The $1/k$-Eulerian polynomials of type $B$

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Abstract

In this paper, we define the $1/k$-Eulerian polynomials of type $B$. Properties of these polynomials, including combinatorial interpretations, recurrence relations and $\gamma$-positivity are studied. In particular, we show that the $1/k$-Eulerian polynomials of type $B$ are $\gamma$-positive when $k > 0$. Moreover, we define the $1/k$-derangement polynomials of type $B$, denoted $d^B_n(x; k)$. We show that the polynomials $d^B_n(x; k)$ are bi-$\gamma$-positive when $k \geq 1/2$. In particular, we get a symmetric decomposition of the polynomials $d^B_n(x; 1/2)$ in terms of the classical derangement polynomials.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

Throughout this paper, we always let $k$ be a fixed positive number. Following Savage and Viswanathan [23], the $1/k$-Eulerian polynomials $A_n^{(k)}(x)$ are defined by

$$
\sum_{n=0}^{\infty} A_n^{(k)}(x) \frac{z^n}{n!} = \left( \frac{1 - x}{e^{kz(x-1)} - x} \right)^{1/k}.
$$

(1)
When \( k = 1 \), the polynomial \( A_n^{(k)}(x) \) reduces to the classical Eulerian polynomial \( A_n(x) \). Savage and Viswanathan [23] showed that

\[
A_n^{(k)}(x) = \sum_{e \in I_{n,k}} x^{\text{asc}(e)},
\]

where \( I_{n,k} = \{ e \mid 0 \leq e_i \leq (i-1)k \} \) is the set of \( n \)-dimensional \( k \)-inversion sequences with \( e = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n \) and

\[
\text{asc}(e) = \# \left\{ i : 1 \leq i \leq n - 1 \mid \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1} \right\}.
\]

In the following, we first recall the other combinatorial interpretations of \( A_n^{(k)}(x) \), and then we define the \( 1/k \)-Eulerian polynomials of type \( B \) as well as the \( 1/k \)-derangement polynomials of type \( B \).

Let \( \mathfrak{S}_n \) denote the symmetric group of all permutations of \( [n] = \{1, 2, \ldots, n\} \) and let \( \pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n \). A descent (resp. ascent, excedance) of \( \pi \) is an index \( i \in [n-1] \) such that \( \pi(i) > \pi(i+1) \) (resp. \( \pi(i) < \pi(i+1) \), \( \pi(i) > i \)). Let \( \text{des}(\pi) \) (resp. \( \text{asc}(\pi) \), \( \text{exc}(\pi) \)) denote the number of descents (resp. ascents, excedances) of \( \pi \). It is well known that the statistics \( \text{des}(\pi), \text{asc}(\pi) \) and \( \text{exc}(\pi) \) are equidistributed over \( \mathfrak{S}_n \), and their common enumerative polynomial is the Eulerian polynomial \( A_n(x) \), i.e.,

\[
A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{asc}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)}.
\]

In [13], Foata and Schützenberger introduced a \( q \)-analog of \( A_n(x) \) defined by

\[
A_n(x, q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)},
\]

where \( \text{cyc}(\pi) \) is the number of cycles of \( \pi \). Brenti [6] showed that some crucial properties of Eulerian polynomials have nice \( q \)-analogues for the polynomials \( A_n(x, q) \). According to [6, Proposition 7.3], we have

\[
\sum_{n=0}^{\infty} A_n(x, q) \frac{z^n}{n!} = \left( \frac{1 - x}{e^z(x-1) - x} \right)^q.
\]

By comparing this with (1), one can immediately get that

\[
A_n^{(k)}(x) = k^n A_n(x, 1/k) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} k^{n-\text{cyc}(\pi)}.
\]

A left-to-right minimum in \( \pi \) is an index \( i \) such that \( \pi(i) < \pi(j) \) for any \( j < i \) or \( i = 1 \). Let \( \text{lrmin}(\pi) \) denote the number of left-to-right minima of \( \pi \). By using the fundamental
transformation of Foata and Schützenberger [13], the pairs of statistics (exc, cyc) and (asc, lrmin) are equidistributed over $\mathfrak{S}_n$. Thus

$$\sum_{\pi \in \mathfrak{D}_n} x^{\text{asc} (\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{asc} (\pi)} k^{n-\text{lrmin} (\pi)}. \quad (2)$$

A bijective proof of (2) was recently given in [7]. According to [18, Theorem 2], the $1/k$-Eulerian polynomial $A_n^{(k)}(x)$ is also the longest ascent plateau polynomial of $k$-Stirling permutations of order $n$.

Let $f(x) = \sum_{i=0}^n f_i x^i$ be a symmetric polynomial, i.e., $f_i = f_{n-i}$ for any $0 \leq i \leq n$. Then $f(x)$ can be expanded uniquely as $f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1 + x)^{n-2k}$, and it is said to be $\gamma$-positive if $\gamma_k \geq 0$ for $0 \leq k \leq \lfloor n/2 \rfloor$ (see [15, 16]). The $\gamma$-positivity of $f(x)$ implies unimodality of $f(x)$. We refer the reader to Athanasiadis’s survey article [1] for details. The $\gamma$-positivity of Eulerian polynomials was first obtained by Foata and Schützenberger [13]. Subsequently, Foata and Strehl [14] proved the $\gamma$-positivity of Eulerian polynomials by using a group action. Using the theory of enriched $P$-partitions, Stembridge [28, Remark 4.8] showed that

$$A_n(x) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 4^i P(n, i) x^i (1 + x)^{n-2i},$$

where $P(n, i)$ is the number of permutations in $\mathfrak{S}_n$ with $i$ interior peaks, i.e., the indices $i \in \{2, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. It should be noted that if $k \neq 1$, then the polynomial $A_n^{(k)}(x)$ is not symmetric, and so it is not $\gamma$-positive.

A permutation $\pi \in \mathfrak{S}_n$ is a derangement if it has no fixed points, i.e., $\pi(i) \neq i$ for all $i \in [n]$. Let $\mathfrak{D}_n$ be the set of derangements in $\mathfrak{S}_n$, and let $d_n(x) = \sum_{\pi \in \mathfrak{D}_n} x^{\text{exc} (\pi)}$ be the derangement polynomials. It is well known that the generating function of $d_n(x)$ is given as follows (see [4, Proposition 6]):

$$d(x, z) = \sum_{n \geq 0} d_n(x) \frac{z^n}{n!} = \frac{1 - x}{e^{xz} - xe^z}. \quad (3)$$

Using continued fractions, Shin and Zeng [24, Theorem 11] obtained the following result.

