Exact minimum codegree thresholds for K_4^--covering and K_5^--covering

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Abstract

Given two 3-graphs F and H, an F-covering of H is a collection of copies of F in H such that each vertex of H is contained in at least one copy of them. Let $c_2(n, F)$ be the minimum integer t such that every 3-graph with minimum codegree greater than t has an F-covering. In this note, we answer an open problem of Falgas-Ravry and Zhao (SIAM J. Discrete Math., 2016) by determining the exact value of $c_2(n, K_4^-)$ and $c_2(n, K_5^-)$, where K_t^- is the complete 3-graph on t vertices with one edge removed.

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1 Introduction

Given a set V and a positive integer k, let $\binom{V}{k}$ be the collection of k-element subsets of V. A *simple k-uniform hypergraph* (or k-graph for short) $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$. We write graph for 2-graph for short. For a set $S \subseteq V(H)$, the *neighbourhood* $N_H(S)$ of S is $\{T \subseteq V(H)\backslash S : T \cup S \in E(H)\}$ and the *degree* of S is $d_H(S) = |N_H(S)|$. The *minimum (resp. maximum) s-degree* of H, denoted by $\delta_s(H)$ (resp. $\Delta_s(H)$), is the minimum (resp. maximum) $d_H(S)$ taken over all s-element sets

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of $V(H)$. $\delta_{k-1}(H)$ and $\delta_1(H)$ are usually called the *minimum codegree* and the *minimum degree* of H, respectively. An r-graph H is called an *r-partite r-graph* if the vertex set of H can be partitioned into r parts such that each edge of H intersects each part exactly one vertex. Given disjoint sets $V_1, V_2, \cdots, V_r$, let $K(V_1, V_2, \ldots, V_r)$ be the complete r-partite r-graph with vertex classes $V_1, V_2, \ldots, V_r$.

Given a k-graph F, we say a k-graph H has an *F-covering* if each vertex of H is contained in some copy of F. For $0 \leqslant i < k$, define

$$c_i(n, F) = \max\{\delta_i(H) : H \text{ is a } k\text{-graph on } n \text{ vertices with no } F\text{-covering}\}.$$

We call $c_{k-1}(n, F)$ the *minimum codegree threshold* for F-covering.

There are two well studied extremal problems related to the covering problem. Given a k-graph F, a k-graph H is F-free if H does not contain a copy of F as a subgraph. For $0 \leqslant i < k$, define

$$\mathrm{ex}_i(n, F) = \max\{\delta_i(H) : |V(H)| = n \text{ and } H \text{ is } F\text{-free}\}.$$

The quantity $\mathrm{ex}_0(n, F)$ is known as the Turán number of F, and $\mathrm{ex}_{k-1}(n, F)$ was studied by Mubayi and Zhao [7]. For an overview of the Turán problem for hypergraphs, one can see a survey given by Keevash [5]. Given two k-graphs H and F with $|V(H)|$ is divisible by $|V(F)|$, a perfect F-tiling (or an F-factor) in H is a spanning collection of vertex-disjoint copies of F. For $0 \leqslant i < k$ and n divisible by $|V(F)|$, define

$$t_i(n, F) = \max\{\delta_i(H) : |V(H)| = n \text{ and } H \text{ does not have an } F\text{-factor}\}.$$

The study of the tiling problem also has a long history. For detailed discussion of the area, one can refer to the surveys due to Rödl and Ruciński [8] and Zhao [10]. Trivially, for $0 \leqslant i < k$, we have

$$\mathrm{ex}_i(n, F) \leqslant c_i(n, F) \leqslant t_i(n, F).$$

So the covering problem is an intermediate but distinct problem from the well-studied Turán and tiling problems. This is also partial motivation for the study of covering problems.

