A combinatorial model for the decomposition of multivariate polynomial rings as S_n-modules

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Abstract

We consider the symmetric group S_n-module of the polynomial ring with m sets of n commuting variables and m' sets of n anti-commuting variables and show that the multiplicity of an irreducible indexed by the partition λ (a partition of n) is the number of multiset tableaux of shape λ satisfying certain column and row strict conditions. We also present a finite generating set for the ring of S_n invariant polynomials of this ring.

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Contents

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1 Introduction

Let m and n be positive integers. The multivariate polynomial ring of m sets of n commuting variables is a $GL_n \times GL_m$-module that is familiar in the combinatorial representation theory literature. Denote this module by

$$\mathbb{C}[X_{n\times m}] := \mathbb{C}[x_{ij} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m]$$

then it is well known (e.g. [GW] Theorem 5.6.7) that the space decomposes as

$$\mathbb{C}[X_{n\times m}] \simeq \bigoplus_{\lambda} W_n^{\lambda} \otimes W_m^{\lambda}$$

where the direct sum is over all partitions with length less than or equal to $\min(m, n)$ and W_n^{λ} is a polynomial irreducible GL_n-module indexed by the partition λ. More precisely, as a GL_n-module, the multiplicity of the irreducible module W_n^{λ} is equal to the dimension of W_m^{λ}. This dimension is equal to the number of column strict tableaux of shape λ and content in the entries $\{1, 2, \ldots, m\}$. The actions of GL_n and GL_m commute with each other and this decomposition is a consequence of the double centralizer theorem.

For a sequence $(a_1, a_2, \ldots, a_m)$ of non-negative integers the span of the monomials such that, for each i between 1 and m, the degree in the variables $x_{1i}, x_{2i}, \ldots, x_{ni}$ is equal to a_i is a GL_n submodule of $\mathbb{C}[X_{n\times m}]$. This homogeneous submodule has character $h_{a_1}[X_n]h_{a_2}[X_n] \cdots h_{a_m}[X_n]$ where the $h_r[X_n]$ are the complete homogenous symmetric functions.

The symmetric group S_n, realized as permutation matrices, is a subgroup of GL_n and so this subspace is also a S_n-module. The multiplicity of the irreducible S_n-module indexed by the partition λ in this module can be expressed in terms of plethysm [Lit, ST] of symmetric functions,

$$\langle h_{a_1} h_{a_2} \cdots h_{a_m}, s_{\lambda}[1 + h_1 + h_2 + \cdots]\rangle \ . \tag{1}$$

While there are no general techniques for computing plethysm multiplicities, it is possible to give a combinatorial interpretation for this particular expression (e.g. [LR, Theorem 10], [LW]).

We extend the module under consideration by looking at polynomial rings in m sets of commuting variables and m' sets of anticommuting variables. That is, let

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}] := \mathbb{C}[x_{ij}, \theta_{ij'} : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant j' \leqslant m']$$

where the variables x_{ij} commute and commute with the $\theta_{ij'}$ variables and $\theta_{ij}\theta_{ab} = -\theta_{ab}\theta_{ij}$ if either $i \neq a$ or $j \neq b$, and $\theta_{ij}^2 = 0$. There is a $GL_n \times GL_m \times GL_{m'}$ action on this space; however, in this paper we concentrate on the restriction of the GL_n action to the subgroup of permutation matrices. This copy of the symmetric group acts on the first indices of the variables. We are particularly interested in the decomposition of the subspaces of fixed homogeneous degree. In this case, it is also possible to give an interpretation for the multiplicity of an irreducible module in terms of plethysm. However, in this case, we are not aware of general techniques for finding a combinatorial interpretation for this multiplicity. The main goal of this paper is to give a combinatorial interpretation for this multiplicity in terms of tableaux.

In Section 2, we give definitions and introduce the notation used in this paper. Then, in Section 3 and Section 4 we present the two main results:

- A combinatorial interpretation for the multiplicity of an irreducible symmetric group representation in a homogeneous component of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ in terms of certain multiset tableaux (see Theorem 3).
- A finite set of algebraic generators for the S_n invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ (see Theorem 17).

An interesting consequence of the combinatorial interpretation is that it shows that the symmetric group submodule of fixed homogeneous degree is representation stable in the sense defined in [CF, CEF].

An important application of the main theorem of this paper and the double centralizer theorem [CR, GW] is to give an interpretation to the dimensions of the irreducible representations of the centralizer of $\mathbb{C}S_n$ when it acts on multivariate polynomial rings (see [NPS, OZ3]).

2 Notation and Preliminaries

Let $X_n^{(i)}$ represent a collection of commuting variables $x_{1i}, x_{2i}, \ldots, x_{ni}$ on which the symmetric group S_n acts by permutation of the first index. That is, $\sigma(x_{ri}) = x_{\sigma(r)i}$ for all $\sigma \in S_n$. The notation $\Theta_n^{(i)}$ will be used to represent a collection of anti-commuting variables (*Grassmannian variables*) $\theta_{1i}, \theta_{2i}, \ldots, \theta_{ni}$ (again, on which the symmetric group acts on the first index). Now denote the polynomial ring in m sets of the commuting variables and m' sets of anti-commuting variables by

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}] := \mathbb{C}[X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(m)}, \Theta_n^{(1)}, \Theta_n^{(2)}, \ldots, \Theta_n^{(m')}]$$

where the product satisfies the relations

$$\theta_{ri}\theta_{sj} = -\theta_{sj}\theta_{ri} \text{ if } r \neq s \text{ or } i \neq j \qquad \text{and} \qquad \theta_{ri}^2 = 0$$

$$x_{rk}x_{sd} = x_{sd}x_{rk} \qquad \text{and} \qquad x_{rk}\theta_{sj} = \theta_{sj}x_{rk}$$

for $1 \leqslant r, s \leqslant n$, $1 \leqslant i < j \leqslant m'$ and $1 \leqslant k \leqslant d \leqslant m$.

A monomial in $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ is said to be of *degree* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ in the commuting variables and $\beta = (\beta_1, \beta_2, \ldots, \beta_{m'})$ in the Grassmannian variables if the total degree in the variables $X_n^{(k)}$ is α_k and the total degree of the monomial in the variables $\Theta_n^{(i)}$ is β_i for $1 \leqslant i \leqslant m'$ and $1 \leqslant k \leqslant m$. The homogeneous subspace spanned by all monomials of degree $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ in the commuting variables and $\beta = (\beta_1, \beta_2, \ldots, \beta_{m'})$ in the Grassmannian variables is an S_n submodule of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$.