**Theorem 1.** For $n \geq 2$, we have

$$\sum_{\pi \in \mathfrak{D}_n} x^{\text{exc} (\pi)} q^{\text{cyc} (\pi)} = \sum_{i=1}^{\lfloor n/2 \rfloor} c_{n,k}(q) x^k (1 + x)^{n-2k},$$

where $c_{n,k}(q) = \sum_{\pi \in \mathfrak{D}_n(k)} q^{\text{cyc} (\pi)}$ and $\mathfrak{D}_n(k)$ is the subset of derangements in $\mathfrak{D}_n$ with exactly $k$ cyclic valleys and without cyclic double descents.

Let $\pm [n] = [n] \cup \{-1, \ldots, -n\}$. Let $B_n$ be the hyperoctahedral group of rank $n$ and let $w = w(1)w(2) \cdots w(n) \in B_n$. Elements of $B_n$ are permutations of $\pm [n]$ with the property that $w(-i) = -w(i)$ for all $i \in [n]$. Let

$$\text{des}_B (w) = \# \{ i \in \{0, 1, \ldots, n-1\} \mid w(i) > w(i+1) \},$$

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where \( w(0) = 0 \). As usual, we denote by \( \bar{i} \) the negative element \(-i\). We say that \( i \in [n] \) is a weak excedance of \( w \) if \( w(i) = i \) or \( w(|w(i)|) > w(i) \) (see [5, p. 431]). An excedance of \( w \) is an index \( i \in [n] \) such that \( w(|w(i)|) > w(i) \). A fixed point (resp. singleton) of \( w \) is an index \( i \in [n] \) such that \( w(i) = i \) (resp. \( w(i) = \bar{i} \)). Let \( \text{wexc} (w) \) (resp. \( \text{exc} (w) \), \( \text{fix} (w) \), \( \text{single} (\pi) \)) denote the number of weak excedances (resp. excedances, fixed points, singletons) of \( w \). By definition, we have \( \text{wexc} (w) = \text{exc} (w) + \text{fix} (w) \). According to [5, Theorem 3.15], the statistics \( \text{des}_B(w) \) and \( \text{wexc} (w) \) have the same distribution over \( B_n \), and their common enumerative polynomial is the Eulerian polynomial of type \( B \):

\[
B_n(x) = \sum_{w \in B_n} x^{\text{des}_B(w)} = \sum_{w \in B_n} x^{\text{wexc}(w)}.
\]

A left peak of \( \pi \in \mathfrak{S}_n \) is an index \( i \in [n-1] \) such that \( \pi(i-1) < \pi(i) > \pi(i+1) \), where we take \( \pi(0) = 0 \). Let \( \text{lpk}(\pi) \) be the number of left peaks in \( \pi \). Let \( Q(n,i) \) be the number of permutations in \( \mathfrak{S}_n \) with \( i \) left peaks. By using the theory of enriched \( P \)-partitions, Petersen [21, Proposition 4.15] obtained the following result.

**Theorem 2.** We have \( B_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} 4^i Q(n,i) x^i (1+ x)^{n-2i} \).

In recent years, various refinements of Theorem 2 have been studied by several authors, see [16, 25, 29] and references therein.

For \( w \in B_n \), we say that \( w \) is a type \( B \) derangement if \( \text{fix}(w) = 0 \). Let \( D_B^n \) be the set of all type \( B \) derangements in \( B_n \). Clearly, \( \text{wexc} (w) = \text{exc} (w) \) for \( w \in D_B^n \). The type \( B \) derangement polynomials \( d_n^B(x) \) are defined by

\[
d_n^B(x) = \sum_{\pi \in D_B^n} x^{\text{exc}(\pi)},
\]

which have been studied by Chen et al. [9] in a slightly different form. According to [11, Theorem 3.2], the generating function of \( d_n^B(x) \) is given as follows:

\[
\sum_{n=0}^{\infty} d_n^B(x) \frac{z^n}{n!} = \frac{(1-x)e^z}{e^{2xz} - xe^{2xz}}.
\] (4)

Combining (3) and (4), we obtain \( d_n^B(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i} 2^i d_i(x) \).

The type \( B \) \( 1/k \)-Eulerian polynomials \( B_{n}^{(k)}(x) \) and the type \( B \) \( 1/k \)-derangement polynomials \( d_{n}^{B}(x;k) \) are defined by using the following generating functions:

\[
\sum_{n=0}^{\infty} B_{n}^{(k)}(x) \frac{z^n}{n!} = \left( \frac{(1-x)e^{kz(1-x)}}{1-xe^{2kz(1-x)}} \right)^{\frac{1}{k}},
\] (5)

\[
\sum_{n=0}^{\infty} d_{n}^{B}(x;k) \frac{z^n}{n!} = \left( \frac{(1-x)e^{kz}}{e^{2kxz} - xe^{2kxz}} \right)^{\frac{1}{k}}.
\] (6)
In particular, \( B_n^{(1)}(x) = B_n(x) \) and \( d_n^B(x; 1) = d_n^B(x) \). Comparing (5) with (6), we have
\[
B_n^{(k)}(x) = \sum_{i=0}^{n} \binom{n}{i} d_n^B(x; k)x^{n-i}.
\]

This paper is organized as follows. In the next section, we present the main results. In particular, we show that the type \( B \frac{1}{k} \)-Eulerian polynomials \( B_n^{(k)}(x) \) are \( \gamma \)-positive when \( k \) is positive and the type \( B \frac{1}{k} \)-derangement polynomials \( d_n^B(x; k) \) are bi-\( \gamma \)-positive when \( k \geq 1/2 \). In Sections 3 and 4, we respectively prove Theorem 4 and Theorem 9.

2 Main results

2.1 The \( 1/k \)-Eulerian polynomials of type \( B \)

An element is a left-to-right maximum of \( \pi \in S_n \) if it is larger than or equal to all the elements to its left. We always assume that \( \pi(1) \) is a left-to-right maximum. Let \( \text{lrmax}(\pi) \) be the number of left-to-right maxima of \( \pi \). We can write \( \pi \in S_n \) in standard cycle decomposition, where each cycle is written with its largest entry first and the cycles are written in increasing order of their largest entry. A cycle peak of \( \pi \) is an index \( i \) such that \( \pi^{-1}(i) < i > \pi(i) \). Let \( \text{cpk}(\pi) \) be the number of cycle peaks of \( \pi \). By using the fundamental transformation of Foata and Schützenberger [13], it is easy to verify that
\[
\sum_{\pi \in S_n} x^{\text{cpk}(\pi)} y^{\text{cyc}(\pi)} = \sum_{\pi \in S_n} x^{\text{lpk}(\pi)} y^{\text{lrmax}(\pi)}.
\]

In the following discussion, we always write \( w \in B_n \) by using its standard cycle decomposition, in which each cycle is written with its largest entry last and the cycles are written in ascending order of their last entry. It should be noted that the \( n \) letters appearing in the cycle notation of \( w \in B_n \) are the letters \( w(1), w(2), \ldots, w(n) \). Let \( \text{cyc}(w) \) be the number of cycles of \( w \).

