Exact minimum codegree thresholds for $K_4^-$-covering and $K_5^-$-covering

Lei Yu$^a$ Xinmin Hou$^{a,b,}$* Yue Ma$^a$ Boyuan Liu$^a$ 

$^a$School of Mathematical Sciences  
University of Science and Technology of China  
Hefei, Anhui, 230026, PR China  
{yu1lei,mymy,lby1055}@mail.ustc.edu.cn

$^b$CAS Wu Wen-Tsun Key Laboratory of Mathematics  
University of Science and Technology of China  
Hefei, Anhui, 230026, PR China  
xmhou@ustc.edu.cn

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Abstract

Given two 3-graphs $F$ and $H$, an $F$-covering of $H$ is a collection of copies of $F$ in $H$ such that each vertex of $H$ is contained in at least one copy of them. Let $c_2(n,F)$ be the minimum integer $t$ such that every 3-graph with minimum codegree greater than $t$ has an $F$-covering. In this note, we answer an open problem of Falgas-Ravry and Zhao (SIAM J. Discrete Math., 2016) by determining the exact value of $c_2(n,K_4^-)$ and $c_2(n,K_5^-)$, where $K_t^-$ is the complete 3-graph on $t$ vertices with one edge removed.

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1 Introduction

Given a set $V$ and a positive integer $k$, let $\binom{V}{k}$ be the collection of $k$-element subsets of $V$. A simple $k$-uniform hypergraph (or $k$-graph for short) $H = (V,E)$ consists of a vertex set $V$ and an edge set $E \subseteq \binom{V}{k}$. We write graph for 2-graph for short. For a set $S \subseteq V(H)$, the neighbourhood $N_H(S)$ of $S$ is $\{ T \subseteq V(H) \setminus S : T \cup S \in E(H) \}$ and the degree of $S$ is $d_H(S) = |N_H(S)|$. The minimum (resp. maximum) $s$-degree of $H$, denoted by $\delta_s(H)$ (resp. $\Delta_s(H)$), is the minimum (resp. maximum ) $d_H(S)$ taken over all $s$-element sets

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of $V(H)$. $\delta_{k-1}(H)$ and $\delta_1(H)$ are usually called the minimum codegree and the minimum degree of $H$, respectively. An $r$-graph $H$ is called an $r$-partite $r$-graph if the vertex set of $H$ can be partitioned into $r$ parts such that each edge of $H$ intersects each part exactly one vertex. Given disjoint sets $V_1, V_2, \ldots, V_r$, let $K(V_1, V_2, \ldots, V_r)$ be the complete $r$-partite $r$-graph with vertex classes $V_1, V_2, \ldots, V_r$.

Given a $k$-graph $F$, we say a $k$-graph $H$ has an $F$-covering if each vertex of $H$ is contained in some copy of $F$. For $0 \leq i < k$, define

$$c_i(n, F) = \max \{ \delta_i(H) : H \text{ is a } k \text{-graph on } n \text{ vertices with no } F \text{-covering} \}.$$  

We call $c_{k-1}(n, F)$ the minimum codegree threshold for $F$-covering.

There are two well studied extremal problems related to the covering problem. Given a $k$-graph $F$, a $k$-graph $H$ is $F$-free if $H$ does not contain a copy of $F$ as a subgraph. For $0 \leq i < k$, define

$$ex_i(n, F) = \max \{ \delta_i(H) : |V(H)| = n \text{ and } H \text{ is } F \text{-free} \}.$$  

The quantity $ex_0(n, F)$ is known as the Turán number of $F$, and $ex_{k-1}(n, F)$ was studied by Mubayi and Zhao [7]. For an overview of the Turán problem for hypergraphs, one can see a survey given by Keevash [5]. Given two $k$-graphs $H$ and $F$ with $|V(H)|$ is divisible by $|V(F)|$, a perfect $F$-tiling (or an $F$-factor) in $H$ is a spanning collection of vertex-disjoint copies of $F$. For $0 \leq i < k$ and $n$ divisible by $|V(F)|$, define

$$t_i(n, F) = \max \{ \delta_i(H) : |V(H)| = n \text{ and } H \text{ does not have an } F \text{-factor} \}.$$  

The study of the tiling problem also has a long history. For detailed discussion of the area, one can refer to the surveys due to Rödl and Ruciński [8] and Zhao [10]. Trivially, for $0 \leq i < k$, we have

$$ex_i(n, F) \leq c_i(n, F) \leq t_i(n, F).$$  

So the covering problem is an intermediate but distinct problem from the well-studied Turán and tiling problems. This is also partial motivation for the study of covering problems.