There is another notation for this symmetric group module that is worth mentioning in terms of the symmetric tensor $S^r(V)$ and antisymmetric tensor $\bigwedge^{r'}(V)$. If V_n is a vector space of dimension n with a basis $\{v_1, v_2, \ldots, v_n\}$. We note that as S_n-modules,

$$\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}] \simeq \bigoplus_{r,r'\geqslant 0} S^r(V_n \otimes V_m) \otimes \bigwedge^{r'}(V_n \otimes V_{m'})$$

where the symmetric group S_n acts on the vector space V_n in this expression.

2.1 Combinatorial definitions

A *partition* of an integer n is a sequence of positive weakly decreasing integers whose terms sum to n. The notation $\lambda \vdash n$ denotes that λ is a partition of n and $\ell(\lambda)$ denotes the number of terms in the sequence. We use $|\lambda|$ to denote the sum of the terms of the partition λ. The *cells* of a partition λ is the set of pairs $\{(i, j) : 1 \leqslant i \leqslant \ell(\lambda), 1 \leqslant j \leqslant \lambda_i\}$. In this paper, the cells will be graphically represented by displaying them in the first quadrant using French notation with the largest row of the partition on the bottom.

A *multiset* is a collection of objects where the entries are allowed to repeat. Multisets will be indicated by enclosing the collection of elements with $\{\!\{, \}\!\}$ to indicate that the structure keeps the multiplicity of the elements. When the multiset has entries which are integers between 1 and m, the *content vector* of the multiset will be a vector $(a_1, a_2, \ldots, a_m)$ where $a_i \geqslant 0$ is the number of times that i appears in the multiset.

A *multiset partition* is a multiset of multisets. That is, $\pi = \{\!\{S_1, S_2, \ldots, S_r\}\!\}$ where each of the S_i is a multiset. The entries in π are referred to as the *parts* of π and the *length* of π is the number of (non-empty) parts of π. The *content* of a multiset partition π is the disjoint union of the entries of π, that is, $\biguplus_{i=1}^{\ell(\pi)} S_i$. The multiset partitions that appear in this paper will be in two different alphabets. Fix two non-negative integers m and m'. The multiset partitions that are considered here will have entries in $[m] \cup [\overline{m'}] := \{1, 2, \ldots, m\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{m'}\}$ where barred entries are allowed to occur at most once in each part of the multiset partition (but may occur in several parts of a given multiset partition). The notation $\pi \Vdash S$ will be used to indicate that π is a multiset partition with content S. The condition that barred entries are not allowed to occur twice in any given part of a multiset partition is imposed by the algebraic relation that the Grassmannian variables square to zero and that this data structure is used to encode these algebraic objects. *Throughout this paper it will be assumed that barred entries may not repeat within a single multiset, a part of a multiset partition, or within a single cell of a tableau or filling.*

Multisets are, by definition, an unordered structure, but it will be necessary to specify an order on multisets for constructing multiset tableaux. The results of this paper are independent of the order we choose; however, the order chosen will need to be consistent with the algebra. For instance, in the proof of 3 when multiplying monomials and in Section 4 the basis elements will inherit a sign from the order. We assume that our alphabet is totally ordered by $1 < 2 < \cdots < m < \overline{1} < \overline{2} < \cdots < \overline{m'}$ In this paper, the multisets will be ordered in *reverse lexicographic order*. For multisets S and S', we say $S < S'$ if $max(S) < max(S')$, and if $max(S) = max(S')$ a comparison is made by removing one instance of $max(S)$ from each multiset. The empty multiset is considered to be the smallest multiset in this order. (See Examples 1 and 11 for an illustration of the use of this multiset order).

Let λ be a partition and S be a multiset of entries from $[m] \cup \overline{[m']}$. A *multiset tableau* of shape λ and content S is a map, T, from the cells of λ to multisets of entries of $[m] \cup \overline{[m']}$ satisfying the following conditions.

1. For each $c \in \lambda$, $T(c)$ is a multiset (which could be empty) in barred and unbarred entries, $[m] \cup \overline{[m']}$.

2. The cells are weakly increasing in both the rows and columns with respect to the chosen order on multisets. That is, $T(c_1, c_2) \leqslant T(c_1+1, c_2)$ and $T(c_1, c_2) \leqslant T(c_1, c_2+1)$ whenever both cells are in the partition.

3. If a multiset label contains an even number of barred entries, then no two cells that are labelled with that multiset may occur in the same column (i.e. the cells with the same multiset label that have an even number of barred entries form a horizontal strip). That is, if S_i is a multiset with an even number of barred entries and (c_1, c_2), (c_1', c_2') are cells of λ such that $T(c_1, c_2) = T(c_1', c_2') = S_i$, then either $(c_1, c_2) = (c_1', c_2')$ or $c_2 \neq c_2'$.

4. If a multiset contains an odd number of barred entries, then no two cells that are labelled with that multiset may occur in the same row (i.e. the cells with the same multiset label that have an odd number of barred entries form a vertical strip). That is, if S_i is a multiset with an odd number of barred entries and (c_1, c_2), (c_1', c_2') are cells of λ such that $T(c_1, c_2) = T(c_1', c_2') = S_i$, then either $(c_1, c_2) = (c_1', c_2')$ or $c_1 \neq c_1'$.

The *content* of a tableau is the multiset union of the content of the cells of the tableau.

Example 1. Let $\lambda = (7, 3, 2, 2, 1)$, then the following is an example of a multiset tableau of content $S = \{\!\{1^3, 2^4, \overline{1}^6, \overline{2}^6\}\!\}$ satisfying the conditions of the definition. Since $\{\!\{\}\!\} < \{\!\{2\}\!\} < \{\!\{1, \overline{1}\}\!\} < \{\!\{\overline{2}\}\!\} < \{\!\{\overline{1}, \overline{2}\}\!\}$.