Example 3. The signed permutation \( w = 351724689 \) can be written as
\[
(9)(3, 1)(2, 5)(4, 7, 6)(8).
\]
Moreover, \( w \) has only one singleton 9, one fixed point 8, \( \text{cyc}(w) = 5 \) and \( \text{exc}(w) = 3 \).

We can now present the first main result of this paper.

Theorem 4. (i) For \( n \geq 1 \), we have
\[
B_n^{(k)}(x) = \sum_{w \in B_n} x^{\text{wexc}(w)} n^{n-\text{cyc}(w)},
\]
and the polynomials \( B_n^{(k)}(x) \) satisfy the recurrence relation
\[
B_{n+1}^{(k)}(x) = (1 + x + 2knx)B_n^{(k)}(x) + 2kx(1 - x) \frac{d}{dx} B_n^{(k)}(x), \tag{7}
\]
with the initial conditions $B_0^{(k)}(x) = 1$ and $B_1^{(k)}(x) = 1 + x$;

(ii) When $k > 0$, the polynomial $B_n^{(k)}(x)$ is $\gamma$-positive. More precisely, we have

$$B_n^{(k)}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=i}^{n-1} b_{n,i,j} x^i (1 + x)^{n-2i},$$

where the numbers $b_{n,i,j}$ satisfy the recurrence relation

$$b_{n+1,i,j} = b_{n,i,j} + 2ib_{n,i,j-1} + (n - 2i + 2)b_{n,i-1,j-1},$$

with $b_{1,0,0} = 1$ and $b_{1,i,j} = 0$ for $(i, j) \neq (0, 0)$;

(iii) For $n \geq 1$, we define

$$b_n(x, q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=i}^{n-1} b_{n,i,j} x^i q^j.$$ 

Set $b_0(x, q) = 1$. Then the generating function of $b_n(x, q)$ is given as follows:

$$b(x, q, z) = \sum_{n=0}^{\infty} b_n(x, q) \frac{z^n}{n!} = \left( \frac{\sqrt{1-x}}{\sqrt{1-x} \cosh(qz\sqrt{1-x}) - \sinh(qz\sqrt{1-x})} \right)^{\frac{1}{q}};$$

(iv) For $n \geq 1$, we have

$$b_n(x, q) = \sum_{\pi \in S_n} x^{\text{cpk}(\pi)} q^{n-\text{cyc}(\pi)};$$

(v) For $n \geq 1$, we have

$$B_{n+1}^{(k)}(x) = (1 + x)B_n^{(k)}(x) + x \sum_{i=0}^{n-1} \binom{n}{i} 2^{n+1-i} k^{n-i} B_i^{(k)}(x) A_{n-i}(x).$$

When $q = 1$, the generating function $b(x, q, z)$ reduces to the generating function of the polynomials $\sum_{i=0}^{\lfloor n/2 \rfloor} Q(n,i)x^i$, which is due to Gessel [26, A008971]. Thus the polynomial $b_n(x, q)$ can be called the $1/q$-left peak polynomial. From the explicit formula of $b(x, q, z)$, it is routine to verify the following result.

**Corollary 5.** For $n \geq 1$, we have

$$\sum_{\pi \in S_n} x^{\text{cpk}(\pi)} (-1)^{n-\text{cyc}(\pi)} = (1 - x)^{\lfloor n/2 \rfloor}.$$ 

Let

$$T(n, i) = \frac{n!}{i!(n-2i)!2^i}$$

be the Bessel number, which is the number of involutions of $[n]$ with $i$ pairs. In other words, the number $T(n, i)$ counts involutions of $[n]$ with $n-i$ cycles. Note that

$$b_{n,i,i} = \# \{ \pi \in S_n : \text{cpk}(\pi) = i, \text{cyc}(\pi) = n-i \}.$$ 

So the following corollary is immediate.
Corollary 6. For $0 \leq i \leq \lfloor n/2 \rfloor$, we have $b_{n,i,i} = T(n,i)$.

It follows from (9) that

$$b_{n+1,i,n} = 2ib_{n,i,n} + (n - 2i + 2)b_{n,j-1,n-1}. \quad (12)$$

Let $\pi \in \mathfrak{S}_{n+1}$ with $\text{cpk} (\pi) = i + 1$ and $\text{cyc} (\pi) = 1$. We write $\pi$ in standard cycle decomposition. If $\pi'$ is obtained from $\pi$ by deleting its parentheses and the element $n + 1$, then $\pi'$ is a permutation in $\mathfrak{S}_n$ with $i$ interior peaks. So we get the following corollary.

Corollary 7. For $0 \leq i \leq \lfloor (n - 1)/2 \rfloor$, the number $b_{n+1,i+1,n}$ equals the number of permutations in $\mathfrak{S}_n$ with $i$ interior peaks.

2.2 The $1/k$-derangement polynomials of type $B$

Let $p(x) = \sum_{i=0}^d p_i x^i$. There is a unique decomposition: $p(x) = a(x) + xb(x)$, where

$$a(x) = \frac{p(x) - x^{d+1}p(1/x)}{1 - x}, \quad b(x) = \frac{x^d p(1/x) - p(x)}{1 - x}. \quad (13)$$

It is clear that $a(x)$ and $b(x)$ are symmetric polynomials satisfying $a(x) = x^d a(\frac{1}{x})$ and $b(x) = x^{d-1} b(\frac{1}{x})$. We call the ordered pair of polynomials $(a(x), b(x))$ the symmetric decomposition of $p(x)$ (see [2]).

Definition 8. Let $(a(x), b(x))$ be the symmetric decomposition of $p(x)$. If $a(x)$ and $b(x)$ are both $\gamma$-positive, then we say that $p(x)$ is bi-$\gamma$-positive.

We say that $p(x)$ is alternatingly increasing if $p_0 \leq p_d \leq p_{d-1} \leq \cdots \leq p_{\lfloor \frac{d+1}{2} \rfloor}$.

As pointed out by Brändén and Solus [3], the polynomial $p(x)$ is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have nonnegative coefficients. Thus the bi-$\gamma$-positivity of $p(x)$ implies that $p(x)$ is alternatingly increasing.

We now present a counterpart of Theorem 1.

Theorem 9. For $n \geq 1$, we have

$$d_n^B(x; k) = \sum_{\pi \in D_n^B} x^{\text{exc} (\pi)} k^{n - \text{cyc} (\pi)}. \quad (14)$$

When $k \geq 1/2$, the polynomials $d_n^B(x; k)$ are bi-$\gamma$-positive. More precisely, we have

$$d_n^B(x; k) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} p(n,j;k)x^j(1+x)^{n-2j} + \sum_{j=0}^{\lfloor n/2 \rfloor} q(n,j;k)x^j(1+x)^{n-2j}, \quad (15)$$
where the numbers \( p(n,j;k) \) and \( q(n,j;k) \) satisfy the following recurrence system:

\[
p(n + 1, j; k) = (1 + 2kj)p(n, j; k) + 4k(n - 2j + 1)p(n, j-1; k) + 2kn\sqrt{p(n-1, j-1; k)} + q(n, j; k),
\]

\[
q(n + 1, j; k) = 2kjq(n, j; k) + 4k(n - 2j + 2)q(n, j-1; k) + 2knq(n-1, j-1; k) + (2k - 1)p(n, j-1; k),
\]

with the initial conditions \( q(0,0; k) = 1, \ q(0, j; k) = 0 \) for \( j \neq 0, \ p(0, j; k) = 0 \) for any \( j \).

