Rainbow matchings of size m in graphs with total color degree at least $2mn$

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Abstract

The existence of a rainbow matching given a minimum color degree, proper coloring, or triangle-free host graph has been studied extensively. This paper generalizes these problems to edge colored graphs with given total color degree. In particular, we find that if a graph G has total color degree $2mn$ and satisfies some other properties, then G contains a matching of size m. These other properties include G being triangle-free, C_4-free, properly colored, or large enough.

Mathematics Subject Classifications: 05C15, 05C70

1 Introduction

Given a graph G, let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. If $S \subseteq V$, then $G[S]$ denotes the subgraph induced by the vertices in S. A graph G is an m-matching if G contains exactly m edges, $2m$ vertices, and $e \cap e' = \{\}$ for all edges $e \neq e'$ in $E(G)$. An edge coloring $c : E(G) \to [r] = \{1, \ldots, r\}$ is an assignment of colors to edges. A proper edge coloring of a graph is an edge coloring such that $c(e) \neq c(e')$ whenever $e \cap e' \neq \varnothing$ and $e \neq e'$. The colors used on a graph will be denoted $c(G)$, and R will denote a generic color class. If $X, Y \subseteq V(G)$, then $c(X, Y)$ will denote the set of colors used on edges of the form xy, where $x \in X$, $y \in Y$. A graph G is rainbow under c if c is injective on $E(G)$. In particular, a rainbow matching is a matching where each edge receives a unique color within the matching. The color degree of a vertex v is denoted $\hat{d}_G(v)$, which is the number of colors c assigns to edges incident upon v in G; when it is clear from the context what G is, we will drop the subscript. Let $\hat{d}^R(v)$ denote the

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number of R colored edges incident upon v. The total color degree of G with respect to c is the sum of all the color degrees in the graph and denoted

$$\hat{d}(G) = \sum_{v \in V(G)} \hat{d}(v).$$

The average color degree of a graph G is obtained by dividing the total color degree by $|V(G)|$, and is an equivalent notion. The minimum color degree of G is denoted $\hat{\delta}(G)$. Finally, let $G - v$ denote the graph G with the vertex v deleted, and $G - R$ denote the graph G with the edges in color class R deleted. When convenient, we will let $c(e)$ denote a color class so that $G - c(e)$ denotes the graph G without the edges in color class containing the edge e.

Rainbow matchings in graphs were originally studied in connection to transversals of Latin squares [9, 10]. However, the existence of rainbow matchings has also been studied in its own right. In [6], Li and Wang conjectured that any graph with $\hat{\delta}(G) \geqslant m \geqslant 4$ contains a rainbow matching of size $\lceil \frac{m}{2} \rceil$. This conjecture was partially confirmed in [5], and fully confirmed in [4].

Wang asked for a function f such that any properly edge colored graph G with $|V(G)| \geqslant f(\hat{\delta}(G))$ contains a rainbow matching of size $\hat{\delta}(G)$ [11]. Diemunsch et al. determined that $|V(G)| \geqslant \frac{98}{23}\hat{\delta}(G)$ is sufficient [1]. This problem was generalized to find a function f such that any edge colored graph G with $|V(G)| \geqslant f(\hat{\delta}(G))$ contains a rainbow matching of size $\hat{\delta}(G)$. The authors of [3] found that $|V(G)| \geqslant \frac{17}{4}\hat{\delta}(G)^2$ sufficed. This was improved to $4\hat{\delta}(G) - 4$ for $\hat{\delta}(G) \geqslant 4$ in [2] and [8] independently.

Local Anti-Ramsey theory asks Anti-Ramsey type questions with assumptions about the local structure of the host graph. In particular, Local Anti-Ramsey theory is about the minimum k such that any coloring of K_n with $\hat{\delta}(G) \geqslant k$ contains a rainbow copy of H. In this vein, Wang's question can be posed as follows: given k, what is the smallest N such that any properly edge colored graph G with $|V(G)| \geqslant N$ and $\hat{\delta}(G) \geqslant k$ contains a rainbow matching of size k? Furthermore, proper edge-coloring and triangle-free properties play similar roles in restricting the structure of a host graph.

