Codegree threshold for tiling balanced complete 3-partite 3-graphs and generalized 4-cycles

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Abstract

Given two k-graphs F and H, a perfect F-tiling (also called an F-factor) in H is a set of vertex-disjoint copies of F that together cover the vertex set of H. Let $t_{k-1}(n,F)$ be the smallest integer t such that every k-graph H on n vertices with minimum codegree at least t contains a perfect F-tiling. Mycroft (JCTA, 2016) determined the asymptotic values of $t_{k-1}(n,F)$ for k-partite k-graphs F and conjectured that the error terms $o(n)$ in $t_{k-1}(n,F)$ can be replaced by a constant that depends only on F. In this paper, we determine the exact value of $t_2(n,K^3_{m,m})$, where $K^3_{m,m}$ (defined by Mubayi and Verstraëte, JCTA, 2004) is the 3-graph obtained from the complete bipartite graph $K_{m,m}$ by replacing each vertex in one part by a 2-elements set. Note that $K^3_{2,2}$ is the well known generalized 4-cycle C^3_4 (the 3-graph on six vertices and four distinct edges A,B,C,D with $A\cup B=C\cup D$ and $A\cap B=C\cap D=\emptyset$). The result confirms Mycroft's conjecture for $K^3_{m,m}$. Moreover, we improve the error term $o(n)$ to a sub-linear term when $F=K^3(m)$ and show that the sub-linear term is tight for $K^3(2)$, where $K^3(m)$ is the complete 3-partite 3-graph with each part of size m.

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1 Introduction

A *k-graph* H is a pair $H = (V, E)$ where V is a set of elements called vertices, and E is a collection of subsets of V with uniform size k called edges. We call $|V|$ the *order* of H and $|E|$ the *size* of H, also denoted by $|H|$ or $e(H)$. We write graph for 2-graph for short. Given two k-graphs F and H, an *F-tiling* in H is a collection of vertex-disjoint copies of F in H. An F-tiling is *perfect* if it covers every vertex of H, also known as an *F-factor*. If F is a single edge then an F-factor in H is a perfect matching in H. As for matchings, a natural question for tiling is to determine the minimum degree threshold for finding a perfect F-tiling. Given $S \subseteq V(H)$, the degree of S, denote by $d_H(S)$, is the number of edges of H containing S. The minimum s-degree $\delta_s(H)$ of H is the minimum of $d_H(S)$ over all $S \subseteq V(H)$ of size s. For integer n divisible by $|V(F)|$, define $t_s(n, F)$ to be the smallest integer t such that every k-graph H on n vertices with $\delta_s(H) \geqslant t$ contains a perfect F-tiling. For $n \in \mathbb{N}$, write $[n]$ for the set $\{1, \ldots, n\}$, and $r\mathbb{N}$ for the set of positive integers divisible by integer r

Tiling problems have been widely studied for graphs. The celebrated Hajnal-Szemerédi Theorem [8] states that $t_1(n, K_r) = (1-1/r)n$ for $n \in r\mathbb{N}$. Alon and Yuster [1] generalized the Hajnal–Szemerédi Theorem to $t_1(n, H) \leqslant (1 - 1/\chi(H))n + o(n)$ for every H with chromatic number $\chi(H)$ and $n \in h\mathbb{N}$; later, Komlós, Sárközy, and Szemerédi [15] proved that the error term $o(n)$ can be replaced by a constant $C = C(H)$. In [19], Kühn and Osthus improved Alon–Yuster's result to $t_1(n, H) = (1 - 1/\chi^*(H))n + O(1)$, where $\chi^*(H)$ depends on the relative sizes of the colour classes in the optimal colourings of H and satisfies $\chi(H) - 1 \leqslant \chi^*(H) \leqslant \chi(H)$. See [18] for a survey on graph tiling.

For hypergraphs, we know much less and tiling problems become much harder. There are a number of research results on perfect matching problem, see [26, 28] for surveys.