$1\overline{1}$						
$1\overline{1}$	$\overline{2}$					
$1\overline{1}$	$\overline{2}$					
2	2	$\overline{1}\overline{2}$				
		2	2	$\overline{2}$	$\overline{1}\overline{2}$	$\overline{1}\overline{2}$

2.2 Symmetric Functions

The proof of the main result will require the use of well known identities and notation in symmetric functions. We will mainly follow the notation which is common to references in this area [Mac, Sag, Sta] with a single addition that we describe below. The ring of symmetric functions is the polynomial algebra in generators p_i (*the power sum generators*) for $i \geqslant 1$ where the degree p_i is i. That is,

$$\Lambda = \mathbb{Q}[p_1, p_2, p_3, \dots] \ .$$

The *elementary* $\{e_i\}_{i\geqslant 1}$ and *homogeneous* generators $\{h_i\}_{i\geqslant 1}$ are related to the power sum generators by the equations

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r} \qquad \text{and} \qquad nh_n = \sum_{r=1}^{n} p_r h_{n-r} \ .$$

To allow for simpler notation, let $h_0 = e_0 = p_0 = 1$ and $h_{-r} = e_{-r} = p_{-r} = 0$ for $r > 0$. For an integer vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$, products of the generators will be represented by the shorthand

$$p_\alpha := p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_\ell}, \qquad e_\alpha := e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_\ell} \qquad \text{and} \qquad h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_\ell} \ .$$

For a partition λ of n, denote the irreducible character of the representation of the symmetric group S_n indexed by the partition λ by χ^λ and the value of this character at a permutation of cycle type μ by $\chi^\lambda(\mu)$. The *Schur symmetric functions* are defined as

$$s_\lambda = \sum_\mu \chi^\lambda(\mu) \frac{p_\mu}{z_\mu}$$

where $z_\mu = \prod_{i\geqslant 1} m_i(\mu)! i^{m_i(\mu)}$ and $m_i(\mu)$ is the number of times that i occurs in the partition μ.

The *Hall scalar product* on symmetric functions is defined for the power sum basis as

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} z_\mu & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \ .$$

There is a combinatorial rule for multiplying an elementary or homogeneous generator and a Schur function that is known as the *Pieri rule*. It says

$$h_r s_\lambda = \sum_\mu s_\mu \qquad \text{and} \qquad e_r s_\lambda = \sum_\gamma s_\gamma \tag{2}$$

where the sum on the left is over partitions μ such that $\lambda_i \leqslant \mu_i$ and for all cells (c_1, c_2) and (c_1', c_2') in μ which are not also in λ, either $(c_1, c_2) = (c_1', c_2')$ or $c_2 \neq c_2'$. The sum on the right is over partitions γ such that $\lambda_i \leqslant \gamma_i$ and for all cells (c_1, c_2) and (c_1', c_2') in γ which are not also in λ, either $(c_1, c_2) = (c_1', c_2')$ or $c_1 \neq c_1'$.

Symmetric functions play a role both as generating functions for characters of the symmetric group, and also as polynomial characters of GL_n representations. The *character of a GL_n representation* is the trace of the representation when it is evaluated at a diagonal matrix with eigenvalues $x_1, x_2, \dots, x_n$. It will always be the case that this character, as a function of the eigenvalues, will be equal to a symmetric function $f \in \Lambda$ where f is expanded in the power sum generators and p_k is replaced by $x_1^k + x_2^k + \cdots + x_n^k$. Denote this character by $f[X_n]$.

The symmetric group may be realized as the subgroup of permutation matrices inside of GL_n. For each permutation σ, let A_σ represent the corresponding permutation matrix. To compute the value of the character of an S_n representation with character equal to $f[X_n]$, the variables of $f[X_n]$ are replaced by the eigenvalues of A_σ. Let μ be a partition of n and σ a permutation of cycle structure μ. Up to reordering, the eigenvalues of A_σ are dependent only on the cycle structure of the permutation σ (that is, on the partition μ). We will denote the *evaluation of $f[X_n]$ at the eigenvalues of a permutation matrix* of cycle structure μ by $f[\Xi_\mu]$ and this is a character value of the S_n representation.

The reason that the computation of the character is important for this problem is that the symmetric group character characterizes an S_n representation up to isomorphism. That is, let X be a GL_n representation with character $f[X_n]$ as a function of the eigenvalues of a permutation matrix. The *Frobenius image of the character* [Mac, Ch I.7, equation (7.2)], [Sag, Ch 4.7] is the generating function

$$\phi(f[X_n]) = \sum_{\mu \vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu} \tag{3}$$

and the multiplicity of an irreducible S_n representaton indexed by the partition λ in X is equal to the coefficient of s_λ in $\phi(f[X_n])$.

Example 2. Note that $s_2[X_3] = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$ is the character of the symmetric group representation corresponding to the module $S^2(V_3)$. To compute the value of the character at the permutations of cycle type $\mu = (1,1,1)$ with eigenvalues $\{1,1,1\}$, cycle type $(2,1)$ with eigenvalues $\{1,-1,1\}$ and cycle type (3) with eigenvalues $\{1, e^{2\pi i/3}, e^{4\pi i/3}\}$. The character values are the evaluations:

$$s_2[\Xi_{(1,1,1)}] = 6, \qquad s_2[\Xi_{(2,1)}] = 2, \qquad \text{and} \qquad s_2[\Xi_{(3)}] = 0\ .$$

It follows that

$$\phi(s_2[X_3]) = 6\frac{p_{111}}{6} + 2\frac{p_{21}}{2} + 0\frac{p_3}{3} = 2s_3 + 2s_{21}.$$

This implies that $S^2(V_3)$ decomposes into 4 irreducible S_n components.

3 A combinatorial model for the S_n-decomposition of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$

One of the main results of this paper is the following combinatorial model for the decomposition of the multivariate polynomial ring as an S_n-module.

Theorem 3. *The multiplicity of the symmetric group irreducible indexed by the partition $\lambda \vdash n$ in the subspace of degree $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ in the commuting variables and degree $\beta = (\beta_1, \beta_2, \ldots, \beta_{m'})$ in the Grassmannian variables is equal to the number of multiset tableaux (see the definition in Section 2.1) of content $\{\!\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, m^{\alpha_m}, \overline{1}^{\beta_1}, \ldots, \overline{m'}^{\beta_{m'}}\}\!\}$ and of shape λ.*

We present an example of this theorem in Example 12 at the end of this section. All of the hard combinatorial effort for proving this theorem appears in two recent references of the authors [OZ, OZ2]. By referring the reader to the combinatorial interpretations in those papers we can present a relatively short proof of this result, but there is a part which is admittedly not completely self contained.