For \( n \geq 1 \), we define

\[
P_n(x; k) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} p(n, j; k)x^j(1 + x)^{n-1-2j},
\]

\[
Q_n(x; k) = \sum_{j=0}^{\lfloor n/2 \rfloor} q(n, j; k)x^j(1 + x)^{n-2j}.
\]

The first few \( P_n(x; k) \) and \( Q_n(x; k) \) are given as follows:

\[
P_1(x; k) = 1, \ P_2(x; k) = 1 + x, \ P_3(x; k) = 1 + (1 + 12k)x + x^2,
\]

\[
Q_1(x; k) = 0, \ Q_2(x; k) = (4k - 1)x, \ Q_3(x; k) = (8k^2 - 1)x(1 + x).
\]

**Corollary 10.** The polynomials \( P_n(x; k) \) and \( Q_n(x; k) \) satisfy the recurrence system

\[
P_{n+1}(x; k) = (1 + (2kn - 2k + 1)x)P_n(x; k) + 2kx(1 - x)P_n'(x; k) + 2knXP_{n-1}(x; k) + Q_n(x; k),
\]

\[
Q_{n+1}(x; k) = 2knxQ_n(x; k) + 2kx(1 - x)Q_n'(x; k) + 2knxQ_{n-1}(x; k) + (2k - 1)xP_n(x; k),
\]

with the initial conditions \( P_0(x; k) = 0, \ P_1(x; k) = 1, Q_0(x; k) = 1 \) and \( Q_1(x; k) = 0 \).

**Proof.** For \( n \geq 1 \), we define

\[
p_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} p(n, j; k)x^j, \ q_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} q(n, j; k)x^j.
\]

Multiplying both sides of the recurrence system of the numbers \( p(n,j;k) \) and \( q(n,j;k) \) by \( x^j \) and summing over all \( j \), we get the following recurrence system:

\[
p_{n+1}(x) = (1 + 4k(n - 1)x)p_n(x) + 2kx(1 - 4x)p_n'(x) + 2knxp_{n-1}(x) + q_n(x),
\]

\[
q_{n+1}(x) = 4knxq_n(x) + 2kx(1 - 4x)q_n'(x) + 2knxq_{n-1}(x) + (2k - 1)xp_n(x),
\]
with the initial conditions $p_0(x) = 0$, $p_1(x) = 1$, $p_2(x) = 1$, $q_0(x) = 1$, $q_1(x) = 0$ and $q_2(x) = (4k - 1)x$. For $n \geq 1$, we have

$$P_n(x; k) = (1 + x)^{n-1} p_n \left( \frac{x}{(1 + x)^2} \right),$$

$$Q_n(x; k) = (1 + x)^n q_n \left( \frac{x}{(1 + x)^2} \right).$$

Substituting $x \to x/(1 + x)^2$ into the recurrence system of the polynomials $p_n(x)$ and $q_n(x)$ and simplifying some terms leads to the desired result.

A *succession* of $\pi \in \mathfrak{S}_n$ is an index $i$ such that $\pi(i + 1) = \pi(i) + 1$, where $i \in [n - 1]$. Let $\mathfrak{S}_n^*$ denote the set of permutations in $\mathfrak{S}_n$ with no successions. We can now give the following result.

**Theorem 11.** For $n \geq 1$, we have

$$d_n^{B}(x; 1/2) = \frac{1}{x} d_{n+1}(x) + d_n(x),$$

where $d_n(x)$ is the derangement polynomial. Moreover, we have

$$d_n^{B}(x; 1/2) = \sum_{\pi \in \mathfrak{S}_{n+1}^*} x^{\text{asc}(\pi)}. \quad (16)$$

**Proof.** Let $P_n(x) = P_n(x; 1/2)$ and $Q_n(x) = Q_n(x; 1/2)$. It follows from Theorem 9 that $d_n^{B}(x; 1/2) = P_n(x) + Q_n(x)$. By using Corollary 10, we see that the polynomials $P_n(x)$ and $Q_n(x)$ satisfy the following recurrence system:

$$P_{n+1}(x) = (1 + nx) P_n(x) + x(1 - x) P'_n(x) + n x P_{n-1}(x) + Q_n(x),$$

$$Q_{n+1}(x) = n x Q_n(x) + x(1 - x) Q'_n(x) + n x Q_{n-1}(x),$$

with the initial conditions $P_0(x) = 0$, $P_1(x) = 1$, $Q_0(x) = 1$ and $Q_1(x) = 0$. According to [17, Eq. (3.2)], the polynomials $Q_n(x)$ satisfy the same recurrence relation and initial conditions as $d_n(x)$, so they agree. We now prove that

$$P_n(x) = \frac{1}{x} d_{n+1}(x).$$

Clearly, it holds for $n = 0, 1, 2$. Assume it holds for $n$. Then we get

$$P_{n+1}(x) = \frac{1 + nx}{x} d_{n+1}(x) + \frac{x(1 - x)}{x^2} (x d_{n+1}'(x) - d_{n+1}(x)) + \frac{n x}{x} d_n(x) + d_n(x)$$

$$= (n + 1) d_{n+1}(x) + (1 - x) d_{n+1}'(x) + (n + 1) d_n(x)$$

$$= \frac{1}{x} d_{n+2}(x),$$

as desired. The combinatorial interpretation (16) follows immediately from [22, Eq. (3.8)]. This completes the proof.
Let

\[ S_n(x) = \sum_{\pi \in S_n} x^{\text{asc}(\pi)}. \]

Combining (6) and (16), we obtain

\[ \sum_{n=0}^{\infty} S_n(x) \frac{z^n}{n!} = e^z \left( \frac{1 - x}{ez - xe^z} \right)^2. \]  

(17)

Combining (1), (3), (4) and (17), we get the following result.