For graphs F, the F-covering problem was solved asymptotically in [9] by showing that $c_1(n, F) = (\frac{\chi(F)-2}{\chi(F)-1} + o(1))n$, where $\chi(F)$ is the chromatic number of F. For general k-graphs, the function $c_i(n, F)$ was determined for some special families of k-graphs F. For example, Han, Lo, and Sanhueza-Matamala [3] proved that $c_{k-1}(n, C_s^{(k,k-1)}) \leqslant (\frac{1}{2} + o(1))n$ for $k \geqslant 3, s \geqslant 2k^2$ and the result is asymptotically tight if k and s satisfy some special constrains, where $C_s^{(k,\ell)}$ $(1 \leqslant \ell < k)$ is the k-graph on s vertices such that its vertices can be ordered cyclicly so that every edge consists of k consecutive vertices under this order and two consecutive edges intersect in exactly ℓ vertices. Han, Zang, and Zhao showed in [4] that $c_1(n, K) = (6 - 4\sqrt{2} + o(1))\binom{n}{2}$, where K is a complete 3-partite 3-graph with at least two vertices in each part. Let K_t denote the complete 3-graph on t vertices and let K_t^- denote the 3-graph obtained from K_t by removing one edge. Recently, Falgas-Ravry, Markström, and Zhao [1] asymptotically determined $c_1(n, K_4^-)$ and gave close to optimal bounds for $c_1(n, K_4^-)$. In this note, we focus on the problem to determine the exact value of $c_2(n, K_t^-)$ when $t = 4$ and 5. Falgas-Ravry and Zhao [2] determined the exact value of $c_2(n, K_4)$ for $n > 98$ and gave lower and upper bounds of $c_2(n, K_4^-)$ and $c_2(n, K_5^-)$. More specifically, they proved the following theorem.

Theorem 1.1 (Theorem 1.2 in [2])**.** *Suppose $n = 6m + r$ for some $r \in \{0, 1, 2, 3, 4, 5\}$ and $m \in \mathbb{N}$ with $n \geqslant 7$. Then*

$$c_2(n, K_4^-) = \begin{cases} 2m - 1 \text{ or } 2m & \text{if } r = 0, \\ 2m & \text{if } r \in \{1, 2\}, \\ 2m \text{ or } 2m + 1 & \text{if } r \in \{3, 4\}, \\ 2m + 1 & \text{if } r = 5. \end{cases}$$

Theorem 1.2 (Theorem 1.4 in [2])**.** $\lfloor \frac{2n-5}{3} \rfloor \leqslant c_2(n, K_5^-) \leqslant \lfloor \frac{2n-2}{3} \rfloor$.

Falgas-Ravry and Zhao [2] also conjectured that the gap between the upper and lower bounds for $c_2(n, K_4^-)$ could be closed and left this as an open problem.

Problem 1.3 ([2])**.** *Determine the exact value of $c_2(n, K_4^-)$ in the case $n \equiv 0, 3, 4 \pmod 6$.*

In this note, we determine not only the exact value of $c_2(n, K_4^-)$ but also the exact value of $c_2(n, K_5^-)$, thereby resolving Problem 1.3 and sharpening Theorem 1.5.

Theorem 1.4. $c_2(n, K_4^-) = \lfloor \frac{n}{3} \rfloor$.

Theorem 1.5. $c_2(n, K_5^-) = \lfloor \frac{2n-2}{3} \rfloor$.

The following are some definitions and notation used in our proofs. For a k-graph H and $x \in V(H)$, the *link graph* of x, denoted by $H(x)$, is the $(k-1)$-graph with vertex set $V(H) \setminus \{x\}$ and edge set $N_H(x)$. Given a graph G and a positive integer vector $\mathbf{k} \in Z_+^{V(G)}$, the $\mathbf{k}$-*blowup* of G, denoted by $G^{(\mathbf{k})}$, is the graph obtained by replacing every vertex v of G with an independent $\mathbf{k}(v)$-set X_v, and placing a complete bipartite graph between X_u and X_v whenever u and v are adjacent in G. We call the independent set X_v in $G^{(\mathbf{k})}$ the blowup of v in G. When there is no confusion, we write ab and abc as a shorthand for $\{a, b\}$ and $\{a, b, c\}$, respectively. Given a positive integer n, write $[n]$ for the set $\{1, 2, \ldots, n\}$.