Remark 4. The case of $m = 1$ and $m' = 0$ or $m = 0$ and $m' = 1$ is a well known result in the theory of symmetric functions due to A. C. Aitkin [Ait1, Ait2] (see [Sta] p.474–5 exercises **7.72** and **7.73**). The case of $m > 0$ and $m' = 0$ follows from a result of Littlewood [Lit, ST] and known techniques for calculating plethysm coefficients. The multivariate version that we present here is a repeated tensor of Aitkin's results. What we hope to convey is the surprising fact that the decomposition of this symmetric group module has a simple description in terms of 'multiset tableaux' and these combinatorial objects specialize to several well known special cases.

The following lemmas and propositions involve finding combinatorial interpretations for algebraic expressions. In particular, we use combinatorial objects to explain coefficients in expressions of symmetric functions. When we began extending our combinatorial results to explain the decomposition of expressions that are no longer bases of the symmetric functions, the multiset tableaux that appear in Theorem 3 were a consequence.

Before we prove the theorem we state the following lemma which is a typical calculation of a computation of a GL_n character.

Lemma 5. *The GL_n character of the subspace of degree $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ in the x_{ij} variables and degree $\beta = (\beta_1, \beta_2, \ldots, \beta_{m'})$ in the θ_{ik} Grassmannian variables is equal to $h_\alpha[X_n]e_\beta[X_n]$.*

The proof will also require a combinatorial interpretation for the evaluation of this character at eigenvalues of a permutation matrix of cycle structure μ because we will explicitly compute the Frobenius character. For this we need the following combinatorial definitions.

Definition 6. (Definition 33 of [OZ]) Let $\mathcal{T}_{\alpha,\mu}$ be the set of fillings of some of the cells of the partition μ with multisets such that the total content of the filling is $\{\!\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, \ell(\alpha)^{\alpha_{\ell(\alpha)}}\}\!\}$ and any number of labels can go into the same cell but all cells in the same row must have the same multiset of labels.

Definition 7. (Definition 5.13 of [OZ2]) For a non-negative integer vector β and a partition μ let $\overline{\mathcal{T}}_{\beta,\mu}$ be the fillings of some of the cells of the diagram of the partition μ with subsets of $\{\overline{1}, \overline{2}, \ldots, \overline{\ell(\beta)}\}$ such that the total content of the filling is $\{\!\{\overline{1}^{\beta_1}, \overline{2}^{\beta_2}, \ldots, \overline{\ell(\beta)}^{\beta_{\ell(\beta)}}\}\!\}$ and such that all cells in the same row have the same subset of entries. For $F \in \overline{\mathcal{T}}_{\beta,\mu}$, we define the *weight* of F, $wt(F)$, to be -1 to the power of the number of filled cells plus the number of rows occupied by the sets of odd size.

These two combinatorial definitions are used to describe the following expressions for symmetric group characters whose characters are given as homogeneous and elementary symmetric functions.

Proposition 8. *(Proposition 27 and Theorem 37 of [OZ]; Proposition 5.6 and Lemma 5.15 of [OZ2]) For non-negative integer vectors α and β and any partition μ,*

$$h_\alpha[\Xi_\mu] = |\mathcal{T}_{\alpha,\mu}| \qquad \text{and} \qquad e_\beta[\Xi_\mu] = \sum_{F\in\overline{\mathcal{T}}_{\beta,\mu}} wt(F)\ . \tag{4}$$

Now a product of these expressions will have terms indexed by elements in $\mathcal{T}_{\alpha,\mu} \times \overline{\mathcal{T}}_{\beta,\mu}$ and combining the objects into multiset fillings of μ establishes that

$$h_\alpha[\Xi_\mu]e_\beta[\Xi_\mu]$$

is equal to the sum over all fillings of the diagram for μ with multisets of content

$$\{\!\!\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, \ell(\alpha)^{\alpha_{\ell(\alpha)}}, \overline{1}^{\beta_1}, \overline{2}^{\beta_2}, \ldots, \overline{\ell(\beta)}^{\beta_{\ell(\beta)}}\}\!\!\}$$

where the cells in any given row must have the same label, and there is a weight of the filling equal to -1 to the power of the number of filled cells plus the number of rows occupied by the multisets with an odd number of barred entries.

Our theorem follows from one final result that we pull from [OZ2] and state here without proof.

Proposition 9. *(Proposition 5.8 [OZ2]) For partitions λ,τ and μ, let $\mathcal{F}^\mu_{\lambda,\tau}$ be the fillings of the diagram for the partition μ with λ_i labels i and τ_j labels j' such that all cells in a row are filled with the same label. For $F \in \mathcal{F}^\mu_{\lambda,\tau}$, the weight of the filling, $wt(F)$ is equal to -1 raised to the number of cells filled with primed labels plus the number of rows occupied by the primed labels, then*

$$\left\langle h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau, p_\mu \right\rangle = \sum_{F\in\mathcal{F}^\mu_{\lambda,\tau}} wt(F)\ . \tag{5}$$

This last proposition indicates that we should assign a 'type' to each filling described above and group the fillings with the same type together.

Definition 10. Let F be a filling of the diagram for μ with multisets such that the multiset union of all of the labels is of content

$$cont_{\alpha,\beta} := \{\!\!\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, \ell(\alpha)^{\alpha_{\ell(\alpha)}}, \overline{1}^{\beta_1}, \overline{2}^{\beta_2}, \ldots, \overline{\ell(\beta)}^{\beta_{\ell(\beta)}}\}\!\!\}\ .$$

We associate the filling to a multiset partition, $\mathrm{MSP}(F)$, which is equal to the multiset collection of the non-empty labels of the cells.

For a given multiset partition, define $\tilde{m}_e(\pi)$ to be the partition whose entries are the multiplicities of the parts of π that have an even number of barred entries and $\tilde{m}_o(\pi)$ be the partition whose entries are the multiplicities of the parts of π that have an odd number of barred entries.