**Theorem 12.** For \( n \geq 0 \), we have

\[ S_n(x) = \frac{1}{2n} \sum_{i=0}^{n} \binom{n}{i} d_i^B(x) d_{n-i}^B(x), \]

\[ S_n(x) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) d_{n-i}(x). \]

3 Proof of Theorem 4

In this section we complete the proof of Theorem 4 by using the theory of context-free grammars. For an alphabet \( V \), let \( \mathbb{Q}[[V]] \) be the rational commutative ring of formal power series in monomials formed from letters in \( V \). A context-free grammar over \( V \) is a function \( G : V \to \mathbb{Q}[[V]] \) that replaces a letter in \( V \) by an element of \( \mathbb{Q}[[V]] \) (see [8, 12]). The formal derivative \( D_G \) is a linear operator defined with respect to a grammar \( G \). In other words, \( D_G \) is the unique derivation satisfying \( D_G(x) = G(x) \) for \( x \in V \), and for any two formal functions \( u \) and \( v \), we have

\[ D_G(u + v) = D_G(u) + D_G(v), \quad D_G(uv) = D_G(u)v +uD_G(v). \]

For a constant \( c \), we have \( D_G(c) = 0 \). It follows from Leibniz’s rule that

\[ D_G^k(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u) D_G^{n-k}(v). \]  

(18)

For example, if \( G = \{ x \to xy, y \to y \} \), then

\[ D_G(x) = xy, \quad D_G(y) = y, \quad D_G^2(x) = D_G(xy) = xy^2 + xy. \]

A grammatical labeling is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar (see [10]). Following [20, Definition 1], a change of grammars is a substitution method in which the original grammars are replaced with functions of other grammars.

In the following discussion, we always write \( w \in B_n \) by its standard cycle decomposition. For \( w \in B_n \), we say that \( i \in [n] \) is an anti-excedance of \( w \) if \( w(i) = i \) or \( w(i) > w(|w(i)|) \). Let \( \text{aexc}(w) \) be the number of anti-excedances of \( w \). It is clear that \( \text{wexc}(w) + \text{aexc}(w) = n \) for \( w \in B_n \). The following lemma is fundamental.
Lemma 13. Let \( G = \{ I \to I(x + y), \ x \to 2kxy, \ y \to 2kxy \} \). We have

\[
D_G^n(I) = I \sum_{w \in B_n} x^{\text{wexc}(w)} y^{\text{awexc}(w)} k^{n-\text{cyc}(w)}. \tag{19}
\]

Proof. We first introduce a grammatical labeling of \( w \in B_n \) as follows:

(L1) If \( w(i) = i \), then put a superscript label \( x \) right after \( i \), i.e., \( (i^x) \);

(L2) If \( w(i) = \tilde{i} \), then put a superscript label \( y \) right after \( \tilde{i} \), i.e., \( (\tilde{i}^y) \);

(L3) If \( w(i) < w(|w(i)|) \), then put a superscript label \( x \) right after \( w(i) \);

(L4) If \( w(i) > w(|w(i)|) \), then put a superscript label \( y \) right after \( w(i) \);

(L5) Put a subscript label \( k \) just before every element of \( w \) except the first element in each cycle;

(L6) Put a subscript label \( I \) right after \( w \).

The weight of \( w \) is the product of its labels. Note that the weight of \( w \) is given by

\[
I x^{\text{wexc}(w)} y^{\text{awexc}(w)} k^{n-\text{cyc}(w)}. \]

Every permutation in \( B_n \) can be obtained from a permutation in \( B_{n-1} \) by inserting \( n \) or \( \overline{n} \). For \( n = 1 \), we have \( B_1 = \{ (1^x)_I, (\overline{1}^y)_I \} \). Note that \( D(I) = I(x + y) \). Then the sum of weights of the elements in \( B_1 \) is given by \( D(I) \). Hence the result holds for \( n = 1 \). We proceed by induction on \( n \). Suppose that we get all labeled permutations in \( w \in B_{n-1} \), where \( n \geq 2 \). Let \( \tilde{w} \) be obtained from \( w \in B_{n-1} \) by inserting \( n \) or \( \overline{n} \). When the inserted \( n \) or \( \overline{n} \) forms a new cycle, the insertion corresponds to the substitution rule \( I \to I(x + y) \). Recall that each cycle of a signed permutation is written with its largest entry last and the cycles are written in ascending order of their last entry. Now we insert \( n \) or \( \overline{n} \) right after \( w(i) \). If \( i \) is a weak excess of \( w \), then the changes of labeling are illustrated as follows:

\[
\cdots (i^x) \cdots \mapsto \cdots (i_k^x n^y) \cdots ; \quad \cdots (i^x) \cdots \mapsto \cdots (\overline{i_k}^{\overline{x}} \overline{n}^y) \cdots ;
\]

\[
\cdots (\cdots w(i)_k^x w(|w(i)|)) \cdots \mapsto \cdots (w(|w(i)|)_k \cdots w(i)_k^x n^y) ; \quad \cdots (\cdots w(i)_k^x w(|w(i)|)) \cdots \mapsto \cdots (\cdots w(i)_k^x \overline{n}^y) ;
\]

If \( i \) is an anti-excess of \( w \), then the changes of labeling are illustrated as follows:

\[
\cdots (\overline{i}^y) \cdots \mapsto \cdots (i_k^x n^y) \cdots ; \quad \cdots (\overline{i}^y) \cdots \mapsto \cdots (\overline{i_k}^{\overline{x}} \overline{n}^y) \cdots ;
\]

\[
\cdots (\cdots w(i)_k^x w(|w(i)|)) \cdots \mapsto \cdots (w(|w(i)|)_k \cdots w(i)_k^x n^y) ; \quad \cdots (\cdots w(i)_k^x w(|w(i)|)) \cdots \mapsto \cdots (\cdots w(i)_k^x \overline{n}^y) ;
\]

In each case, the insertion of \( n \) or \( \overline{n} \) corresponds to one substitution rule in \( G \). By induction, it is routine to check that the action of \( D_G \) on elements of \( B_{n-1} \) generates all elements of \( B_n \). This completes the proof.

\( \square \)
Let
\[ F_n(x, y; k) = \sum_{w \in B_n} x^{\text{we}c(w)} y^{\text{we}c(w)} k^{n - \text{cyc}(w)}. \]

**Lemma 14.** We have
\[ F(x, y, z; k) = \sum_{n=0}^{\infty} F_n(x, y; k) \frac{z^n}{n!} = \left( \frac{(y - x)e^{kz(y-x)}}{y - x e^{2kz(y-x)}} \right)^{\frac{1}{k}}. \]

**Proof.** From Lemma 13, we obtain \( D^n_G(I) = IF_n(x, y; k) \). By using \( D^2_G(I) = D_G(IF_n(x, y; k)) \),
we get that the polynomials \( F_n(x, y; k) \) satisfy the following recurrence relation
\[ F_{n+1}(x, y; k) = (x + y)F_n(x, y; k) + 2kxy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F_n(x, y; k), \tag{20} \]
with the initial conditions \( F_0(x, y; k) = 1 \) and \( F_1(x, y; k) = x + y \). By rewriting (20) in terms of the generating function \( F := F(x, y, z; k) \), we have
\[ \frac{\partial}{\partial z} F = (x + y)F + 2kxy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) F. \tag{21} \]
It is routine to check that the generating function
\[ \tilde{F} := \tilde{F}(x, y, z; k) = \left( \frac{(y - x)e^{kz(y-x)}}{y - x e^{2kz(y-x)}} \right)^{\frac{1}{k}} \]
satisfies (21). Also, this generating function gives \( \tilde{F}(x, y, 0; k) = 1 \). Hence \( F = \tilde{F} \). \qed

**Proof of Theorem 4.** We divide our proof into five parts.

(i) Comparing (5) with Lemma 14, we obtain
\[ F_n(x, y; k) = y^n B_n^{(k)} \left( \frac{x}{y} \right). \tag{22} \]
Therefore,
\[ B_n^{(k)}(x) = \sum_{w \in B_n} x^{\text{we}c(w)} k^{n - \text{cyc}(w)}. \]
Combining (20) and (22), it is routine to verify (7).