The local assumptions in Anti-Ramsey theory are interesting in so far as they highlight the relationship between a local parameter and the target graph. In much of the rainbow matching literature, there are confounding local assumptions. For example, [1], [7], and [11] all consider host graphs that have a prescribed minimum color degree and are properly edge colored. In this case, an intuitive interpretation is that the minimum color degree and proper edge-coloring properties spread the colors apart in the host graph. As one would expect, this makes it easier to find a large rainbow matching. However, it is unclear whether both the minimum color degree and proper edge coloring property are necessary to find a large matching.

The goal of this paper is to shed light on the relationship between local assumptions and rainbow matchings. Rather than considering host graphs with a prescribed minimum color degree, we will consider host graphs with a prescribed average color degree. This is motivated in part by a question posed during the Rocky Mountain and Great Plains Graduate Research Workshop in Combinatorics in 2017.

Question 1. If G is an edge colored graph on n vertices with $\hat{d}(G) \geqslant 2mn$, does G contain a rainbow matching of size m?

Section 2 considers this question for triangle-free and C_4-free host graphs. In the case of triangle-free graphs, we will prove the slightly stronger statement that if G is a graph with $\hat{d}(G) > 2mn$, then there exists a rainbow matching of size $m+1$. Section 3 pertains to properly edge colored host graphs. Finally, Section 4 considers edge colored graphs with total color degree $2mn$, but with no further assumptions.

2 Triangle-free and C_4-free Graphs

In this section, we consider triangle-free and C_4-free graphs.

Theorem 2. *Let G be a triangle-free graph on n vertices. Let c be an edge coloring of G with $\hat{d}(G) > 2mn$. Then c admits a rainbow matching of size $m+1$.*

Proof. For the sake of contradiction, let M be a maximum rainbow matching of size $k \leqslant m$ with edges u_iv_i for $1 \leqslant i \leqslant k$, such that the number of colors appearing on $G[V(G) \setminus V(M)] = H$ is maximized. Without loss of generality, suppose that $c(u_iv_i) = i$. Since G is triangle-free, $\hat{d}(u_i) + \hat{d}(v_i) \leqslant n$ for all $u_iv_i \in E(M)$. If H has an edge e, then $c(e) \in [k]$. Without loss of generality, suppose that $c(H) = [j]$ for some $0 \leqslant j \leqslant k$. Then for all $v \in V(H)$, we have $\hat{d}(v) \leqslant k + j$. Notice that if there exists an edge $e \in H$ with $c(e) = i$, then we can swap e and u_iv_i to conclude that $\hat{d}(u_i) + \hat{d}(v_i) \leqslant 2(j+k)$.

Now consider

$$\begin{aligned}
2mn &< \sum_{i=1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \hat{d}_G(v) \\
&\leqslant \sum_{i=1}^{j} \hat{d}(u_i) + d(v_i) + \sum_{i=j+1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \left(\hat{d}_H(v) + k \right) \\
&\leqslant 2j(k+j) + (k-j)n + (n-2k)(j+k) \\
&= 2jk + 2j^2 + 2nk - 2jk - 2k^2 \\
&\leqslant 2j^2 - 2k^2 + 2nk \\
&\leqslant 2nm.
\end{aligned}$$

This is a contradiction; therefore, $k \geqslant m+1$. □

A key element to the proof of Theorem 2 is the bound $\hat{d}(v) + \hat{d}(u) \leqslant n$ where uv is an edge in a maximal matching. We can obtain a similar bound in C_4-free graphs in order to prove the next theorem.

Theorem 3. *Let G be a C_4-free graph on n vertices. Let c be an edge coloring of G with $\hat{d}(G) \geqslant 2mn$. Then c admits a rainbow matching of size m.*

Proof. For the sake of contradiction, let M be a maximum rainbow matching of size $k < m$ with edges $u_i v_i$ for $1 \leqslant i \leqslant k$, such that the number of colors appearing on $G[V(G) \setminus V(M)] = H$ is maximized. Without loss of generality, suppose that $c(u_i v_i) = i$. Since G is C_4-free, $\hat{d}(u_i) + \hat{d}(v_i) \leqslant n + 1$ for all $u_i v_i \in E(M)$. If H has an edge e, then $c(e) \in [k]$. Without loss of generality, suppose that $c(H) = [j]$ for $0 \leqslant j \leqslant k$.