Example 11. Let $n = 24$ and $\mu = (5, 5, 3, 2, 2, 2, 2, 1, 1, 1)$ and consider the filling F,

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1} 112 \\
\cline{1-1} 22 \\
\cline{1-1} \\
\cline{1-2} 1\overline{1}\overline{2} & 1\overline{1}\overline{2} \\
\cline{1-2} \overline{1} & \overline{1} \\
\cline{1-2} 1\overline{1} & 1\overline{1} \\
\cline{1-2} \overline{1} & \overline{1} \\
\cline{1-3} & & \\
\hline 13 & 13 & 13 & 13 & 13 \\
\hline & & & & \\
\hline
\end{array}$$

The content of this filling is $\{\!\{1^{11}, 2^3, 3^5, \overline{1}^8, \overline{2}^2\}\!\}$ and $\mathrm{MSP}(F) = \{\!\{\{\!\{1, 1, 2\}\!\}, \{\!\{2, 2\}\!\}, \{\!\{1, 3\}\!\}, \{\!\{1, 3\}\!\}, \{\!\{1, 3\}\!\}, \{\!\{1, 3\}\!\}, \{\!\{1, 3\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{1, \overline{1}\}\!\}, \{\!\{1, \overline{1}\}\!\}, \{\!\{1, \overline{1}, \overline{2}\}\!\}, \{\!\{1, \overline{1}, \overline{2}\}\!\}\}\!\}$ where (while a multiset is by definition an unordered structure) we have listed the entries by increasing reverse lexicographic order to be consistent with the order that we will use on tableaux. There are 6 cells which have labels with an odd number of barred entries and they occupy 3 rows so the weight of this filling is -1. If we let $\pi = \mathrm{MSP}(F)$, then $\tilde{m}_e(\pi) = (5, 2, 1, 1)$ and $\tilde{m}_o(\pi) = (4, 2)$.

Proof. (of Theorem 3.1) The GL_n character of the subspace of degree $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ in the x_{ij} variables and degree $\beta = (\beta_1, \beta_2, \ldots, \beta_{m'})$ in the θ_{ik} Grassmannian variables is equal to $h_\alpha[X_n]e_\beta[X_n]$ by Lemma 5. We will use the evaluation of this character at elements of the symmetric group as a subset of elements of GL_n.

With the GL_n character, we can compute the S_n character by evaluating $h_\alpha[X_n]e_\beta[X_n]$ at the eigenvalues of a permutation matrix. Fix a partition μ of n and we will calculate, using the combinatorial gadgets, the character of the subspace at a permutation matrix of cycle structure μ. Proposition 8 implies that $h_\alpha[\Xi_\mu]e_\beta[\Xi_\mu]$ is equal to a sum over fillings of the diagram for μ with multisets of content $cont_{\alpha,\beta}$. We then group all fillings by the associated multiset partition to the filling, $\mathrm{MSP}(F)$, hence

$$h_\alpha[\Xi_\mu]e_\beta[\Xi_\mu] = \sum_\pi \sum_{F:\mathrm{MSP}(F)=\pi} wt(F)$$

where the outer sum is over all multiset partitions π of such that $\pi \Vdash \{\!\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, m^{\alpha_m}, \overline{1}^{\beta_1}, \overline{2}^{\beta_2}, \ldots, \overline{m'}^{\beta_{m'}}\}\!\}$.

We next apply Proposition 9 and consider the labels of the filling of multisets where the 'primed' entries of the filling are those with an odd number of barred entries and the 'unprimed' entries of the filling are those with an even number of barred entries. This implies that

$$h_\alpha[\Xi_\mu]e_\beta[\Xi_\mu] = \sum_\pi \left\langle h_{|\mu|-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|} h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)}, p_\mu \right\rangle$$

where again the sum is over all multiset partitions π of content $cont_{\alpha,\beta}$.

Since we have computed the character of the subspace for each permutation of cycle structure μ, from Equation (3) we can compute the Frobenius image of the character of the subspace as

$$\sum_{\mu\vdash n} h_\alpha[\Xi_\mu] e_\beta[\Xi_\mu] \frac{p_\mu}{z_\mu} = \sum_{\pi}\sum_{\mu\vdash n} \left\langle h_{n-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|} h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)}, p_\mu \right\rangle \frac{p_\mu}{z_\mu}$$
$$= \sum_{\pi} h_{n-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|} h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)} \qquad (6)$$

where the sum is over all multiset partitions π of content $cont_{\alpha,\beta}$.

To conclude the proof of the theorem we need to establish that the multiplicity of a Schur function indexed by a partition λ in this expression agrees with the description stated in the theorem. Here is where the order of the multisets in the tableau plays a role in determining the combinatorial interpretation. Each of the generators in the product $h_{n-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|} h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)}$ will represent the cells in the multiset tableau which are labeled by a fixed multiset. The generator $h_{n-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|}$ represents the blank cells, and those with an even number of barred entries are represented by the generators in the product $h_{\tilde{m}_e(\pi)}$, while those with an odd number of barred entries are represented by the generators in the product $e_{\tilde{m}_o(\pi)}$. Since the multiplication in the ring of symmetric functions is commutative, we can choose to order these terms with respect to the total order that we have placed on these multisets.

To determine the multiplicity of the Schur function s_λ in this expression, we repeatedly apply the Pieri rule. We use tableaux to keep track of the terms in the Schur expansion of

$$h_{n-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|} h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)},$$

where the labels of the tableaux are the multisets represented by the h or e-generators. The Pieri rule implies we will record i cells in a horizontal strip for each product of an h_i generator, while we will record i cells in a vertical strip for each e_i generator.

We provide an example below to ensure that it is clear that the coefficient of a Schur function s_λ in $h_{n-|\tilde{m}_e(\pi)|-|\tilde{m}_o(\pi)|} h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)}$ is equal to the number of multiset tableaux whose entries are the multisets of π. By equation (6), the coefficient of s_λ in the Frobenius image of the subspace has multiplicity equal to the total number of multiset tableaux of shape λ and content $cont_{\alpha,\beta}$. $\square$

Example 12. Let $\pi = \{\!\{ \{\!\{1,1,2\}\!\}, \{\!\{2,2\}\!\}, \{\!\{1,3\}\!\}, \{\!\{1,3\}\!\}, \{\!\{1,3\}\!\}, \{\!\{1,3\}\!\}, \{\!\{1,3\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{\overline{1}\}\!\}, \{\!\{1,\overline{1}\}\!\}, \{\!\{1,\overline{1}\}\!\}, \{\!\{1,\overline{1},\overline{2}\}\!\}, \{\!\{1,\overline{1},\overline{2}\}\!\} \}\!\}$ where we have $\tilde{m}_e(\pi) = (5,2,1,1)$ and $\tilde{m}_o(\pi) = (4,2)$. This implies that $h_{24-9-6} h_{(5,2,1,1)} e_{(4,2)}$ will occur as a summand in Equation (6).