(ii) We now consider a change of the grammar \( G \) given in Lemma 13. Setting \( u = x + y \) and \( v = xy \), we get \( D_G(I) = Iu, \ D_G(u) = 4kv \) and \( D_G(v) = 2kwv \). We define
\[ G_1 = \{ I \rightarrow Iu, \ u \rightarrow 4kv, \ v \rightarrow 2kwv \}. \tag{23} \]
Note that \( D_G^1(I) = Iu \), \( D_G^2(I) = I(u^2 + 4kv) \), \( D_G^3(I) = I(u^3 + (12k + 8k^2)uv) \) and \( D_G^4(I) = I(u^4 + (24k + 32k^2 + 16k^3)u^2v + (48k^2 + 32k^3)v^2) \). By induction, it is routine to verify that there exist nonnegative integers \( b_{n,i,j} \) such that

\[
D^n_G(I) = I \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=i}^{n-1} b_{n,i,j} k^j 4^i v^i u^{n-2i}. \tag{24}
\]

It should be noted that \( b_{n,i,j} = 0 \) if \( i \) and \( j \) are outside the bounds given in (24). Note that

\[
D^{n+1}_G(I) = D_G \left( I \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=i}^{n-1} b_{n,i,j} k^j 4^i v^i u^{n-2i} \right) = \sum_{i,j} b_{n,i,j} k^j 4^i v^i \left( u^{n-2i+1} + 2kiu^{n-2i+1} + 4k(n - 2i)vu^{n-2i-1} \right).
\]

Equating the coefficients of \( k^j 4^i v^i u^{n+1-2i} \) in both sides of the above expression, we get the recurrence relation (9). Since \( D_G^1(I) = Iu \), we see that \( b_{1,0,0} = 1 \) and \( b_{1,i,j} = 0 \) if \((i,j) \neq (0,0)\). Taking \( u = x + y \), \( v = xy \) in (24) and then setting \( y = 1 \), we get (8).

(iii) Multiplying both sides of (9) by \( x^i q^j \) and summing over all \( i \) and \( j \), we obtain

\[
b_{n+1}(x, q) = (1 + nqx)b_n(x, q) + 2qx(1 - x) \frac{\partial}{\partial x} b_n(x, q). \tag{25}
\]

In particular, \( b_1(x, q) = 1 \) and \( b_2(x, q) = 1 + qx \). Set \( b = b(x, q, z) \). By rewriting (25) in terms of \( b \), we have

\[
(1 - qxz) \frac{\partial b}{\partial z} = b + 2qx(1 - x) \frac{\partial b}{\partial x}. \tag{26}
\]

It is routine to verify that

\[
\tilde{b}(x, q, z) = \left( \frac{1-x}{\sqrt{1-x} \cosh(qz\sqrt{1-x}) - \sinh(qz\sqrt{1-x})} \right)^{1/z}
\]

satisfies (26). Also, this generating function gives \( \tilde{b}(x, q, 0) = 1 \) and \( \tilde{b}(0, q, z) = e^z \). Hence \( \tilde{b}(x, q, z) = b(x, q, z) \).

(iv) Assume that (10) holds for \( n \). Let \( \pi \in S_n \), and let \( \pi_i \) be an element of \( S_{n+1} \) obtained from \( \pi \) by inserting the entry \( n + 1 \) right after \( i \) if \( i \in [n] \) or as a new cycle \((n + 1) \) if \( i = n + 1 \). It is clear that

\[
\text{cyc} (\pi_i) = \begin{cases} 
\text{cyc} (\pi), & \text{if } i \in [n]; \\
\text{cyc} (\pi) + 1, & \text{if } i = n + 1.
\end{cases}
\]
Therefore, we have

\[ b_{n+1}(x, q) = \sum_{i=1}^{n+1} \sum_{\pi \in S_n} x^{cpk(\pi)} q^{n+1-cyc(\pi)} \]

\[ = \sum_{\pi \in S_n} x^{cpk(\pi)} q^{n-cyc(\pi)} + \sum_{i=1}^{n} \sum_{\pi \in S_n} x^{cpk(\pi)} q^{n+1-cyc(\pi)} \]

\[ = b_n(x, q) + \sum_{\pi \in S_n} (2cpk(\pi) x^{cpk(\pi)} + (n - 2cpk(\pi)) x^{cpk(\pi)+1}) q^{n+1-cyc(\pi)} \]

\[ = b_n(x, q) + nqxb_n(x, q) + 2q(1-x) \sum_{\pi \in S_n} cpk(\pi) x^{cpk(\pi)} q^{n-cyc(\pi)}, \]

and (25) follows. Thus (10) holds for \( n + 1. \)

(v) Let \( G \) be the grammar given in Lemma 13. It follows from (19) that

\[ D^n_G(I) = I \sum_{w \in B_n} x^{wexc(w)} y^{n-wexc(w)} k^{n-cyc(w)}. \]

Setting \( y = 1, \) we get \( D^n_G(I)|_{y=1} = IB_n^{(k)}(x). \) Dumont [12] discovered that if

\[ G_2 = \{ x \rightarrow xy, y \rightarrow xy \}, \]

then we have

\[ D^n_{G_2}(x) = x \sum_{\pi \in S_n} x^{exc(\pi)} y^{n-exc(\pi)} = xy^n A_n \left( \frac{x}{y} \right). \] (27)

By using (27), it is easy to verify that for \( n \geq 1, \) we have

\[ D^n_G(x + y) = 2^{n+1}k^n x \sum_{\pi \in S_n} x^{exc(\pi)} y^{n-exc(\pi)} = 2^{n+1}k^n xy^n A_n \left( \frac{x}{y} \right). \]

It follows from Leibniz’s rule (18) that for \( n \geq 1, \) we have

\[ D^{n+1}_G(I) = \sum_{i=0}^{n} \binom{n}{i} D^i_G(I) D^{n-i}_G(x + y) \]

\[ = (x + y) D^n_G(I) + \sum_{i=0}^{n-1} \binom{n}{i} D^i_G(I) D^{n-i}_G(x + y). \]

Setting \( y = 1 \) in both sides of the above expression, we immediately get (11).  \( \square \)
4 Proof of Theorem 9

In this section we complete the proof of Theorem 9. A grammatical interpretation of the polynomial $d_n^B(x)$ was given by [19, Theorem 11]. We now give a refinement of Lemma 13.