For graphs $F$, the $F$-covering problem was solved asymptotically in [9] by showing that $c_1(n, F) = (\chi(F)-2 + o(1))n$, where $\chi(F)$ is the chromatic number of $F$. For general $k$-graphs, the function $c_1(n, F)$ was determined for some special families of $k$-graphs $F$. For example, Han, Lo, and Sanhueza-Matamala [3] proved that $c_{k-1}(n, C_s^{(k,k-1)}) \leq (\frac{k}{2} + o(1))n$ for $k \geq 3, s \geq 2k^2$ and the result is asymptotically tight if $k$ and $s$ satisfy some special constrains, where $C_s^{(k,\ell)} (1 \leq \ell < k)$ is the $k$-graph on $s$ vertices such that its vertices can be ordered cyclicly so that every edge consists of $k$ consecutive vertices under this order and two consecutive edges intersect in exactly $\ell$ vertices. Han, Zang, and Zhao showed in [4] that $c_1(n, K) = \left(6 - 4\sqrt{2} + o(1)\right)\left(\binom{n}{3}\right)$, where $K$ is a complete 3-partite 3-graph with at least two vertices in each part. Let $K^-$ denote the complete 3-graph on $t$ vertices and let $K^-$ denote the 3-graph obtained from $K$ by removing one edge. Recently, Falgas-Ravry, Markström, and Zhao [1] asymptotically determined $c_1(n, K^-)$ and gave close to optimal
bounds for $c_1(n, K_4^-)$. In this note, we focus on the problem to determine the exact value of $c_2(n, K_4^-)$ when $t = 4$ and 5. Falgas-Ravry and Zhao [2] determined the exact value of $c_2(n, K_4)$ for $n > 98$ and gave lower and upper bounds of $c_2(n, K_4^-)$ and $c_2(n, K_5^-)$. More specifically, they proved the following theorem.

**Theorem 1.1** (Theorem 1.2 in [2]). Suppose $n = 6m + r$ for some $r \in \{0, 1, 2, 3, 4, 5\}$ and $m \in \mathbb{N}$ with $n \geq 7$. Then

$$c_2(n, K_4^-) = \begin{cases} 2m - 1 \text{ or } 2m & \text{if } r = 0, \\ 2m & \text{if } r \in \{1, 2\}, \\ 2m \text{ or } 2m + 1 & \text{if } r \in \{3, 4\}, \\ 2m + 1 & \text{if } r = 5. \end{cases}$$

**Theorem 1.2** (Theorem 1.4 in [2]). $\lfloor \frac{2n-5}{3} \rfloor \leq c_2(n, K_5^-) \leq \lfloor \frac{2n-2}{3} \rfloor$.

Falgas-Ravry and Zhao [2] also conjectured that the gap between the upper and lower bounds for $c_2(n, K_4^-)$ could be closed and left this as an open problem.

**Problem 1.3** ([2]). Determine the exact value of $c_2(n, K_4^-)$ in the case $n \equiv 0, 3, 4 \pmod{6}$.

In this note, we determine not only the exact value of $c_2(n, K_4^-)$ but also the exact value of $c_2(n, K_5^-)$, thereby resolving Problem 1.3 and sharpening Theorem 1.5.

**Theorem 1.4.** $c_2(n, K_4^-) = \lfloor \frac{n}{3} \rfloor$.

**Theorem 1.5.** $c_2(n, K_5^-) = \lfloor \frac{2n-2}{3} \rfloor$.

The following are some definitions and notation used in our proofs. For a $k$-graph $H$ and $x \in V(H)$, the link graph of $x$, denoted by $H(x)$, is the $(k-1)$-graph with vertex set $V(H) \setminus \{x\}$ and edge set $N_H(x)$. Given a graph $G$ and a positive integer vector $k \in \mathbb{Z}_+^{|V(G)|}$, the $k$-blowup of $G$, denoted by $G^{(k)}$, is the graph obtained by replacing every vertex $v$ of $G$ with an independent $k(v)$-set $X_v$, and placing a complete bipartite graph between $X_u$ and $X_v$ whenever $u$ and $v$ are adjacent in $G$. We call the independent set $X_v$ in $G^{(k)}$ the blowup of $v$ in $G$. When there is no confusion, we write $ab$ and $abc$ as a shorthand for $\{a,b\}$ and $\{a,b,c\}$, respectively. Given a positive integer $n$, write $[n]$ for the set $\{1, 2, \ldots, n\}$.