For complete k-graphs and related, the research focus on the case $k = 3$. Let K_4^3 be the complete 3-graph on four vertices, and $K_4^3 - \ell e$ be the 3-graphs obtained from K_4^3 by deleting ℓ edges. Kühn and Osthus [17] showed that $t_2(n, K_4^3 - 2e) = (1/4 + o(1))n$, and Czygrinow, DeBiasio and Nagle [3] determined its exact value for large n. Lo and Markström [20] proved that $t_2(n, K_4^3 - e) = (1/2 + o(1))n$ and the exact value was determined for large n by Han, Lo, Treglown and Zhao [10] recently. Lo and Markström [21] also proved that $t_2(n, K_4^3) = (3/4 + o(1))n$, and the exact value was determined for large n by Keevash and Mycroft [14].

A (k, ℓ)-cycle $C_s^{(k,\ell)}$ is a k-graph on s vertices so that whose vertices can be ordered cyclically in such a way that the edges are sets of consecutive k vertices and every two consecutive edges share exactly ℓ vertices. Gao and Han [6] and Czygrinow [2] determined the exact value of $t_2(n, C_6^{(3,1)})$ and $t_2(n, C_s^{(3,1)})(s \geqslant 6)$, respectively, and Gao, Han and Zhao [7] determined $t_{k-1}(n, C_s^{(k,1)})$ for $k \geqslant 4$. Han, Lo, and Sanhueza-Matamala [11] proved $t_{k-1}(n, C_s^{(k,k-1)}) \leqslant (1/2 + 1/(2s) + o(1))n$ where $k \geqslant 3$ and $s \geqslant 5k^2$ and this bound is asymptotically best possible for infinitely many pairs of s and k.

In the study of tiling problems, another family of hypergraphs which was well studied are k-partite k-graphs. A k-graph F on vertex set V is said to be *k-partite* if V can be partitioned into vertex classes $V_1, \ldots, V_k$ so that for any $e \in F$ and $1 \leqslant j \leqslant k$ we have

$|e \cap V_j| = 1$. The partition $V_1, \ldots, V_k$ of V is called a *k-partite realisation* of V. Define

$$\mathcal{S}(F) := \bigcup_{\chi} \{|V_1|, \ldots, |V_k|\} \text{ and } \mathcal{D}(F) := \bigcup_{\chi} \{|V_i| - |V_j| : i, j \in [k]\},$$

where in each case the union is taken over all k-partite realisations $\chi = \{V_1, \ldots, V_k\}$ of V. The *greatest common divisor* of F, denoted by $\gcd(F)$, is then defined to be the greatest common divisor of the set $\mathcal{D}(F)$ (if $\mathcal{D}(F) = \{0\}$ then $\gcd(F)$is undefined). The *smallest class ratio* of F, denoted by $\sigma(F)$, is defined by

$$\sigma(F) := \min_{S \in \mathcal{S}(F)} \frac{S}{|V(F)|}.$$

Note in particular that $\sigma(F) \leqslant 1/k$, with equality if and only if $|V_1| = |V_2| = \ldots = |V_k|$ for any k-partite realisation $\chi = \{V_1, V_2, \ldots, V_k\}$. A *complete k-partite k-graph* with vertex classes $V_1, \ldots, V_k$ is a k-graph on $V = V_1 \cup \ldots \cup V_k$ and edge set $E = \{e : |e \cap V_i| = 1 \text{ for each } i \in [k]\}$. Observe that a complete k-partite k-graph has only one k-partite realisation up to permutations of the vertex classes $V_1, \ldots, V_k$. Hence, we write $K^k(V_1, \ldots, V_k)$ for a complete k-partite k-graph with vertex classes $V_1, \ldots, V_k$ and if the sizes of V_i are emphasized, we write $K^k(|V_1|, \ldots, |V_k|)$ for $K^k(V_1, \ldots, V_k)$, if $|V_1| = \cdots = |V_k| = m$ we write $K^k(m)$ for $K^k(V_1, \ldots, V_k)$ and call $K^k(m)$ the *balanced complete k-partite k-graph.* Mycroft [23] proved a general result on tiling k-partite k-graphs.