For example, to compute the tableaux with entries in π and of shape $\lambda = (10,8,5,1)$, we first order the generators with respect to the order on multisets mentioned in Section 2.1, $\{\!\{\}\!\} < \{\!\{1,1,2\}\!\} < \{\!\{2,2\}\!\} < \{\!\{1,3\}\!\} < \{\!\{\overline{1}\}\!\} < \{\!\{1,\overline{1}\}\!\} < \{\!\{1,\overline{1},\overline{2}\}\!\}$. The number of multiset tableaux with these entries will be the coefficient of the Schur function in the symmetric function $h_9h_1h_1h_5e_4e_2h_2$. The three tableaux with this filling are:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\cline{1-1} \bar{1} \\
\cline{1-5} 22 & \bar{1} & 1\bar{1} & 1\bar{1}\bar{2} & 1\bar{1}\bar{2} \\
\cline{1-8} 112 & 13 & 13 & 13 & 13 & 13 & \bar{1} & 1\bar{1} \\
\hline & & & & & & & & & \bar{1} \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\cline{1-1} \bar{1} \\
\cline{1-5} 13 & 13 & \bar{1} & 1\bar{1} & 1\bar{1}\bar{2} \\
\cline{1-8} 112 & 22 & 13 & 13 & 13 & \bar{1} & 1\bar{1} & 1\bar{1}\bar{2} \\
\hline & & & & & & & & & \bar{1} \\
\hline
\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\cline{1-1} \bar{1} \\
\cline{1-5} 13 & \bar{1} & 1\bar{1} & 1\bar{1}\bar{2} & 1\bar{1}\bar{2} \\
\cline{1-8} 112 & 22 & 13 & 13 & 13 & 13 & \bar{1} & 1\bar{1} \\
\hline & & & & & & & & & \bar{1} \\
\hline
\end{array} .$$

4 The ring of S_n-invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$

Since the multiplicity of an irreducible representation indexed by a partition λ in the S_n-module of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ is equal to the number of multiset tableaux of shape λ, then the multiplicity of S_n invariants (the irreducible indexed by (n)) is equal to the number of single row multiset tableaux. Single row multiset tableaux are in bijection with multiset partitions as defined in Section 2 with one additional condition imposed by the construction on tableau.

Say that π is a *super multiset partition* of $[m] \cup [\overline{m'}]$ if the parts with an odd number of barred entries appear at most once in the multiset partition.

Corollary 13. *A basis for the ring of S_n invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ is indexed by super multiset partitions of $[m] \cup [\overline{m'}]$ of length less than or equal to n.*

The following special cases of these rings of invariants are examples that we are aware of that are considered in the algebraic combinatorics literature.

- If $m = 1$ and $m' = 0$, then the ring of invariants of $\mathbb{C}[X_n]$ are known as symmetric polynomials and are equal to the span of the polynomials $Sym_n := \bigoplus_{k\geqslant 0}\{p_\lambda[X_n] : \lambda \vdash k\}$. A basis for the ring of invariants in this case is indexed by partitions which have length of $\lambda \leqslant n$.

  A well known result [C] states that as an S_n-module,

  $$\mathbb{C}[X_n] \simeq Sym_n \otimes \mathbb{C}[X_n]/I$$

  where I is the ideal $\langle p_k[X_n] : 1 \leqslant k \leqslant n\rangle$ and this is equal to the ideal generated by symmetric polynomials with non-constant term. The quotient $\mathbb{C}[X_n]/I$ are often referred to as the *coinvariants* and the inverse system corresponding to that quotient are the *harmonics*.

- If $m = 2$ and $m' = 0$, the quotient of $\mathbb{C}[X_{n\times 2}]$ by the ideal generated by the invariants of the ring was defined by Haiman and is known as the ring of *diagonal coinvariants* [Hai94]. A combinatorial formula for the monomial expansion of the Frobenius characteristic of this S_n-module was known as *the shuffle conjecture* [HHLRU, CM15].

- For $m > 2$ and $m' = 0$, F. Bergeron and L.-F. Préville-Ratelle [Ber, BPR] considered quotients and harmonics in multivariate polynomial spaces and their general linear group and symmetric group characters.

- If $m = 2$ and $m' = 1$, then the second author [Zab19] recently proposed the quotient $\mathbb{C}[X_{n\times 2}; \Theta_n]$ by the ideal generated by the invariants as a representation theoretic model for a generalization of the shuffle conjecture known as *the delta conjecture* [HRW].

- If $m > 1$ and $m' = 0$, then the ring of invariants of $\mathbb{C}[X_{n\times m}]$ is known as *MacMahon symmetric functions* [Ges, Ros] (MacMahon [MacM] called these *symmetric functions in several systems of parameters*). MacMahon indexed the basis of this space of symmetric functions by vector partitions instead of multiset partitions.

- If $m = 0$ and $m' = 2$, then Kim and Rhoades [KR] give the standard monomial basis of the quotient of $\mathbb{C}[\Theta_{n\times 2}]$ by the ideal generated by the invariants of the ring. They (indirectly) used the same set of generators we define in this section for this special case.

- If $m = 1$ and $m' = 1$, then the ring of invariants are known as *symmetric functions in superspace* and was studied by Desrosiers, Lapointe and Mathieu [DLM]. There the invariants are indexed by objects called *superpartitions*, which are pairs of the form $(\Lambda^a; \Lambda^s)$, where Λ^a is a strict partition and Λ^s is a partition. L. Solomon [Sol] proved that if $\{f_1, f_2, \ldots, f_n\}$ is a set of free generators for $\mathbb{C}[X_n]$, then $\{f_1, f_2, \ldots, f_n, d(f_1), d(f_2), \ldots, d(f_n)\}$ is a set of free generators for the algebra of invariants of $\mathbb{C}[X_n; \Theta_n]$, where $d : \mathbb{C}[X_n] \to \mathbb{C}[X_n; \Theta_n]$ is an operator on polynomials defined by $d(f) = \sum_{i=1}^{n} \partial_{x_i} f \theta_i$.

