A note on the Poljak-Rödl function

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Abstract

The Poljak-Rödl function is defined as $f(c) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = c\}$. This note proves that $\limsup_{c\to\infty} \frac{f(c)}{c} \leqslant \frac{1}{2}$.

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The *categorical product* (also known as the *tensor product* or the *direct product*) $G \times H$ of G and H has vertex set

$$V(G \times H) = \{(x, y) : x \in V(G), y \in V(H)\}$$

and edge set

$$E(G \times H) = \{(x, y)(x', y') : xx' \in E(G), yy' \in E(H)\}.$$

The Poljak-Rödl function [8] is defined as

$$f(c) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = c\}.$$

A proper colouring ϕ of G induces a proper colouring Φ of $G \times H$ defined as $\Phi(x, y) = \phi(x)$. So $\chi(G \times H) \leqslant \chi(G)$. Therefore $f(c) \leqslant c$ for all positive integers c. Hedetniemi conjectured in 1966 [4] that $f(c) = c$ for all positive integers c. This conjecture received a lot of attention [1, 5, 9, 12, 14, 15] and was confirmed for $c \leqslant 4$ [1]. However the conjecture was disproved by Shitov in [10] in 2019. For a positive integer c, let $[c] = \{1, 2, \ldots, c\}$. For a graph G, the *exponential graph* K_c^G has vertex set

$$V(K_c^G) = \{f : f \text{ is a mapping } V(G) \to [c]\}$$

and in which f and g are adjacent if and only if for any edge xy of G, $f(x) \neq g(y)$. For any graph G and any positive integer c, the mapping $\phi : V(G \times K_c^G) \to [c]$ defined as

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$\phi(x,f) = f(x)$ is a proper c-colouring of $G \times K_c^G$. So $\chi(G \times K_c^G) \leqslant c$. On the other hand, if ϕ is a proper c-colouring of $G \times H$, then the mapping sending $u \in V(H)$ to $f_u \in V(K_c^G)$ defined as $f_u(v) = \phi(v,u)$ is a homomorphism from H to K_c^G and hence $\chi(H) \leqslant \chi(K_c^G)$. Thus Hedetniemi's conjecture is equivalent to the statement that "if $\chi(G) > c$, then K_c^G is c-colourable."

The *lexicographic product* $G[K_q]$ of G and K_q is the graph obtained from G by replacing each vertex of G with a q-clique. Thus $G[K_q]$ has vertex set $\{(x,i) : x \in V(G), i \in V(K_q)\}$ and $(x,i) \sim (y,j)$ if $x \sim y$ or $x = y$ and $i \neq j$. The *fractional chromatic number* of G is defined as

$$\chi_f(G) = \inf\{\frac{\chi(G[K_q])}{q} : q \in \mathbb{N}\}.$$

Shitov disproved Hedetniemi's conjecture by showing the following result:

Theorem 1 (Shitov). *Assume $|V(G)| = p$, $\chi_f(G) \geqslant 3.1$, $\mathrm{girth}(G) \geqslant 6$, q is sufficiently large and $c = \lceil 3.1q \rceil$. Then $\chi(G[K_q]) > c$ and $\chi\left(K_c^{G[K_q]}\right) > c$.*

As a consequence of this result, we have $f(c) < c$ for sufficiently large c. Using Shitov's result, Tardif and Zhu [13] showed that for sufficiently large c, $f(c) \leqslant c - (\log c)^{1/4-o(1)}$. Tardif and Zhu also asked if there is a constant $\epsilon > 0$ such that $f(c) \leqslant (1-\epsilon)c$ for sufficiently large c and showed that if a special case of Stahl's conjecture in [11] on the multi-chromatic number of Kneser graphs is true, then $\limsup_{c\to\infty} f(c)/c \leqslant 1/2$. The above question was answered by He and Wigderson [3], who showed that for $\epsilon \approx 10^{-9}$, $f(c) \leqslant (1-\epsilon)c$ for sufficiently large c.

On the other hand, the problem whether $f(c)$ is bounded by a constant remains a challenging open problem. It is known [7, 8, 14] that $f(c)$ is either bounded by 9 to goes to infinity.

This note shows that $\limsup_{c\to\infty} f(c)/c \leqslant 1/2$ without assuming Stahl's conjecture.

Theorem 2. *For $d \geqslant 1$, let G be a graph of girth 6 and with $\chi_f(G) \geqslant 6.3d$. Let $p = |V(G)|$, q is sufficiently large and $c = \lceil 3.1q \rceil$. Then $\chi(G[K_q]) \geqslant 2dc - 2c + 2$ and $\chi(K_{dc}^{G[K_q]}) \geqslant 2dc - 2c + 2$. Consequently, $f(2dc - 2c + 2) \leqslant dc$.*

Proof. It is well-known that $\chi(G[K_q]) \geqslant \chi_f(G)q \geqslant 6.3dq \geqslant 2dc > 2dc - 2c + 2$. Now we show that $\chi(K_{dc}^{G[K_q]}) \geqslant 2dc - 2c + 2$.