Theorem 1.1 (Theorem 1.1, 1.2, 1.3 in [23]). Let F be a k-partite k-graph. Then for any $\alpha > 0$ there exists n_0 such that if H is a k-graph on $n \geqslant n_0$ vertices for which $|V(F)|$ divides n and

$$\delta_{k-1}(H) \geqslant \begin{cases} n/2 + \alpha n & \text{if } \mathcal{S}(F) = \{1\} \text{ or } \gcd(S(F)) > 1; \\ \sigma(F)n + \alpha n & \text{if } \gcd(F) = 1; \\ \max\{\sigma(F)n, \frac{n}{p}\} + \alpha n & \text{if } \gcd(\mathcal{S}(F)) = 1 \text{ and } \gcd(F) > 1, \end{cases} \tag{1}$$

then H contains a perfect F-tiling, where p is the smallest prime factor of $\gcd(F)$. Moreover, (1) is asymptotically best possible for a large class of k-partite k-graphs including complete k-partite k-graphs.

Furthermore, Mycroft also conjectured that the error terms in (1) can be replaced by a (sufficiently large) constant that depends only on F.

Conjecture 1.2 ([23]). Let F be a k-partite k-graph. Then there exists a constant $C = C(F)$ such that the error terms in (1) can be replaced by C.

Gao, Han and Zhao [7] improved the error term for complete k-partite k-graphs $F = K^k(a_1, \ldots, a_{k-1}, a_k)$ with $\gcd(F) = 1$ and disproved Conjecture 1.2 for all complete k-partite k-graphs F with $\gcd(F) = 1$ and $a_{k-1} \geqslant 2$ (remark: in the updated version of [7], the authors constructed more counterexamples for the conjecture of Mycroft). Han, Zang, and Zhao [13] determined $t_1(n, K)$ asymptotically for all complete 3-partite 3-graphs K. In this paper, we focus on balanced complete 3-partite 3-graphs. One of our main results is the following.

Theorem 1.3. Let $m \geqslant 2$ be an integer. There exists an integer $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph on $n \geqslant n_0$ vertices with $n \in 3m\mathbb{N}$. If $\delta_2(H) \geqslant n/2 + m^{\frac{1}{m}} n^{1-\frac{1}{m}}$ then H contains a $K^3(m)$-factor.

For $K^3(2)$, we show that the lower bound of $\delta_2(H)$ is tight up to a factor.

Proposition 1. There exists an integer $n_1 \in \mathbb{N}$. For every $n \geqslant n_1$, there exists a 3-graph H on n vertices with $\delta_2(H) \geqslant n/2 + \sqrt{2n}/5 - 3$ containing no $K^3(2)$-factor.

Clearly, Theorem 1.3 improves the error term αn in (1) to $Cn^{1-1/m}$ when $F = K^3(m)$, and Proposition 1 shows that the error term $C\sqrt{n}$ can not be replaced by a constant for $F = K^3(2)$ and henceforth for $F = K^3(2m)$, which gives a new family of counterexamples for Conjecture 1.2 (As mentioned in the end of [7], $K^3(2)$ is not included in the family of counterexamples given by Gao, Han and Zhao).

Given integer k, let $\mathcal{C}_4^k$ be the family of k-graphs which contains four distinct edges A, B, C, D with $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$, which was first introduced by Erdős [4], and is also called the generalized 4-cycles. For $k = 2$ or 3, we write C_4^k for $\mathcal{C}_4^k$ instead because there is only one graph, up to isomorphism, in $\mathcal{C}_4^k$ in these cases. Note that C_4^3 is a supported subgraph of $K^3(2)$.