- If $m = 1$ and $m' \geqslant 2$, then the ring of invariants was studied by Alarie-Vézina, Lapointe and Mathieu [ALM]. In that work, the invariants are indexed by generalizations of superpartitions.

The 'super' prefix of the name super multiset partition was borrowed from the references [DLM, ALM] mentioned above, The main result of the rest of this section is to establish a finite list of algebraic generators for the ring of S_n invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ (the analogue of the power sums) in Theorem 17.

Analogues of the elementary and complete homogeneous generators exist and we will hint, but not explicitly state, how to define them in terms of generating functions. The generators of this ring are not 'free' because they will satisfy relations coming from the Grassmannian variables.

Let $\pi = \{\!\{S_1, S_2, \ldots, S_{\ell(\pi)}\}\!\}$ be a multiset partition of length less than or equal to n whose entries are in $[m] \cup [\overline{m'}]$. Furthermore let us assume that the parts of the multisets S_i are ordered in weakly increasing order and $S_i = \{\!\{1^{a_{i1}}, 2^{a_{i2}}, \ldots, m^{a_{im}}, \overline{s}_{i1}, \overline{s}_{i2}, \ldots, \overline{s}_{i\ell_i}\}\!\}$. The *monomial symmetric polynomial* indexed by π is denoted m_π and it is defined as the polynomial in $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ that equals the sum of the distinct S_n orbits of

$$(X; \Theta)^\pi := \prod_{i=1}^{\ell(\pi)} x_{i1}^{a_{i1}} x_{i2}^{a_{i2}} \cdots x_{im}^{a_{im}} \theta_{is_{i1}} \theta_{is_{i2}} \cdots \theta_{is_{i\ell_i}}$$

where the product follows the order on the multiset partition π and the entries in the multiset are in increasing order. Recall that S_n acts on the first indices of the variables x_{ij} and θ_{ij}. If $\ell(\pi) > n$ then the monomial symmetric polynomial is 0.

Looking carefully at this set of symmetric group invariants, we notice that if $S_i = S_{i+1}$ and ℓ_i is odd (there are an odd number of barred elements in S_i), then $\sigma(i, i+1)(X;\Theta)^\pi = -\sigma(X;\Theta)^\pi$, for $\sigma \in S_n$. As a consequence, $m_\pi = 0$ whenever π contains two equal parts with an odd number of barred entries. Therefore the indexing set for the monomial basis are the super multiset partitions.

Let S be a multiset and let $S = T \cup \overline{T}$ where $T = \{\!\{1^{a_1}, 2^{a_2}, \ldots, m^{a_m}\}\!\}$ is a multiset with entries in $[m]$, and $\overline{T} = \{\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_k\}$ is a subset of $[\overline{m'}]$. The *power sum generators* are defined by

$$p_S := \sum_{r=1}^{n} x_{r1}^{a_1} x_{r2}^{a_2} \cdots x_{rm}^{a_m} \theta_{rs_1} \theta_{rs_2} \cdots \theta_{rs_k} \ . \tag{7}$$

For a multiset partition $\pi = \{\!\{S_1, S_2, \ldots, S_{\ell(\pi)}\}\!\}$, the *power symmetric polynomials* are

$$p_\pi := p_{S_1} p_{S_2} \cdots p_{S_{\ell(\pi)}}$$

where the product of the generators are in increasing order with respect to reverse lexicographic order.

Example 14. Let $\pi = \{\!\{\{\!\{1, 1, \overline{1}\}\!\}, \{\!\{1, 1, \overline{2}\}\!\}, \{\!\{\overline{1}, \overline{2}\}\!\}, \{\!\{\overline{1}, \overline{2}\}\!\}\}\!\}$. As long as $n \geqslant 4$,

$$m_\pi = \sum_{(a,b,c,d)} x_{a1}^2 \theta_{a1} x_{b1}^2 \theta_{b2} \theta_{c1} \theta_{c2} \theta_{d1} \theta_{d2}$$

where the sum is over all sequences (a, b, c, d) of distinct entries with $c < d$ (since $\theta_{c1}\theta_{c2}\theta_{d1}\theta_{d2} = \theta_{d1}\theta_{d2}\theta_{c1}\theta_{c2}$). The following power sum generators are

$$\begin{aligned} p_{\{\!\{1,1,\overline{1}\}\!\}} &= x_{11}^2\theta_{11} + x_{21}^2\theta_{21} + \cdots + x_{n1}^2\theta_{n1} \\ p_{\{\!\{1,1,\overline{2}\}\!\}} &= x_{11}^2\theta_{12} + x_{21}^2\theta_{22} + \cdots + x_{n1}^2\theta_{n2} \\ p_{\{\!\{\overline{1},\overline{2}\}\!\}} &= \theta_{11}\theta_{12} + \theta_{21}\theta_{22} + \cdots + \theta_{n1}\theta_{n2} \ . \end{aligned}$$

Some explicit expansion of the products of the power sum generators shows that

$$p_{\{\!\{1,1,\overline{1}\}\!\}} p_{\{\!\{1,1,\overline{2}\}\!\}} p_{\{\!\{\overline{1},\overline{2}\}\!\}}^2 = m_{\{\!\{\{\!\{1,1,\overline{1}\}\!\},\{\!\{1,1,\overline{2}\}\!\},\{\!\{\overline{1},\overline{2}\}\!\},\{\!\{\overline{1},\overline{2}\}\!\}\}\!\}} + m_{\{\!\{\{\!\{1,1,1,1,\overline{1},\overline{2}\}\!\},\{\!\{\overline{1},\overline{2}\}\!\},\{\!\{\overline{1},\overline{2}\}\!\}\}\!\}} \ .$$

Lemma 15. *The power symmetric polynomials $\{p_\pi\}$ are a basis for the ring of S_n invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ where the π run over all super multiset partitions with $\ell(\pi) \leqslant n$.*

Proof. The ring of S_n invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$ are clearly spanned by the monomial symmetric polynomials since they are the S_n orbits of a single monomial in the polynomial ring. This basis is indexed by the super multiset partitions with length less than or equal to n.