In the rest of the note, we give proofs of Theorems 1.4 and 1.5.

2 Proof of Theorems 1.4 and 1.5

We will construct extremal 3-graphs for K_4^- and K_5^- with minimum codegree matching the upper bounds in Theorems 1.1 and 1.2, respectively.

2.1 Proof of Theorem 1.4

We first give an observation, which can be verified directly from the definitions.

Observation 1. *Let H be a 3-graph and $x \in V(H)$. x is not covered by a copy of K_4^- if and only if (i) $H(x)$ is triangle-free, and (ii) every edge in H induces at most one edge in $H(x)$.*

By Theorem 1.1, to show Theorem 1.4, it is sufficient to construct 3-graphs H on n vertices for $n \equiv 0, 3, 4 \pmod 6$ and with $\delta_2(H) = \lfloor \frac{n}{3} \rfloor$ such that H has no K_4^--covering. In the proof we distinguish three cases. Let C_6 be the 6-cycle $v_1v_2v_3v_4v_5v_6v_1$ and let $12\ldots k1$ be a k-cycle on vertices $1, 2, \ldots, k$ for some positive integer k.

Construction A: Let G_1 be the graph obtained from C_6 and the 5-cycle 123451 by adding the edges $1v_1, 1v_3, 2v_2, 2v_5, 3v_4, 3v_6, 4v_3, 4v_5, 5v_2, 5v_6$.

Construction B: Let G_2 be the graph obtained from C_6 and the 8-cycle 123456781 by adding the edges $1v_1, 1v_3, 2v_2, 2v_6, 3v_1, 3v_5, 4v_3, 4v_6, 5v_2, 5v_4, 6v_3, 6v_5, 7v_4, 7v_6, 8v_2, 8v_5$.

Construction C: Let G_3 be the graph obtained from C_6 and the 8-cycle 123456781 by adding a new vertex 9 and the edges $19, 39, 79$, $1v_1, 1v_3, 2v_2, 2v_6, 3v_1, 3v_4, 4v_3, 4v_5, 5v_4, 5v_6, 6v_1, 6v_5, 7v_3, 7v_6, 8v_2, 8v_4, 9v_2, 9v_5$.

It can be checked that G_1, G_2, G_3 are triangle-free (see Fig.1); therefore, so are their blowups.

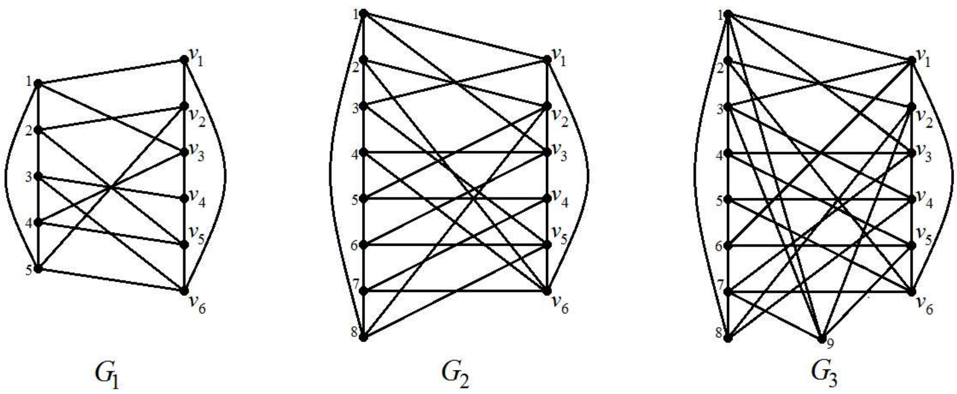

Figure 1: The graphs G_1, G_2 and G_3

Case 1. $n = 6m$ for some integer $m \geqslant 1$.