In the rest of the note, we give proofs of Theorems 1.4 and 1.5.

**2 Proof of Theorems 1.4 and 1.5**

We will construct extremal 3-graphs for $K_4^-$ and $K_5^-$ with minimum codegree matching the upper bounds in Theorems 1.1 and 1.2, respectively.
2.1 Proof of Theorem 1.4

We first give an observation, which can be verified directly from the definitions.

**Observation 1.** Let $H$ be a 3-graph and $x \in V(H)$. $x$ is not covered by a copy of $K_4^-$ if and only if (i) $H(x)$ is triangle-free, and (ii) every edge in $H$ induces at most one edge in $H(x)$.

By Theorem 1.1, to show Theorem 1.4, it is sufficient to construct 3-graphs $H$ on $n$ vertices for $n \equiv 0, 3, 4 \pmod{6}$ and with $\delta_2(H) = \lfloor \frac{n}{3} \rfloor$ such that $H$ has no $K_4^-$-covering.

In the proof we distinguish three cases. Let $C_6$ be the 6-cycle $v_1v_2v_3v_4v_5v_6v_1$ and let $12\ldots k1$ be a $k$-cycle on vertices $1, 2, \ldots, k$ for some positive integer $k$.

**Construction A:** Let $G_1$ be the graph obtained from $C_6$ and the 5-cycle $123451$ by adding the edges $1v_1, 1v_3, 2v_2, 2v_6, 3v_4, 3v_6, 4v_3, 4v_5, 5v_2, 5v_6$.

**Construction B:** Let $G_2$ be the graph obtained from $C_6$ and the 8-cycle $123456781$ by adding the edges $1v_1, 1v_3, 2v_2, 2v_6, 3v_4, 3v_6, 4v_5, 5v_2, 5v_4, 6v_3, 6v_5, 7v_4, 7v_6, 8v_2, 8v_5$.

**Construction C:** Let $G_3$ be the graph obtained from $C_6$ and the 8-cycle $123456781$ by adding a new vertex $9$ and the edges $19, 39, 79, 1v_1, 1v_3, 2v_2, 2v_6, 3v_4, 3v_6, 4v_5, 4v_4, 5v_2, 5v_4, 5v_6, 6v_1, 6v_5, 7v_3, 7v_6, 8v_2, 8v_4, 9v_2, 9v_5$.

It can be checked that $G_1, G_2, G_3$ are triangle-free (see Fig.1); therefore, so are their blowups.

![Figure 1: The graphs $G_1, G_2$ and $G_3$](image-url)
(1) The link graph of $x$, $H_1(x)$, consists of the $k_1$-blowup of $G_1$ by replacing $v_i$ by $V_i$ for $i \in \{6\}$ and adding a perfect matching between $V_1$ and $V_4$.

(2) A triple $abc$ with $x \in \{a, b, c\}$ belongs to $E(H_1)$ if and only if it is $P_2$-free in $H_1(x)$.

Claim 1. $H_1$ contains no $K_4^+$-covering and $\delta_2(H_1) = 2m = \lfloor \frac{2}{3} \rfloor$.

Proof of Claim 1. By the definition of $G_1$, $v_1$ and $v_4$ have no common neighbor. So by (1) of Construction 1, $H_1(x)$ is triangle-free. By (2) of Construction 1, any two incident edges of $H_1(x)$ are not contained in one edge of $H_1$. By Observation 1, $x$ is contained in no copy of $K_4^+$ in $H_1$. So $H_1$ has no $K_4^+$-covering.

By (1) of Construction 1, one can check that $H_1(x)$ is $2m$-regular. So $d_{H_1}(x, a) = 2m$ for all $a \in V \setminus \{x\}$. Now we consider the degree of the pair $\{a, b\}$ with $x \notin \{a, b\}$. If $ab \in E(H_1(x))$, then by (2) of Construction 1, $N_{H_1}(x, a) \cap N_{H_1}(a, b) = \emptyset$, $N_{H_1}(x, b) \cap N_{H_1}(a, b) = \emptyset$, and $N_{H_1}(x, a) \cap N_{H_1}(x, b) = \emptyset$; or equivalently, for any $c \notin N_{H_1}(x, a) \cup N_{H_1}(x, b)$, $\{a, b, c\}$ forms an edge of $H_1$. So $d_{H_1}(a, b) = 6m - 2 \times 2m = 2m$. If $ab \notin E(H_1(x))$ then $x \notin N_{H_1}(a, b)$. By (2) of the construction of $H_1$, $N_{H_1}(x, a) \cap N_{H_1}(x, b) \cap N_{H_1}(a, b) = \emptyset$; or equivalently, for any $c \notin (N_{H_1}(x, a) \cap N_{H_1}(x, b)) \cup \{x, a, b\}$, we have $abc \in E(H_1)$. So $d_{H_1}(a, b) = 6m - 3 - |N_{H_1}(a, x) \cap N_{H_1}(b, x)| \geq 4m - 3 \geq 2m$ if $m > 1$. If $m = 1$, then $H_1(x)$ is the 5-cycle 123451, one can check that $d_{H_1}(a, b) \geq 2 = 2m$.