**Lemma 15.** If $G_4 = \{ I \rightarrow I(y + u), x \rightarrow 2k xy, y \rightarrow 2k xy, u \rightarrow 2k xy \}$, then

$$D_{G_4}^n(I) = I \sum_{w \in B_n} x^{\text{exc}(w)} y^{\text{aexc}(w)} u^{\text{fix}(w)} k^{n - \text{cyc}(w)}. \quad (28)$$

**Proof.** We now introduce a grammatical labeling of $w \in B_n$ as follows:

(L1) If $w(i) < w(|w(i)|)$, then put a superscript label $x$ right after $w(i)$;

(L2) If $w(i) > w(|w(i)|)$ or $w(i) = \bar{i}$, then put a superscript label $y$ right after $w(i)$;

(L3) If $w(i) = i$, then put a superscript label $z$ right after $w(i)$, i.e., $(i^n)$;

(L4) Put a subscript label $I$ right after $w$;

(L5) Put a subscript label $k$ just before every element of $w$ except the first element in each cycle.

Then the weight of $w$ is given by

$$I x^{\text{exc}(w)} y^{\text{aexc}(w)} u^{\text{fix}(w)} k^{n - \text{cyc}(w)}. $$

For $n = 1$, we have $B_1 = \{(1^n)_{I_1}, (\bar{1^n})_{I_1}\}$. Note that $D_{G_4}(I) = I(y + u)$. Thus the sum of weights of the elements in $B_1$ is given by $D_{G_4}(I)$. Hence the result holds for $n = 1$. We proceed by induction on $n$. Suppose that we get all labeled permutations in $B_{n-1}$, where $n \geq 2$. Let $\tilde{w}$ be obtained from $w \in B_{n-1}$ by inserting $n$ or $\bar{n}$. When the inserted $n$ or $\bar{n}$ forms a new cycle, the insertion corresponds to the substitution rule $I \rightarrow I(y + u)$. If $i$ is a weak excedance of $w$, then the changes of labeling are illustrated as follows:

$$\cdots (i^y) \cdots \mapsto \cdots (i^n) \cdots \mapsto \cdots (i^y n^n) \cdots ;$$

$$\cdots (\cdots w(i)^k w(|w(i)|) \cdots) \cdots \mapsto \cdots (w(|w(i)|) \cdots w(i)^n n^n) \cdots ;$$

$$\cdots (\cdots w(i)^y w(|w(i)|) \cdots) \cdots \mapsto \cdots (\cdots w(i)^y \pi^k_w w(|w(i)|) \cdots) \cdots ;$$

If $i$ is an anti-excedance of $w$, then the changes of labeling are illustrated as follows:

$$\cdots (\bar{i}^y) \cdots \mapsto \cdots (\bar{i}^n) \cdots \mapsto \cdots (\bar{i}^y n^n) \cdots ;$$

$$\cdots (\cdots w(i)^k w(|w(i)|) \cdots) \cdots \mapsto \cdots (w(|w(i)|) \cdots w(i)^n n^n) \cdots ;$$

$$\cdots (\cdots w(i)^y w(|w(i)|) \cdots) \cdots \mapsto \cdots (\cdots w(i)^y \pi^k_w w(|w(i)|) \cdots) \cdots .$$

In each case, the insertion of $n$ or $\bar{n}$ corresponds to one substitution rule in $G_4$. By induction, it is routine to check that the action of $D_{G_4}$ on elements of $B_{n-1}$ generates all elements of $B_n$. This completes the proof. \hfill \Box
We define
\[ H_n(x, y, u; k) = \sum_{w \in B_n} x^{\text{exc}(w)} y^{\text{exc}(w)} u^{\text{fix}(w)} k^{n-\text{cyc}(w)}, \]
\[ H := H(x, y, u, z; k) = \sum_{n=0}^{\infty} H_n(x, y, u; k) \frac{z^n}{n!}. \]

Lemma 16. We have
\[ H(x, y, u, z; k) = \left( \frac{(y - x)e^{kz(y+u-2x)}}{y - xe^{2kz(y-x)}} \right)^{\frac{1}{k}}. \]  

Proof. Since \( D_{G_4}^{n+1}(I) = D_{G_4}(IH_n(x, y, u; k)) \), it follows that
\[ D_{G_4}^{n+1}(I) = I(y + u)H_n(x, y, u; k) + 2kxyI \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial u} \right) H_n(x, y, u; k). \]
Thus \( H_{n+1}(x, y, u; k) = (y + u)H_n(x, y, u; k) + 2kxy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial u} \right) H_n(x, y, u; k) \). By rewriting this recurrence relation in terms of the generating function \( H \), we have
\[ \frac{\partial}{\partial z} H = (y + u)H + 2kxy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial u} \right) H. \]

It is routine to check that the generating function
\[ \tilde{H}(x, y, u, z; k) = \left( \frac{(y - x)e^{kz(y+u-2x)}}{y - xe^{2kz(y-x)}} \right)^{\frac{1}{k}} \]
satisfies (30). Note that \( \tilde{H}(x, y, u, 0; k) = 1, \tilde{H}(0, y, u, z; k) = e^{u+y} \) and \( \tilde{H}(x, 0, u, z; k) = e^{uz} \). Hence \( \tilde{H}(x, y, u, z; k) = H(x, y, u, z; k) \). \( \square \)

Let \( w \in B_n \) with exactly one fixed point. Suppose that \( \text{cyc}(w) = k \) and \( w(\ell) = \ell \), i.e., \( \ell \) is the fixed point of \( w \). Then the standard form of \( w \) can be written as \( w = C_1C_2 \cdots C_k \), where \( C_i = (c_{i1}, \cdots, c_{ij}) \), \( 1 \leq i \leq k \) and \( 1 \leq j \leq n \). The reduction of \( w \) is defined by
\[ \text{red}(w) = \text{red}(C_1)\text{red}(C_2) \cdots \text{red}(C_k). \]

If \( C_i = (\ell) \), then \( \text{red}(C_i) = \emptyset \), i.e., we delete the fixed point of \( w \). If \( \# C_i \geq 2 \), then let \( \text{red}(C_i) = (\bar{c}_{i1}, \cdots, \bar{c}_{ij}) \). For \( 1 \leq s \leq j \), the elements \( \bar{c}_{is} \) are defined as follows:

- If \( |c_{is}| < \ell \), then \( \bar{c}_{is} = c_{is} \);
- If \( c_{is} > \ell \), then \( \bar{c}_{is} = c_{is} - 1 \);
- If \( c_{is} < 0 \) and \( |c_{is}| > \ell \), then \( \bar{c}_{is} = c_{is} + 1 \).