Claim 4. *If $xy \in E(H)$ with $c(xy) = i \leqslant j$, then $\hat{d}(u_i) + \hat{d}(v_i) \leqslant 2j + 2k$.*

Notice that x, y each see at most j colors in H. Since xy can share at most two edges with any edge in M without creating a C_4 subgraph, we have $|c(\{u_i, v_i\}, xy)| \leqslant 2$ for every $1 \leqslant i \leqslant k$. Thus, $\hat{d}(x) + \hat{d}(y) \leqslant 2j + 2k$. By swapping $u_i v_i$ and xy, we obtain the desired bound on $\hat{d}(u_i) + \hat{d}(v_i)$.

Furthermore, $\sum_{v \in H} \hat{d}_G(v) \leqslant (n - 2k)(j + k) + k$. The $(n - 2k)j$ term comes from the fact that H has $n - 2k$ vertices, each of which can see every color in $[j]$. We will show that there are at most $(n - 2k)k + k$ color degrees in H that do not come from a color in $[j]$ by contradiction. Suppose that there are $(n - 2k)k + k + 1$ edges from H to M. By the pigeon hole principle, there exists an edge $u_i v_i \in M$ that receives at least $n - 2k + 2$ edges from H. Notice that each vertex in H can send at most two edges to $u_i v_i$. Therefore, there must exist two vertices in H that each send two edges to $u_i v_i$, witnessing a C_4 subgraph; this is a contradiction.

Now consider

$$\begin{aligned}
2mn &\leqslant \sum_{i=1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \hat{d}_G(v) \\
&\leqslant \sum_{i=1}^{j} \hat{d}(u_i) + d(v_i) + \sum_{i=j+1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \left(\hat{d}_H(v) + k \right) \\
&\leqslant j(2k + 2j) + (k - j)(n + 1) + (n - 2k)(j + k) + k \\
&= 2kj + 2j^2 + nk + k - nj - j + nj + nk - 2kj - 2k^2 + k \\
&\leqslant 2j^2 + 2nk - j + 2k - 2k^2 \\
&\leqslant 2j^2 - 2k^2 + 2k - j - 2n + 2mn \\
&< 2mn.
\end{aligned}$$

This is a contradiction; therefore, $k \geqslant m$. $\square$

3 Properly Edge Colored Graphs

In this section, we consider properly edge colored graphs. The idea to analyze a greedy algorithm that constructs a matching appears in [1] and [3]. The algorithm employed in this section is similar, with some adjustments to take into account the weaker degree assumption.

Theorem 5. *Let c be a proper edge coloring of G with $n \geqslant 8m$ and $\hat{d}(G) \geqslant 2mn$. Then c admits a rainbow matching of size m.*

Proof. Assume that G is an edge minimal counter example to Theorem 5. Consider the following algorithm:

1. set $G_0 := G$
2. if there exists $v \in V(G_{i-1})$ with $\hat{d}(v) \geqslant 3(m-i)+1$, then $G_i = G_{i-1} - v$ and return to 2
3. else, if there exists color class R with $|R| \geqslant 2(m-i)+1$ in G_{i-1}, then $G_i = G_{i-1} - R$ and return to 2
4. else, if there exists $uv \in E(G_{i-1})$, then $G_i = G_{i-1} - u - v - c(uv)$ and return to 2
5. return $i-1$

Claim 6. *Suppose the algorithm returns $k \leqslant m$. Then G_i contains a matching of size $k-i$ for $0 \leqslant i \leqslant k$*

We will prove the claim by reverse induction on i. If $i = k$, then G_i is empty, and the claim is true. Assume that the claim is true for i. We will prove the claim for $i-1$. By the induction hypothesis, there exists a matching $M \subseteq G_i$ of size $k-i$. There are three cases:

Case 1: Assume $G_i = G_{i-1} - v$ where $\hat{d}(v) \geqslant 3(m-i)+1$. By construction, $v \notin V(M)$. Since $\hat{d}(v) \geqslant 3(m-i)+1$, there exists $u \in N(v)$, such that $u \notin V(M)$ and $c(uv) \notin c(M)$. Then $M' = M \cup \{uv\}$ is a rainbow matching of size $k-i+1$.

Case 2: Assume $G_i = G_{i-1} - R$ for some color R with $|R| \geqslant 2(m-i)+1$. This implies that $c(e) \neq R$ for all $e \in E(M)$. Since c is a proper coloring and $|R| \geqslant 2(m-i)+1$, there exist $e \in G_{i-1}$ such that $c(e) = R$ and $M' = M \cup \{e\}$ is a rainbow matching.

