step 3. This means that the following monomials will belong to $\mathcal{M}$ upon the completion of Algorithm 17

$$
\begin{array}{cccccc}
x_4^3 & & x_4 x_5^2 & & x_6^3 & x_4^2 x_6 & x_4^2 x_5 \\
x_4^2 & x_4 x_5 & x_5^2 & x_6^2 & x_4 x_6 \\
x_4 & & x_5 & & x_6 \\
1
\end{array}
$$

Now we can observe that $\mathcal{M} \setminus \{x_4^3, x_4^2 x_5\}$ is a pure multicomplex whose F-vector is $h(\Delta)$.

Case 3: Vertices v_5 and v_6 have type B in $\mathcal{C}(\Delta')$.

In Step 1 of Algorithm 17, the monomials $\{x_5, x_4 x_5 x_5^2, x_4 x_5^2\}$ and $\{x_6, x_4 x_6 x_6^2, x_4 x_6^2\}$ are added to the initial set of monomials $\{1, x_4, x_4^2, x_4^3\}$ in $\mathcal{M}$. Because $\eta_E > 0$, the monomials $\{x_5 x_6, x_5^2 x_6\}$ will be added to $\mathcal{M}$ in Step 2 of Algorithm 17. Finally, because $\eta_F > 0$, the monomial $x_4^2 x_5$ will be added in Step 3. This means that the following monomials will belong to $\mathcal{M}$ upon the completion of Algorithm 17

$$
\begin{array}{cccccc}
x_4^3 & & x_4 x_5^2 & & x_4 x_6^2 & & x_5^2 x_6 & x_4^2 x_5 \\
x_4^2 & x_4 x_5 & x_5^2 & x_4 x_6 & x_6^2 & x_5 x_6 \\
x_4 & & x_5 & & x_6 \\
1
\end{array}
$$

Now we can observe that $\mathcal{M} \setminus \{x_4^3, x_4 x_5^2\}$ is a pure multicomplex whose F-vector is $h(\Delta)$. $\qquad\square$

Together, Theorem 24, Theorem 27, Proposition 28, and Theorems 30 and 31 exhaust all possibilities when $\Sigma_0 = \partial \sigma^1 * \partial \sigma^1 * \partial \sigma^1$. This completes the proof of the main theorem.

Theorem 32. *Let* $\Delta = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_t$ *be a 2-dimensional PS ear-decomposable simplicial complex with* $\Sigma_0 = \partial\sigma^1 * \partial\sigma^1 * \partial\sigma^1$. *Then* $h(\Delta)$ *is a pure* $\mathcal{O}$-*sequence.*

Acknowledgments

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