It should be noted that \( \text{red}(w) \in B_{n-1} \) with no fixed points and the reduction map of \( w \) does not change the numbers of excedances and anti-excedances of \( w \).
Proof of Theorem 9. Comparing (6) with (29), we immediately get (14). In the following, we shall prove (15). We first consider a change of the grammar given in Lemma 15. Note that $D_{G_4}(I) = Iy + Iu$ and
\[
D_{G_4}(Iy) = I(y^2 + yu + 2kxy) = Iy(x + y) + Iyu + (2k - 1)Ixy.
\]
Setting $a = xy$, $b = x + y$ and $c = Iy$, we get $D_{G_4}(a) = 2kab, D_{G_4}(b) = 4ka$,
\[
D_{G_4}(I) = c + uI, D_{G_4}(c) = (b + u)c + (2k - 1)aI, D_{G_4}(u) = 2ka.
\]

Consider the grammar
\[ G_5 = \{ I \rightarrow c + uI, c \rightarrow (b + u)c + (2k - 1)aI, u \rightarrow 2ka, a \rightarrow 2kab, b \rightarrow 4ka \}. \]

Note that $D_{G_5}(I) = c + Iu$ and $D_{G_5}^2(I) = (b + 2u)c + (u^2 + (4k - 1)a)I$. By induction, it is routine to verify that there exist nonnegative integers $p(n, i, j; k)$ and $q(n, i, j; k)$ such that
\[
D_{G_5}^n(I) = \sum_{i=0}^{n} u^i \left( \sum_{j=0}^{\frac{n-1}{2}} p(n, i, j; k)a^j b^{n-1-i-2j} + \sum_{j=0}^{\frac{n-1}{2}} q(n, i, j; k)a^j b^{n-1-i-2j} I \right). \tag{31}
\]
Combining (31) and Lemma 15, we immediately obtain
\[
\sum_{w \in B_n} x^{\text{exc}(w)} y^{\text{aexc}(w)} u^{\text{fix}(w)} k^{n - \text{cyc}(w)} = \sum_{i=0}^{n} u^i \sum_{j=0}^{\frac{n-1}{2}} p(n, i, j; k)(xy)^j(x + y)^{n-1-i-2j}y + \sum_{i=0}^{n} u^i \sum_{j=0}^{\frac{n-1}{2}} q(n, i, j; k)(xy)^j(x + y)^{n-1-i-2j}.
\]
Since $D_{G_5}(I) = c + Iu$, we have $p(1, 0, 0; k) = q(1, 1, 0; k) = 1$, $p(1, i, j; k) = 0$ if $(i, j) \neq (0, 0)$ and $q(1, i, j; k) = 0$ if $(i, j) \neq (1, 0)$. By induction, it is routine to verify that $p(n, i, j; k) = q(n, i, j; k) = 0$ if $i$ and $j$ are outside the bounds given in (31).

Extracting the coefficients of $a^j b^{n-2j}c$ and $a^j b^{n+1-2j}I$ on both sides of the expression
\[
D_{G_5}^{n+1}(I) = D_{G_5} \left( \sum_{i,j} p(n, i, j; k)u^i a^j b^{n-1-i-2j}c + \sum_{i,j} q(n, i, j; k)u^i a^j b^{n-1-i-2j}I \right),
\]
we obtain the following recurrence system:
\[
p(n + 1, 0, j; k) = (1 + 2kj)p(n, 0, j; k) + 4k(n - 2j + 1)p(n, 0, j - 1; k) + 2kp(n, 1, j - 1; k) + q(n, 0, j; k),
\]
\[
q(n + 1, 0, j; k) = 2kj q(n, 0, j; k) + 4k(n - 2j + 2)q(n, 0, j - 1; k) + 2kq(n, 1, j - 1; k) + (2k - 1)p(n, 0, j - 1; k).
\]
Let \( w \in B_n \) and \( \text{fix}(w) = 1 \). Then \( \text{exc}(w) + a\text{exc}(w) = n - \text{fix}(w) = n - 1 \). Recall that we always write \( w \) in standard cycle decomposition. Since \( w \) has only one fixed point, there are \( n \) choices for the fixed point of \( w \). For the numbers \( p(n, i, j; k) \) and \( q(n, i, j; k) \), by comparing (28) with (31), we see that the index \( i \) only marks the number of fixed points, and the index \( j \) only depends on the numbers of excedances and anti-excedances. Let \( w' \) be the reduction of \( w \). Then \( w' \in B_{n-1} \), \( \text{cyc}(w') = \text{cyc}(w) - 1 \) and \( \text{fix}(w') = 0 \). Moreover, we have \( \text{exc}(w) = \text{exc}(w') \) and \( a\text{exc}(w) = a\text{exc}(w') \). By using the properties of the reduction map, we get

\[
\sum_{w \in B_n \atop \text{fix}(w) = 1} x^{\text{exc}(w)} y^{a\text{exc}(w)} k^{n-\text{cyc}(w)} = n \sum_{w' \in B_{n-1} \atop \text{fix}(w') = 0} x^{\text{exc}(w')} y^{a\text{exc}(w')} k^{n-1-\text{cyc}(w')}.
\]

By using (31), we see that

\[
\sum_{w \in B_n \atop \text{fix}(w) = 1} x^{\text{exc}(w)} y^{a\text{exc}(w)} k^{n-\text{cyc}(w)} = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} p(n, 1, j; k)(xy)^j(x + y)^{n-2j}y + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} q(n, 1, j; k)(xy)^j(x + y)^{n-1-2j},
\]

\[
\sum_{w' \in B_{n-1} \atop \text{fix}(w') = 0} x^{\text{exc}(w')} y^{a\text{exc}(w')} k^{n-1-\text{cyc}(w')} = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} p(n - 1, 0, j; k)(xy)^j(x + y)^{n-2-2j}y + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} q(n - 1, 0, j; k)(xy)^j(x + y)^{n-1-2j}.
\]

Replacing \( j \) with \( j - 1 \) while adjusting the bounds of summation, and then equating appropriate coefficients yields the following relations:

\[
p(n, 1, j - 1; k) = np(n - 1, 0, j - 1; k), \quad q(n, 1, j - 1; k) = nq(n - 1, 0, j - 1; k).
\]

Therefore, by setting \( p(n, 0, j; k) = p(n, j; k) \) and \( q(n, 0, j; k) = q(n, j; k) \), we obtain the recurrence system of the numbers \( p(n, j; k) \) and \( q(n, j; k) \).

Setting \( u = 0 \) in (31) and then taking \( a = x \), \( b = 1 + x \) and \( c = I \), we get the symmetric decomposition of the polynomials \( d_n^B(x; k) \). Clearly, when \( k \geq 1/2 \), the numbers \( p(n, j; k) \) and \( q(n, j; k) \) are nonnegative, and so the polynomials \( d_n^B(x; k) \) are bi-\( \gamma \)-positive. This completes the proof. 

\[\square\]

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References