Case 1 follows directly from Claim 1.

Case 2: $n = 6m + 3$ for some integer $m \geq 1$.

Define a positive integer vector $k_2 \in Z^+(G_2)$ by $k_2(v_i) = m - 1$ for $i \in \{6\}$ and $k_2(i) = 1$ for $i \in \{8\}$.

Construction 2. Let $V_1, \ldots, V_6$ be six disjoint sets of the same size $m - 1$ and let $x$ be a specific vertex. Define the 3-graph $H_2$ on vertex set $\{x\} \cup \{8\} \cup (\cup_{i=1}^6 V_i)$ such that the following holds:

(1) The link graph of $x$, $H_2(x)$, consists of the $k_2$-blowup of $G_2$ by replacing $v_i$ with $V_i$ for $1 \leq i \leq 6$, and adding a perfect matching between $V_1$ and $V_4$ and a matching $\{15, 26, 37, 48\}$.

(2) A triple $abc$ with $x \in \{a, b, c\}$ belongs to $E(H_1)$ if and only if it is $P_2$-free in $H_1(x)$.

Claim 2. $H_2$ contains no $K_4^+$-covering and $\delta_2(H_2) = 2m + 1 = \lfloor \frac{2}{3} \rfloor$.

Proof of Claim 2. By the definition of $G_2$, $N_{G_2}(v_1) \cap N_{G_2}(v_4) = \emptyset$ and $N_{G_2}(1) \cap N_{G_2}(5) = N_{G_2}(2) \cap N_{G_2}(6) = N_{G_2}(3) \cap N_{G_2}(7) = N_{G_2}(4) \cap N_{G_2}(8) = \emptyset$. So by (1) of Construction 2, $H_2(x)$ is triangle-free, too; and by (2) of Construction 2, any two incident edges of $H_2(x)$ are not contained in one edge of $H_2$. By Observation 1, $x$ is contained in no copy of $K_4^+$ in $H_2$. So $H_2$ has no $K_4^+$-covering.

By (1) of Construction 2, $H_2(x)$ is $(2m + 1)$-regular. So $d_{H_2}(x, a) = 2m + 1$ for all $a \in V(H_2) \setminus \{x\}$. Now assume $\{a, b\} \subseteq V(H_2) \setminus \{x\}$. If $ab \in E(H_2(x))$, then by (2) of Construction 2, $N_{H_2}(x, a) \cap N_{H_2}(a, b) = \emptyset$, $N_{H_2}(x, b) \cap N_{H_2}(a, b) = \emptyset$ and $N_{H_2}(x, a) \cap N_{H_2}(x, b) = \emptyset$; or equivalently, for any $c \notin N_{H_2}(x, a) \cup N_{H_2}(x, b)$, $\{a, b, c\}$ forms an edge
of $H_2$. So $d_{H_2}(a,b) = 6m + 3 - 2(2m + 1) = 2m + 1$. If $ab \notin E(H_2(x))$ then $x \notin N_{H_2}(a,b)$. By (2) of the construction of $H_2$, $N_{H_2}(x,a) \cap N_{H_2}(x,b) \cap N_{H_2}(a,b) = \emptyset$; or equivalently, for any $c \notin (N_{H_2}(x,a) \cap N_{H_2}(x,b)) \cup \{x,a,b\}$, $abc \in E(H_2)$. So we have $d_{H_2}(a,b) = 6m + 3 - 3 - |N_{H_2}(a,x) \cap N_{H_2}(b,x)| \geq 4m - 1 \geq 2m + 1$.

Case 2 follows from Claim 2.

Case 3: $n = 6m + 4$ for some integer $m \geq 1$.