Case 3: Assume that $G_i = G_{i-1} - v - u - c(uv)$ for some $uv \in E(G_{i-1})$. By construction $N[u] \cup N[v]$ is disjoint from $V(M)$ and $c(e) \neq c(uv)$ for all $e \in M$. Therefore, $M' = M \cup \{uv\}$ is a rainbow matching.

This concludes the proof of the claim. Since G is an edge minimal counter example, the algorithm applied to G will return $k < m$. We will now derive a contradiction.

Let $W(G_i)$ denote the difference of total color degree between G_i and G_{i-1} under c.

Claim 7. *For all $1 \leqslant i \leqslant k$, we have $W(G_i) \leqslant 2n$.*

Case 1: Assume $G_i = G_{i-1} - v$ where $\hat{d}(v) \geqslant 3(m-i)+1$. Notice that v is incident to at most $n-1$ edges. Therefore, deleting v will remove at most $2(n-1)$ color degrees.

Case 2: Assume $G_i = G_{i-1} - R$ for some color R with $|R| \geqslant 2(m-i)+1$. Because c is proper, $|R| \leqslant \lfloor n/2 \rfloor$. Deleting all edges of color R reduces the total color degree by at most n.

Case 3: Assume that $G_i = G_{i-1} - v - u - c(uv)$ for some $uv \in E(G_{i-1})$. Since G_i is not constructed by step 2, we know that $\hat{d}(u), \hat{d}(v) \leqslant 3(m-i)$. Furthermore, since G_i is not constructed by step 3, we know that $|c(uv)| \leqslant 2(m-i)$. This implies that

$$\begin{aligned} W(G_i) &= 2(\hat{d}(v) + \hat{d}(u)) + 2|c(uv)| \\ &\leqslant 16(m-i) \\ &\leqslant 2n. \end{aligned}$$

This concludes the proof of the claim. Now we have

$$2nm \leqslant \hat{d}(G) = \sum_{i=1}^{k} W(G_i) \leqslant 2nk,$$

which is a contradiction since $k < m$. Therefore, the theorem is proven. □

4 General Edge-Colored Graphs

Theorem 8 provides contrast for Theorems 2, 3, and 5. The proof of Theorem 8 is similar to the proof of Theorem 5. However, the greedy algorithm has been modified to accommodate graphs that are not properly colored.

Theorem 8. *Let c be an edge coloring of G be a graph with $\hat{d}(G) \geqslant 2mn$ and $n \geqslant 12m^2 + 4m$. Then c admits a rainbow matching of size m.*

Proof. Assume that G is an edge minimal counter example to Theorem 8. Since G is edge minimal, no color class can induce a P_4 (path on 4 vertices) or a triangle. This follows from the fact that if a color class R induces a P_4 or triangle, then an edge can be deleted without reducing the total color degree of the graph. Therefore, each color class in G induces a forest of stars. Let $s(R)$ denote the number of components induced by the color class R. Consider the following algorithm:

1. set $G_0 := G$
2. if there exists $v \in V(G_{i-1})$ with $\hat{d}(v) \geqslant 3(m-i)+1$, then $G_i = G_{i-1} - v$ and return to 2
3. else, if there exists color R with $s(R) \geqslant 2(m-i)+1$ in G_{i-1}, then $G_i = G_{i-1} - R$ and return to 2
4. else, if there exists a vertex v and a color R such that $\hat{d}^R(v) \geqslant 3(m-i)+1$ in G_{i-1}, then $G_i = G_{i-1} - v - R$ and return to 2
5. else, if there exists $uv \in E(G_{i-1})$, then $G_i = G_{i-1} - u - v - c(uv)$ and return to 2
6. return $i-1$

Since this algorithm is so similar to the algorithm featured in the proof of Theorem 5, the only things that remain to be checked are that step 4 lets us extend a matching, and that the bounds on steps 4 and 5 are still good.