Let $X_1, X_2, \ldots, X_t$ be t pairwise disjoint sets of size $k-1$ and let Y be a set of s elements disjoint from $\cup_{i\in[t]} X_i$. Define $K_{s,t}^k$ be the k-graph with vertex set $(\cup_{i\in[t]} X_i) \cup Y$ and edge set $\{X_i \cup \{y\} : i \in [t], y \in Y\}$. In [25], Mubayi and Verstraëte investigated the Turán number of $K_{s,t}^3$. We show that Conjecture 1.2 is valid for $K_{m,m}^3$, in particular for generalized 4-cycle since $K_{2,2}^3 = C_4^3$. More precisely, we prove the following theorem.

Theorem 1.4. For any integer m, there exists an integer N such that for all $n \in 3m\mathbb{N}$ and $n \geqslant N$,

$$t_2(n, K_{m,m}^3) = \begin{cases} \lfloor n/2 \rfloor - 1, & \text{if } n \equiv 1 \pmod 4 \\ \lceil n/2 \rceil - 1, & \text{otherwise} \end{cases} \tag{2}$$

To show the lower bound of $t_2(n, K_{m,m}^3)$ in Theorem 1.4 is tight, we give a construction of extremal 3-graph for $K_{m,m}^3$.

Construction 1. Given two disjoint sets A, B, let $\mathcal{B}[A, B]$ be the 3-graph with vertex set $A \cup B$ and edge set $E = \{e : |e| = 3 \text{ and } |e \cap A| = 1 \text{ or } 3\}$.

Clearly, $\delta_2(\mathcal{B}[A, B]) = \min\{|A| - 2, |B| - 1\}$, and each copy of $K_{m,m}^3$ intersects B with an even number of vertices and hence $\mathcal{B}[A, B]$ does not contain a $K_{m,m}^3$-factor provided that $|B|$ is odd. Now, suppose that $n \in 3m\mathbb{N}$. Choose $|A| = n/2+1, |B| = n/2-1$ if $n \equiv 0 \pmod 4$; $|A| = \lfloor n/2 \rfloor, |B| = \lceil n/2 \rceil$ if $n \equiv 1 \pmod 4$; $|A| = |B| = n/2$ if $n \equiv 2 \pmod 4$; and $|A| = \lceil n/2 \rceil, |B| = \lfloor n/2 \rfloor$ if $n \equiv 3 \pmod 4$. We have $\delta_2(\mathcal{B}[A, B]) = \lfloor n/2 \rfloor - 2$ if $n \equiv 1 \pmod 4$, and $\delta_2(\mathcal{B}[A, B]) = \lceil n/2 \rceil - 2$, otherwise. But $\mathcal{B}[A, B]$ does not contain a $K_{m,m}^3$-factor. The extremal 3-graph constructed here implies that (2) is tight.

In the following we give some notation used in this paper. For a k-graph $H = (V, E)$ and a vertex set $U \subseteq V$, write $H[U]$ for the subgraph of H induced by U and $\binom{U}{r}$ for the set of all subsets of size r of U. For an $S \subseteq V$, the *neighbourhood* of S, denoted by $N_H(S)$ or $N(S)$ if there is no confusion from the context, is the set of subsets $T \subseteq V$ such that $S \cup T \in E(H)$, the *link graph* of S, denoted by H_S, is the $(k - |S|)$-graph with vertex set

$V(H) \setminus S$ and edge set $N_H(S)$. For a 3-graph $H = (V, E)$ and $u, v, w \in V$, we write uv and uvw for the sets $\{u, v\}$ and $\{u, v, w\}$, respectively. Let $V_1, \ldots, V_t$ be a partition of $V(H)$. An edge $e = v_1v_2v_3$ is of type $V_{i_1}V_{i_2}V_{i_3}$ if $v_j \in V_{i_j}$ for $j \in [3]$ and $i_j \in [t]$. Write $E(V_{i_1}V_{i_2}V_{i_3})$ for the set of edges of type $V_{i_1}V_{i_2}V_{i_3}$ and $e(V_{i_1}V_{i_2}V_{i_3}) = |E(V_{i_1}V_{i_2}V_{i_3})|$. A subgraph F of H is said to be of type $(t_1, \ldots, t_r)$ if $|V(F) \cap V_i| = t_i$ for each $i \in [t]$. Given two constants α and β, we write $\alpha \ll \beta$ if α is sufficiently small with respect to β.