Assume Ψ is a $(dc+t)$-colouring of $K_{dc}^{G[K_q]}$ with colour set $[dc+t]$. We shall show that $dc + t \geqslant 2dc - 2c + 2$, i.e., $t \geqslant dc - 2c + 2$. Let $S = [dc+t] \setminus [dc]$. The colours in $[dc]$ are called *primary colours* and colours in S are called *secondary colours*.

For $i \in [dc]$, denote by $g_i \in V(K_c^{G[K_q]})$ the constant map $g_i((x,j)) = i$ for all $(x,j) \in G[K_q]$. The set $\{g_i : i \in [dc]\}$ induces a dc-clique in $K_{dc}^{G[K_q]}$. Thus we may assume that $\Psi(g_i) = i$ for $i \in [dc]$.

For any $\phi \in V(K_{dc}^{G[K_q]})$, let $Im(\phi) = \{\phi(x,i) : (x,i) \in V(G[K_q])\}$ be the image set of ϕ. If $i \notin Im(\phi)$, then $\phi \sim g_i$. Hence $\Psi(\phi) \neq \Psi(g_i) = i$. Thus for any $\phi \in V(K_{dc}^{G[K_q]})$, $\Psi(\phi) \in Im(\phi) \cup S$.

For positive integers $m \geqslant 2k$, let $K(m,k)$ be the Kneser graph whose vertices are k-susbets of $[m]$, and for two k-subsets A, B of $[m]$, $A \sim B$ if $A \cap B = \emptyset$. It was proved by Lovász in [6] that $\chi(K(m,k)) = m - 2k + 2$.

For a c-subset A of $[cd]$, let H_A be the subgraph of $K_{cd}^{G[K_q]}$ induced by

$$\{\phi \in V(K_{cd}^{G[K_q]}) : Im(\phi) \subseteq A\}.$$

Then H_A is isomorphic to $K_c^{G[K_q]}$. By Theorem 1, $|\Psi(H_A)| \geqslant c+1$. For every $\phi \in V(H_A)$, we have $Im(\phi) \subseteq A$. Since $|\Psi(H_A)| \geqslant c+1 > |A|$, $\Psi(H_A)$ contains at least one secondary colour. Let $\tau(A)$ be an arbitrary secondary colour contained in $\Psi(H_A)$.

If A, B are c-subsets of $[dc]$ and $A \cap B = \emptyset$, then every vertex in H_A is adjacent to every vertex in H_B. Hence $\Psi(H_A) \cap \Psi(H_B) = \emptyset$. In particular, $\tau(A) \neq \tau(B)$. Thus τ is a proper colouring of the Kneser graph $K(dc, c)$. As $\chi(K(dc,c)) = dc - 2c + 2$, we conclude that $t = |S| \geqslant dc - 2c + 2$. This completes the proof of Theorem 2. □

For a positive integer d, let $p = p(d)$ be the minimum number of vertices of a graph G with girth 6 and $\chi_f(G) \geqslant 6.2d$. It follows from Theorem 2 that for any sufficiently large integer q (which depends on p), $f(2(d-1) \times \lceil 3.1q \rceil + 2) \leqslant \lceil 3.1q \rceil d$. As $f(c)$ is non-decreasing, for integers c in the interval $[2(d-1) \times \lceil 3.1q \rceil + 3,\ 2(d-1) \times \lceil 3.1(q+1) \rceil + 2]$, we have $f(c) \leqslant \lceil 3.1(q+1) \rceil d$.

Hence for all integers $c \geqslant 2 \times \lceil 3.1q \rceil (d-1) + 2$,

$$\frac{f(c)}{c} \leqslant \frac{\lceil 3.1(q+1) \rceil d}{2(d-1) \times \lceil 3.1q \rceil + 2}.$$

Note that if $d \to \infty$, then $p = p(d)$ goes to infinity, and hence q goes to infinity. Hence

$$\limsup_{c \to \infty} \frac{f(c)}{c} \leqslant \frac{1}{2}.$$

Remark The number q in Theorem 1 is required to be large enough so that some inequalities in the proof are satisified. A careful analysis of these inequalities shows that $q \geqslant 3^p p^3$ is enough. Let p be the minimum number of vertices of a graph of girth 6 and fractional chromatic number 3.1. The exact value of p is also unknown. A recent computer search by Exoo [2] found a graph on 83 vertices which has odd girth 7 and fractional chromatic number greater than 3.07 (which is enough for Shitov's proof). For $p = 83$, we have $c \geqslant 3^{96}$. So the graphs in Theorem 1 have huge chromatic number. Recently, a relatively small counterexample to Hedetniemi's conjecture was constructed in [16]. It is now known that Hedetniemi's conjecture fails for $c \geqslant 125$. Two graphs G and H were constructed in [16], such that $\chi(G), \chi(H) > 125$ and $\chi(G \times H) \leqslant 125$, and G and H have $3,403$ and $10,501$ vertices respectively.

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