Assume that $G_i = G_{i-1} - v - R$ where $\hat{d}^R(v) \geqslant 3(m-i)+1$. Let M be a rainbow matching of size $k-i$ contained in G_i. Since $v \notin V(G_i)$, $v \notin V(M)$. Furthermore, M does not contain an edge with color R. Since $\hat{d}^R(v) \geqslant 2(m-i)+1$, there exists an edge uv with $c(uv) = R$ and $u \notin M$. Then $M \cup \{uv\}$ is a rainbow matching of size $k-i+1$ contained in G_{i-1}.

If $G_i = G_{i-1} - v - R$ where $\hat{d}^R(v) \geqslant 3(m-i)+1$, then 2 and 3 must have been rejected. The color R contributes at most $n - 3(m-i)$ color using edges that are not incident upon v. Since $\hat{d}(v) \leqslant 3(m-i)$ and $d(v) \leqslant n$, it follows that $W(G_i) \leqslant n - 3(m-i) + \hat{d}(v) + d(v) \leqslant n - 3(m-i) + 3(m-i) + n = 2n$.

Suppose $G_i = G_{i-1} - v - u - c(uv)$. Then steps 2, 3, and 4 must have been rejected. This implies that $\hat{d}(v), \hat{d}(u) \leqslant 3(m-i)$. Furthermore, each color at v, u can be represented at most $3(m-i)$ times. Finally, the edges of color $c(uv)$ can induce at most $2(m-i)$ stars with $3(m-i)$ edges each. Therefore, deleting all $c(uv)$ colored edges reduces the color degree by at most $6m^2 + 2m$. Thus, $W(G_i) \leqslant 24m^2 + 8m \leqslant 2n$.

Suppose that the algorithm terminates in $k < m$ steps. Now we have

$$2nm \leqslant \hat{d}(G) = \sum_{i=1}^{k} W(G_i) \leqslant 2nk,$$

which is a contradiction since $k < m$. Therefore, the theorem is proven. □

5 Future Work

Though we were not able to resolve Question 1 for all graphs, we believe the answer is affirmative:

Conjecture 9. All edge colored graphs G with $\hat{d}(G) \geqslant 2mn$ contain a rainbow matching of size m.

It would also be interesting to know under which conditions there exists a matching of size $m+1$. It seems that a small improvement in the estimates in the proofs of Theorems 2 and 5 could yield this result for edge colored graphs G with $\hat{d}(G) \geqslant 2mn$. In fact, it may be that the proper question to ask is whether any graph G with $\hat{d}(G) \geqslant 2mn$ contains a rainbow matching of size $m+1$.

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References

[1] J. Diemunsch, M. Ferrara, A. Lo, C. Moffatt, F. Pfender, and P. S. Wenger, Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs, *The Electronic Journal of Combinaotrics* **19** (2012), no. 2, #P52.

[2] A. Gyárfás, G.N. Sárközy, Rainbow matchings and cycle-free partial transversals of Latin Squares, *Discrete mathematics* **327** (2014. pp. 96–102.

[3] A. Kostochka, F. Pfender, and M. Yancey, Large Rainbow Matchings in Large Graphs, arXiv:1204.3193 April 14, 2012.

[4] A. Kostochka, and M. Yancey, Large Rainbow Matchings in Edge-Colored Graphs, *Combinatorics, Probability and Computing* **21** (2012). 255–263.

[5] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West, Rainbow Matching in Edge-Colored Graphs, *The Electronic Journal of Combinatorics* **17** (2010). #N26.

[6] H. Li, and G. Wang, Heterochromatic Matchings in Edge-Colored Graph, *The Electronic Journal of Combinatorics* **15** (2008), #R138.

[7] G. Liu, G. Wang, and J. Zhang, Existence of rainbow matchings in properly edge colored graphs, *Frontiers of mathematics in China* **7**(3) (2012). pp 543–550.

[8] A. Lo, and T. S. Tan, A Note on Large Rainbow Matchings in Edge-colored Graphs, *Graphs and Combinatorics* **30**(2) (2014), 389–393.

[9] H. J. Ryser, Neuere Probleme der Kombinatorik, in “Vortrage uber Kombinatorik Oberwolfach,” Mathematisches Forschungsinstitut Oberwolfach, July 1967, 24–29.

[10] S. K. Stein, Transversals of Latin squares and their generalizations, *Pacific Journal of Mathematics* **59** (1975), no. 2, 567–575.

[11] G. Wang, Rainbow matchings in properly edge colored graphs, *The Electronic Journal of Combinatorics* **18** (2011), #P162.