Define a positive integer vector $\mathbf{k}_1 \in Z_+^{V(G_1)}$ by $\mathbf{k}_1(v_i) = m-1$ for $i \in [6]$ and $\mathbf{k}_1(i) = 1$ for $i \in [5]$.

Construction 1. *Let $V_1, \ldots, V_6$ be six disjoint sets of the same size $m-1$ and let x be a specific vertex. Define the 3-graph H_1 on vertex set $\{x\} \cup [5] \cup (\cup_{i=1}^{6} V_i)$ such that the following holds:*

(1) The link graph of x, $H_1(x)$, consists of the $\mathbf{k}_1$-blowup of G_1 by replacing v_i by V_i for $i \in [6]$ and adding a perfect matching between V_1 and V_4.

(2) A triple abc with $x \in \{a, b, c\}$ belongs to $E(H_1)$ if and only if it is P_2-free in $H_1(x)$.

Claim 1. *H_1 contains no K_4^--covering and $\delta_2(H_1) = 2m = \lfloor \frac{n}{3} \rfloor$.*

Proof of Claim 1. By the definition of G_1, v_1 and v_4 have no common neighbor. So by (1) of Construction 1, $H_1(x)$ is triangle-free. By (2) of Construction 1, any two incident edges of $H_1(x)$ are not contained in one edge of H_1. By Observation 1, x is contained in no copy of K_4^- in H_1. So H_1 has no K_4^--covering.

By (1) of Construction 1, one can check that $H_1(x)$ is $2m$-regular. So $d_{H_1}(x, a) = 2m$ for all $a \in V \setminus \{x\}$. Now we consider the degree of the pair $\{a, b\}$ with $x \notin \{a, b\}$. If $ab \in E(H_1(x))$, then by (2) of Construction 1, $N_{H_1}(x, a) \cap N_{H_1}(a, b) = \emptyset$, $N_{H_1}(x, b) \cap N_{H_1}(a, b) = \emptyset$ and $N_{H_1}(x, a) \cap N_{H_1}(x, b) = \emptyset$; or equivalently, for any $c \notin N_{H_1}(x, a) \cup N_{H_1}(x, b)$, $\{a, b, c\}$ forms an edge of H_1. So $d_{H_1}(a, b) = 6m - 2 \times 2m = 2m$. If $ab \notin E(H_1(x))$ then $x \notin N_{H_1}(a, b)$. By (2) of the construction of H_1, $N_{H_1}(x, a) \cap N_{H_1}(x, b) \cap N_{H_1}(a, b) = \emptyset$; or equivalently, for any $c \notin (N_{H_1}(x, a) \cap N_{H_1}(x, b)) \cup \{x, a, b\}$, we have $abc \in E(H_1)$. So $d_{H_1}(a, b) = 6m - 3 - |N_{H_1}(a, x) \cap N_{H_1}(b, x)| \geqslant 4m - 3 \geqslant 2m$ if $m > 1$. If $m = 1$, then $H_1(x)$ is the 5-cycle 123451, one can check that $d_{H_1}(a, b) \geqslant 2 = 2m$.

Case 1 follows directly from Claim 1.

Case 2: $n = 6m + 3$ for some integer $m \geqslant 1$.

Define a positive integer vector $\mathbf{k}_2 \in Z_+^{V(G_2)}$ by $\mathbf{k}_2(v_i) = m - 1$ for $i \in [6]$ and $\mathbf{k}_2(i) = 1$ for $i \in [8]$.

Construction 2. *Let $V_1, \ldots, V_6$ be six disjoint sets of the same size $m - 1$ and let x be a specific vertex. Define the 3-graph H_2 on vertex set $\{x\} \cup [8] \cup (\cup_{i=1}^6 V_i)$ such that the following holds:*

(1) The link graph of x, $H_2(x)$, consists of the $\mathbf{k}_2$-blowup of G_2 by replacing v_i with V_i for $1 \leqslant i \leqslant 6$, and adding a perfect matching between V_1 and V_4 and a matching $\{15, 26, 37, 48\}$.