Consider the power sum symmetric function indexed by a super multiset partition and order the monomials using reverse lexicographic order where the Grassmannian variables are larger than the commutative variables. It follows that $p_\pi = c_\pi m_\pi$ plus terms which are smaller with respect to this order. Therefore the set $\{p_\pi\}$ with $\ell(\pi) \leqslant n$ also spans the same space and is linearly independent. $\square$

Let $\mathbf{q} = q_1, q_2, \ldots, q_m$ be a commuting set of variables and $\mathbf{z} = z_1, z_2, \ldots, z_{m'}$ be an anticommuting set of variables. Define the following generating functions for the generators

$$E(\mathbf{q}, \mathbf{z}) = \prod_{i=1}^{n} \left(1 + \sum_{j=1}^{m} q_j x_{ij} + \sum_{j'=1}^{m'} z_{j'} \theta_{ij'} \right) \text{ and } \tag{8}$$

$$P(\mathbf{q}, \mathbf{z}) = -\sum_{i=1}^{n} \log \left(1 - \left(\sum_{j=1}^{m} q_j x_{ij} + \sum_{j'=1}^{m'} z_{j'} \theta_{ij'} \right) \right) . \tag{9}$$

These generating functions are related by $E(\mathbf{q}, \mathbf{z}) = \exp\left(-P(-\mathbf{q}, -\mathbf{z})\right)$.

The following lemma involves a standard calculation on the generating function using the definitions in Equation (7) and the expansions of the expression in Equation (9). The variables satisfy $z_{j'}^2 = \theta_{ij'}^2 = 0$ and this imposes the condition that the barred entries may not be repeated in the multiset. There is no sign introduced in the expression from the $\theta_{ij'}$ variables because it is cancelled by the sign from the $z_{j'}$ variables.

Lemma 16. *Let $S = T \cup \overline{T}$ where $T = \{\!\{1^{a_1}, 2^{a_2}, \ldots, m^{a_m}\}\!\}$ is a multiset with entries in $[m]$, and $\overline{T} = \{\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_k\}$ is a subset of $[\overline{m'}]$. The coefficient of $q_1^{a_1} q_2^{a_2} \cdots q_m^{a_m} z_{s_1} z_{s_2} \cdots z_{s_k}$ in $P(\mathbf{q}, \mathbf{z})$ is equal to $\frac{1}{|S|} \binom{|S|}{a_1, a_2, \ldots, a_m, 1^k} p_S$.*

Using that expression, we can obtain the expansion of $E(\mathbf{q}, \mathbf{z})$ by calculating

$$\begin{aligned} E(\mathbf{q}, \mathbf{z}) &= \exp\left(-P(-\mathbf{q}, -\mathbf{z})\right) \\ &= \sum_{n \geqslant 0} \frac{1}{n!} \left(\sum_{S} \frac{(-1)^{|S|+1}}{|S|} \binom{|S|}{a_1, a_2, \ldots, a_m, 1^k} q_1^{a_1} q_2^{a_2} \cdots q_m^{a_m} z_{s_1} z_{s_2} \cdots z_{s_k} p_S \right)^n \\ &= \sum_{S} q_1^{a_1} q_2^{a_2} \cdots q_m^{a_m} z_{s_1} z_{s_2} \cdots z_{s_k} \sum_{\pi \Vdash S} (-1)^{|S| + \ell(\pi)} a_\pi p_\pi \end{aligned} \tag{10}$$

where the sum is over multisets $S = \{\!\{1^{a_1}, 2^{a_2}, \ldots, m^{a_m}, \overline{s}_1, \ldots, \overline{s}_k\}\!\}$ and the inner sum is over all super multiset partitions π of content S. The reverse lexicographic order on multiset partitions was chosen so that the parts of the multiset partition π will have the same order as the entries $s_1 < s_2 < \cdots < s_k$ so as not to introduce an additional sign in the equality at Equation (10). For a multiset of this form, let $c(S) = \binom{|S|}{a_1, a_2, \ldots, a_m, 1^k}$. If the multiset partition π is denoted with its multiplicities as $\pi = \{\!\{S_1^{m_1}, S_2^{m_2}, \ldots, S_r^{m_r}\}\!\}$, then the coefficient

$$a_\pi = \frac{c(S_1)^{m_1} c(S_2)^{m_2} \cdots c(S_r)^{m_r}}{|S_1|^{m_1} |S_2|^{m_2} \cdots |S_r|^{m_r} \cdot m_1! m_2! \cdots m_r!} .$$

The coefficients of a monomial $q_1^{a_1} q_2^{a_2} \cdots q_m^{a_m} z_{s_1} z_{s_2} \cdots z_{s_k}$ in Equation (10) give the expansion of the elementary generator in the power sum symmetric polynomial.

In the notation of the following theorem $S = T \cup \overline{T}$ and $S' = T' \cup \overline{T}'$ are two multisets with T, T' are both multisets of $[m]$ and $\overline{T}, \overline{T}'$ are both subsets of $[\overline{m'}]$.

Lemma 15 establishes that products of the power sum generators will span the space of invariants. The next result states that we only need the power sum generators of degree less than or equal to n to generate the space of invariants.

Theorem 17. *The set $\{p_S\}$, running over all possible multisets S and $|S| \leqslant n$, is a finite generating set for the ring of invariants of $\mathbb{C}[X_{n\times m}; \Theta_{n\times m'}]$.*

Proof. Observe that the generating function expression in Equation (8) is a polynomial, so the coefficient from Equation (10) in $E(\mathbf{q}, \mathbf{z})$ is equal to 0 if $a_1 + a_2 + \cdots + a_m + k > n$. This implies that for all $S = \{\!\{1^{a_1}, 2^{a_2}, \ldots, m^{a_m}, \overline{s}_1, \ldots, \overline{s}_k\}\!\}$ with $|S| > n$, then

$$p_S = - \sum_{\substack{\pi \Vdash S \\ \pi \neq \{\!\{S\}\!\}}} (-1)^{|S|+\ell(\pi)} a_\pi p_\pi \ . \tag{11}$$

This implies that any p_π such that π contains a part $S \in \pi$ with $|S| > n$ can be expressed in terms of p_π with all parts smaller than or equal to n by repeatedly applying this relation. $\square$

These generators are not free however and also satisfy the relation

$$p_S p_{S'} = (-1)^{|\overline{T}|\cdot|\overline{T}'|} p_{S'} p_S$$

if S and S' are not equal and $p_S^2 = 0$ if $|\overline{T}|$ is odd.

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