2 Lemmas and proofs of main results

To show Proposition 1, we first revisit a construction of $K^k(1, \ldots, 1, 2, t+1)$-free ($t \geqslant 1$) k-graph G with $e(G) \sim \frac{\sqrt{t}}{k!} n^{k-\frac{1}{2}}$ edges given by Mubayi [24]. We only need the special case that $k = 3$ and $t = 1$. Let q be a prime power and $\mathbf{F}_q$ be the q-element finite field.

Construction 2 ([24]). Let G_q be a 3-graph with vertex set $V(G_q) = (\mathbf{F}_q \setminus \{0\}) \times (\mathbf{F}_q \setminus \{0\})$, a 3-elements set $\{(a_i, a_i') : i \in [3]\}$ forms an edge in G_q if and only if

$$\prod_{i\in[3]} a_i + \prod_{i\in[3]} a_i' = \mathbf{1}_F.$$

As shown in [24], G_q is $K^3(1, 2, 2)$-free and $\delta_2(G_q) \geqslant q - 3$.

Construction 3. Let G_q' be a vertex-disjoint copy of G_q. Define H_q to be a 3-graph on vertex set $V = V(G_q) \cup V(G_q')$ and edge set $E(H_q)$, of which every edge intersects $V(G_q)$ in precisely two vertices, a 3-elements set $\{(a_i, a_i') : i \in [3]\}$ with $(a_i, a_i') \in V(G_q)$ for $i = 1, 2$ and $(a_3, a_3') \in V(G_q')$ forms an edge in H_q if and only if

$$\prod_{i\in[3]} a_i + \prod_{i\in[3]} a_i' = \mathbf{1}_F.$$

For convenience, we use ordered triple (a, b, c) denote an edge of H_q with $a, b \in V(G_q)$ and $c \in V(G_q')$.

Remark. By the constructions of G_q and H_q, we know that an edge $e = abc \in E(G_q)$ corresponds to three edges $e_1 = (a, b, c), e_2 = (a, c, b), e_3 = (b, c, a)$ in H_q, and H_q possibly contains some edges of the form (a, b, a) or (a, b, b). The following fact shows that H_q inherits some properties from G_q.

Fact 1. *H_q is $K^3(1, 2, 2)$-free and $d_{H_q}(ab) \geqslant q - 3$ for all $a \in V(G_q)$, $b \in V(G_q')$.*

Proof. We show that H_q is also $K^3(1, 2, 2)$-free. As shown in [24], for $(p_1, q_1), (p_2, q_2) \in \mathbf{F}_q \setminus \{0\} \times \mathbf{F}_q \setminus \{0\}$, the equation system

$$\begin{cases} p_1x + p_1'y = 1_F \\ p_2x + p_2'y = 1_F \end{cases} \tag{3}$$

has at most one solution (x, y) if $(p_1, p_1') \neq (p_2, p_2')$. Suppose that H_q contains a copy of $K^3(1, 2, 2)$, say $K^3(\{a\}, \{b_1, b_2\}, \{c_1, c_2\})$. Let $a = (u, u')$, $b_1 = (v_1, v_1')$ and $b_2 =$

(v_2, v_2'). Without loss of generality, we may assume $a, b_1, b_2 \in V(G_q)$. Now let $p_1 = uv_1$, $p_1' = u'v_1'$, $p_2 = uv_2$, and $p_2' = u'v_2'$. Since $(v_1, v_1') \neq (v_2, v_2')$, we have $(p_1, p_1') \neq (p_2, p_2')$. So the equation system (3) has at most one solution, this is a contradiction to $K^3(\{a\}, \{b_1, b_2\}, \{c_1, c_2\}) \subseteq H_q$.