(2) A triple abc with $x \in \{a, b, c\}$ belongs to $E(H_1)$ if and only if it is P_2-free in $H_1(x)$.

Claim 2. *H_2 contains no K_4^--covering and $\delta_2(H_2) = 2m + 1 = \lfloor \frac{n}{3} \rfloor$.*

Proof of Claim 2. By the definition of G_2, $N_{G_2}(v_1) \cap N_{G_2}(v_4) = \emptyset$ and $N_{G_2}(1) \cap N_{G_2}(5) = N_{G_2}(2) \cap N_{G_2}(6) = N_{G_2}(3) \cap N_{G_2}(7) = N_{G_2}(4) \cap N_{G_2}(8) = \emptyset$. So by (1) of Construction 2, $H_2(x)$ is triangle-free, too; and by (2) of Construction 2, any two incident edges of $H_2(x)$ are not contained in one edge of H_2. By Observation 1, x is contained in no copy of K_4^- in H_2. So H_2 has no K_4^--covering.

By (1) of Construction 2, $H_2(x)$ is $(2m + 1)$-regular. So $d_{H_2}(x, a) = 2m + 1$ for all $a \in V(H_2) \setminus \{x\}$. Now assume $\{a, b\} \subseteq V(H_2) \setminus \{x\}$. If $ab \in E(H_2(x))$, then by (2) of Construction 2, $N_{H_2}(x, a) \cap N_{H_2}(a, b) = \emptyset$, $N_{H_2}(x, b) \cap N_{H_2}(a, b) = \emptyset$ and $N_{H_2}(x, a) \cap N_{H_2}(x, b) = \emptyset$; or equivalently, for any $c \notin N_{H_2}(x, a) \cup N_{H_2}(x, b)$, $\{a, b, c\}$ forms an edge of H_2. So $d_{H_2}(a,b) = 6m+3-2(2m+1) = 2m+1$. If $ab \notin E(H_2(x))$ then $x \notin N_{H_2}(a,b)$. By (2) of the construction of H_2, $N_{H_2}(x,a) \cap N_{H_2}(x,b) \cap N_{H_2}(a,b) = \emptyset$; or equivalently, for any $c \notin (N_{H_2}(x,a) \cap N_{H_2}(x,b)) \cup \{x,a,b\}$, $abc \in E(H_2)$. So we have $d_{H_2}(a,b) = 6m+3-3-|N_{H_2}(a,x) \cap N_{H_2}(b,x)| \geqslant 4m-1 \geqslant 2m+1$.

Case 2 follows from Claim 2.

Case 3: $n = 6m+4$ for some integer $m \geqslant 1$.

Define a positive integer vector $\mathbf{k}_3 \in Z_+^{V(G_3)}$ by $\mathbf{k}_3(v_i) = m-1$ for $i \in [6]$ and $\mathbf{k}_3(i) = 1$ for $i \in [9]$.

Construction 3. *Let $V_1, \ldots, V_6$ be six disjoint sets of the same size $m-1$ and let x be a specific vertex. Define a 3-graph H_3 on vertex set $\{x\} \cup [9] \cup (\cup_{i=1}^6 V_i)$ such that the following holds:*

(1) The link graph of x, $H_3(x)$, consists of the $\mathbf{k}_3$-blowup of G_3 by replacing v_i by V_i for $1 \leqslant i \leqslant 6$, and adding a matching $\{15, 26, 48\}$.

(2) A triple abc with $x \in \{a,b,c\}$ belongs to $E(H_1)$ if and only if it is P_2-free in $H_1(x)$.