Define a positive integer vector $k_3 \in \mathbb{Z}^V(G_3)$ by $k_3(v_i) = m - 1$ for $i \in [6]$ and $k_3(i) = 1$ for $i \in [9]$.

Construction 3. Let $V_1, \ldots, V_6$ be six disjoint sets of the same size $m - 1$ and let $x$ be a specific vertex. Define a 3-graph $H_3$ on vertex set $\{x\} \cup [9] \cup (\cup_{i=1}^6 V_i)$ such that the following holds:

1. The link graph of $x$, $H_3(x)$, consists of the $k_3$-blowup of $G_3$ by replacing $v_i$ by $V_i$ for $1 \leq i \leq 6$, and adding a matching $\{15, 26, 48\}$.

2. A triple $abc$ with $x \in \{a, b, c\}$ belongs to $E(H_3)$ if and only if it is $P_2$-free in $H_1(x)$.

Claim 3. $H_3$ contains no $K_4^-$-covering and $\delta_2(H_3) = 2m + 1 = \lceil \frac{m}{3} \rceil$.

Proof of Claim 3: By (1) of Construction 3, one can check that $H_3(x)$ is triangle-free; and by (2) of Construction 3, any two incident edges of $H_3(x)$ are not contained in one edge of $H_3$. By Observation 1, $x$ is contained in no copy of $K_4^-$ in $H_3$. So $H_3$ has no $K_4^-$-covering.

By the construction of $H_3(x)$, one can check that $H_3(x)$ is almost $(2m + 1)$-regular, i.e. $d_{H_3}(a) = 2m + 1$ for all vertices $a \in V(H_3) \setminus \{x, 1\}$ and $d_{H_3}(x) = 2m + 2$. So $d_{H_3}(a,b) = 2m + 1$ for all $a \in V(H_3) \setminus \{x, 1\}$ and $d_{H_3}(x) = 2m + 2$. Now assume $\{a, b\} \subseteq V(H_3) \setminus \{x\}$. If $ab \in E(H_3(x))$, by (2) of Construction 3, $N_{H_3}(x,a) \cap N_{H_3}(a,b) = \emptyset$, $N_{H_3}(x,b) \cap N_{H_3}(a,b) = \emptyset$, and for any $c \in V(H_3 \setminus (N_{H_3}(x,a) \cup N_{H_3}(x,b)))$, $\{a, b, c\}$ forms an edge of $H_3$. Since $H_3(x)$ is triangle-free, $N_{H_3}(x,a) \cap N_{H_3}(x,b) = \emptyset$. If $1 \notin \{a, b\}$ then $d_{H_3}(a,b) = |V(H_3)| - |N_{H_3}(x,a)| - |N_{H_3}(x,b)| = 6m + 4 - 2(2m + 1) = 2m + 2$. Now assume $1 \in \{a, b\}$, say $a = 1$. Then $d_{H_3}(1,b) = |V(H_3)| - |N_{H_3}(x,1)| - |N_{H_3}(x,b)| = 6m + 4 - (2m + 2) - (2m + 1) = 2m + 1$. If $ab \notin E(H_3(x))$ then $x \notin N_{H_3}(a,b)$. By (2) of the construction of $H_3$, $N_{H_3}(x,a) \cap N_{H_3}(x,b) \cap N_{H_3}(a,b) = \emptyset$; or equivalently, for any $c \notin (N_{H_3}(x,a) \cap N_{H_3}(x,b)) \cup \{x, a, b\}$, $abc \in E(H_3)$. So we have $d_{H_3}(a,b) = 6m + 4 - 3 - |N_{H_3}(a,x) \cap N_{H_3}(b,x)| \geq 4m \geq 2m + 1$.

Case 3 follows from Claim 3.

Theorem 1.4 follows from Cases 1, 2, 3 and Theorem 1.1.

2.2 Proof of Theorem 1.5

The following theorem is well known in graph theory.

Theorem 2.1 (König [6]). Let $G$ be a bipartite graph with maximum degree $\Delta$. Then $E(G)$ can be partitioned into $M_1, M_2, \ldots, M_\Delta$ so that each $M_i$ ($1 \leq i \leq \Delta$) is a matching in $G$. In particular, if $G$ is $\Delta$-regular then $E(G)$ can be partitioned into $\Delta$ perfect matchings.
Construction 4. Given positive integers $m, \ell$ with $m \leq \ell$ and two disjoint sets $V_1, V_2$ with $|V_1| \leq |V_2| = m$, by Theorem 2.1, the edge set of the complete bipartite graph $K(V_1, V_2)$ has a partition $M_1, M_2, \ldots, M_m$ such that each $M_i$ ($1 \leq i \leq m$) is a matching. Let $T$ be the 3-partite 3-graph with vertex classes $V_1 \cup V_2 \cup [\ell]$ and edge set

$$E(H) = \bigcup_{i=1}^{m}\{e \cup \{i\} : e \in M_i\}.$$  

Proof of Theorem 1.5. We first give the extremal 3-graph for $K_5^−$. 