Claim 3. *H_3 contains no K_4^--covering and $\delta_2(H_3) = 2m+1 = \lfloor \frac{n}{3} \rfloor$.*

Proof of Claim 3: By (1) of Construction 3, one can check that $H_3(x)$ is triangle-free; and by (2) of Construction 3, any two incident edges of $H_3(x)$ are not contained in one edge of H_3. By Observation 1, x is contained in no copy of K_4^- in H_3. So H_3 has no K_4^--covering.

By the construction of $H_3(x)$, one can check that $H_3(x)$ is almost $(2m+1)$-regular, i.e. $d_{H_3(x)}(a) = 2m+1$ for all vertices $a \in V(H_3) \setminus \{x,1\}$ and $d_{H_3(x)}(1) = 2m+2$. So $d_{H_3}(x,a) = 2m+1$ for all $a \in V(H_3) \setminus \{x,1\}$ and $d_{H_3}(x,1) = 2m+2$. Now assume $\{a,b\} \subseteq V(H_3) \setminus \{x\}$. If $ab \in E(H_3(x))$, by (2) of Construction 3, $N_{H_3}(x,a) \cap N_{H_3}(a,b) = \emptyset$, $N_{H_3}(x,b) \cap N_{H_3}(a,b) = \emptyset$, and for any $c \in V(H_3) \setminus (N_{H_3}(x,a) \cup N_{H_3}(x,b))$, $\{a,b,c\}$ forms an edge of H_3. Since $H_3(x)$ is triangle-free, $N_{H_3}(x,a) \cap N_{H_3}(x,b) = \emptyset$. If $1 \notin \{a,b\}$ then $d_{H_3}(a,b) = |V(H_3)| - |N_{H_3}(x,a)| - |N_{H_3}(x,b)| = 6m+4-2(2m+1) = 2m+2$. Now assume $1 \in \{a,b\}$, say $a = 1$. Then $d_{H_3}(1,b) = |V(H_3)| - |N_{H_3}(x,1)| - |N_{H_3}(x,b)| = 6m+4-(2m+2)-(2m+1) = 2m+1$. If $ab \notin E(H_3(x))$ then $x \notin N_{H_3}(a,b)$. By (2) of the construction of H_3, $N_{H_3}(x,a) \cap N_{H_3}(x,b) \cap N_{H_3}(a,b) = \emptyset$; or equivalently, for any $c \notin (N_{H_3}(x,a) \cap N_{H_3}(x,b)) \cup \{x,a,b\}$, $abc \in E(H_3)$. So we have $d_{H_3}(a,b) = 6m+4-3-|N_{H_3}(a,x) \cap N_{H_3}(b,x)| \geqslant 4m \geqslant 2m+1$.

Case 3 follows from Claim 3.

Theorem 1.4 follows from Cases 1,2,3 and Theorem 1.1.

2.2 Proof of Theorem 1.5

The following theorem is well known in graph theory.

Theorem 2.1 (König [6])**.** *Let G be a bipartite graph with maximum degree Δ. Then $E(G)$ can be partitioned into $M_1, M_2, \ldots, M_\Delta$ so that each M_i $(1 \leqslant i \leqslant \Delta)$ is a matching in G. In particular, if G is Δ-regular then $E(G)$ can be partitioned into Δ perfect matchings.*

Construction 4. *Given positive integers m, ℓ with $m \leqslant \ell$ and two disjoint sets V_1, V_2 with $|V_1| \leqslant |V_2| = m$, by Theorem 2.1, the edge set of the complete bipartite graph $K(V_1, V_2)$ has a partition $M_1, M_2, \ldots, M_m$ such that each M_i $(1 \leqslant i \leqslant m)$ is a matching. Let T be the 3-partite 3-graph with vertex classes $V_1 \cup V_2 \cup [\ell]$ and edge set*

$$E(H) = \bigcup_{i=1}^{m} \{e \cup \{i\} : e \in M_i\}.$$

Proof of Theorem 1.5. We first give the extremal 3-graph for K_5^-.

Construction 5. *Given a positive integer m and three disjoint sets V_1, V_2, V_3 such that $m - 1 \leqslant |V_1| \leqslant |V_2| = m \leqslant |V_3| \leqslant m + 1$ and $|V_3| - |V_1| \leqslant 1$. Let $V_3 = [\ell]$. Let T be the 3-partite 3-graph on vertex set $V_1 \cup V_2 \cup V_3$ constructed by Construction 4. Let x be a specific vertex not belonging to $V_1 \cup V_2 \cup V_3$. Define the 3-graph H_4 on vertex set $V_1 \cup V_2 \cup V_3 \cup \{x\}$ such that the following holds.*

(1) The link graph of x, $H_4(x)$, consists of the union of the three complete bipartite graphs $K(V_1, V_2)$, $K(V_1, V_3)$ and $K(V_2, V_3)$.

(2) Each triple $e \notin E(K(V_1, V_2, V_3))$ with $x \notin e$ is an edge of H_4.

(3) $E(H_4) \cap E(K(V_1, V_2, V_3)) = E(T)$.

Remark. *By the definition of Construction 5, we have $3m - 1 \leqslant \sum_{i=1}^{3} |V_i| \leqslant 3m + 1$ and $\ell = m$ or $m + 1$.*

Let $n = |V(H_4)|$. Then $3m \leqslant n \leqslant 3m + 2$.

Claim 4. *H_4 has no K_5^--covering and $\delta_2(H) \geqslant \lfloor \frac{3n-2}{3} \rfloor$.*

Proof of Claim 4: We show that x is contained in no copy of K_5^- in H_4. Choose a 4-set $\{a, b, c, d\} \subseteq V_1 \cup V_2 \cup V_3$. If it contains at least three vertices in the same part V_i or at least two vertices in at least two different parts V_i, V_j, then by (1) of Construction 5, $\{a, b, c, d\}$ spans at least two non-edges in $H_4(x)$. Otherwise, two vertices in $\{a, b, c, d\}$, say a, b, lie in the same part (whence they span a non-edge in $H_4(x)$) while the other two vertices c and d lie one each in the two other parts. Since $\Delta_2(T) \leqslant 1$ (by Construction 4) and (3) of Construction 5, at most one of a and b makes an edge of T (and hence H_4) with cd. Thus in either case, $\{x, a, b, c, d\}$ spans at least two non-edges of H_4 and hence x is not covered by a copy of K_5^-.

Now we compute the minimum codegree of H_4. Choose two distinct vertices $a, b \in V(H_4)$. If $x \in \{a, b\}$, assume $x = a$ and $b \in V_i$, then by (1) of Construction 5,

$$d(x, b) = n - 1 - |V_i| \geqslant n - 1 - \left\lceil \frac{n-1}{3} \right\rceil = \left\lfloor \frac{2n-2}{3} \right\rfloor.$$

If $a, b \in V_i$ for some $1 \leqslant i \leqslant 3$ then, by (2) of Construction 5, $d(a, b) = \sum_{i=1}^{3} |V_i| - 2 = n - 3 \geqslant \lfloor \frac{2n-2}{3} \rfloor$. If $a \in V_i, b \in V_j$ $(i \neq j)$, then

$$d(a, b) = |V_i| + |V_j| - 2 + 1 + d_T(a, b) \geqslant \left\lfloor \frac{2n-2}{3} \right\rfloor,$$

where the inequality holds since $d_T(a,b) = 1$ for $\{i,j\} = \{1,2\}$, $\{i,j\} = \{1,3\}$ with $|V_3| = m$, or $\{i,j\} \subseteq \{1,2,3\}$ with $|V_1| = |V_2| = |V_3| = m$.

This completes the proof of Claim 4.

By Claim 4, we have

$$c_2(n, K_5^-) \geqslant \delta_2(H_4) = \left\lfloor \frac{2n-2}{3} \right\rfloor.$$

By Theorem 1.2, we have Theorem 1.5. □

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