Construction 5. Given a positive integer $m$ and three disjoint sets $V_1, V_2, V_3$ such that $m - 1 \leq |V_1| \leq |V_2| = m \leq |V_3| \leq m + 1$ and $|V_3| - |V_1| \leq 1$. Let $V_3 = [\ell]$. Let $T$ be the 3-partite 3-graph on vertex set $V_1 \cup V_2 \cup V_3$ constructed by Construction 4. Let $x$ be a specific vertex not belonging to $V_1 \cup V_2 \cup V_3$. Define the 3-graph $H_4$ on vertex set $V_1 \cup V_2 \cup V_3 \cup \{x\}$ such that the following holds.

1. The link graph of $x$, $H_4(x)$, consists of the union of the three complete bipartite graphs $K(V_1, V_2)$, $K(V_1, V_3)$ and $K(V_2, V_3)$.
2. Each triple $e \notin E(K(V_1, V_2, V_3))$ with $x \notin e$ is an edge of $H_4$.
3. $E(H_4) \cap E(K(V_1, V_2, V_3)) = E(T)$.

Remark. By the definition of Construction 5, we have $3m - 1 \leq \sum_{i=1}^{3}|V_i| \leq 3m + 1$ and $\ell = m$ or $m + 1$.

Let $n = |V(H_4)|$. Then $3m \leq n \leq 3m + 2$.

Claim 4. $H_4$ has no $K_5^−$-covering and $\delta_2(H) \geq \left\lceil \frac{3n-2}{3} \right\rceil$.

Proof of Claim 4: We show that $x$ is contained in no copy of $K_5^−$ in $H_4$. Choose a 4-set $\{a, b, c, d\} \subseteq V_1 \cup V_2 \cup V_3$. If it contains at least three vertices in the same part $V_i$ or at least two vertices in at least two different parts $V_i, V_j$, then by (1) of Construction 5, $\{a, b, c, d\}$ spans at least two non-edges in $H_4(x)$. Otherwise, two vertices in $\{a, b, c, d\}$, say $a, b$, lie in the same part (whence they span a non-edge in $H_4(x)$) while the other two vertices $c$ and $d$ lie one each in the two other parts. Since $\Delta_2(T) \leq 1$ (by Construction 4) and (3) of Construction 5, at most one of $a$ and $b$ makes an edge of $T$ (and hence $H_4$) with $cd$. Thus in either case, $\{x, a, b, c, d\}$ spans at least two non-edges of $H_4$ and hence $x$ is not covered by a copy of $K_5^−$.

Now we compute the minimum codegree of $H_4$. Choose two distinct vertices $a, b \in V(H_4)$. If $x \in \{a, b\}$, assume $x = a$ and $b \in V_i$, then by (1) of Construction 5,

$$d(x, b) = n - 1 - |V_i| \geq n - 1 - \left\lfloor \frac{n-1}{3} \right\rfloor = \left\lfloor \frac{2n-2}{3} \right\rfloor.$$  

If $a, b \in V_i$ for some $1 \leq i \leq 3$ then, by (2) of Construction 5, $d(a, b) = \sum_{i=1}^{3}|V_i| - 2 = n - 3 \geq \left\lceil \frac{2n-2}{3} \right\rceil$. If $a \in V_i, b \in V_j$ ($i \neq j$), then

$$d(a, b) = |V_i| + |V_j| - 2 + a_T(a, b) \geq \left\lceil \frac{2n-2}{3} \right\rceil,$$  

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where the inequality holds since $d_T(a, b) = 1$ for $\{i, j\} = \{1, 2\}$, $\{i, j\} = \{1, 3\}$ with $|V_3| = m$, or $\{i, j\} \subseteq \{1, 2, 3\}$ with $|V_1| = |V_2| = |V_3| = m$.

This completes the proof of Claim 4.

By Claim 4, we have
\[
c_2(n, K_5) \geq \delta_2(H_4) = \left\lfloor \frac{2n - 2}{3} \right\rfloor.
\]

By Theorem 1.2, we have Theorem 1.5. □

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