For $a \in V(G_q)$, $b \in V(G_q')$, $d_{H_q}(ab) \geqslant q-3$ is clearly true since the determinate equation $ax + by = 1_{\mathbf{F}}$ has exactly q solutions in $\mathbf{F}_q$ for any non-zero pair $(a, b) \in (\mathbf{F}_q \setminus \{0\}) \times (\mathbf{F}_q \setminus \{0\})$. ☐

Proof of Proposition 1: For sufficiently large n, without loss of generality, we may assume $n \in 6\mathbb{N}$, choose an odd prime power q and $n_0 = (q-1)^2$ such that $n/2 + \frac{2}{5}\sqrt{n/2} \leqslant n_0 \leqslant n/2 + \frac{1}{2}\sqrt{n/2}$. Let $\mathbf{F}_q$ be the q-element finite field and let A and B be the sets obtained by deleting any one element and $2n_0 - n - 1$ elements from $(\mathbf{F}_q \setminus \{0\}) \times (\mathbf{F}_q \setminus \{0\})$, respectively. Then $|A| = n_0 - 1$ and $|B| = n - n_0 + 1$, both of them are odd. Let H' be the subgraph of H_q induced by $A \cup B$ with $A \subset V(G_q)$ and $B \subset V(G_q')$. By Fact 1, H' is $K^3(1,2,2)$-free and $d_{H'}(ab) \geqslant q-4$ for all $a \in A, b \in B$. Let $H = \mathcal{B}[A, B] \cup H'$. Then $\delta_2(H) \geqslant \min\{|A|-2, |B|-1+\sqrt{n_0}-3\} \geqslant n/2 + \frac{2}{5}\sqrt{n/2} - 3$. We claim that H does not contain a $K^3(2)$-factor. Suppose to the contrary that H contains a $K^3(2)$-factor. Since $|A|$ is odd, H must contain a copy of $K^3(2)$ such that $|V(K^3(2)) \cap A|$ is odd. Such a copy of $K^3(2)$ must be of type $(5,1)$ or $(3,3)$. Note that copies of $K^3(2)$ in $\mathcal{B}[A, B]$ must intersect A in an even number of vertices. It is an easy task to check that a copy of $K^3(2)$ of type $(5,1)$ or $(3,3)$ forces a copy of $K^3(1,2,2)$ in H', a contradiction. ☐

The proof of Theorems 1.3 and 1.4 are separated into *non-extremal* case and *extremal* case. For the non-extremal case, we use the standard absorbing method, which has been introduced by Rödl, Ruciński and Szemerédi in [27] and widely used in different research papers for example in [3, 13, 21].

Roughly speaking, our proof follows two steps: first, we use an "absorbing lemma" to find a small absorbing set $W \subset V(H)$ with the property that given any "sufficiently small" set $U \subset V(H) \setminus W$, both $H[W]$ and $H[W \cup U]$ contain $K^3(m)$-factors; second, we use an"almost tiling lemma" to find a $K^3(m)$-tiling in $H \setminus W$ that covers all but at most $o(n)$ vertices. The first step will be completed in Lemma 2.1 and the second step has been done by an almost tiling lemma given by Mycroft in [23], we restate it in Lemma 2.2.

Given $\gamma > 0$, H and G are two 3-graphs on the same vertex set V. We say that H *γ-contains* G if $|E(G) \setminus E(H)| \leqslant \gamma |V|^3$, and H is called *γ-extremal* if there is a partition of $V = A \cup B$ such that $|A| \leqslant |B| \leqslant |A| + 1$ and H γ-contains $\mathcal{B}[A, B]$.

Lemma 2.1 (Absorption lemma). Let $0 < \epsilon_2 \ll \epsilon_1 \ll \gamma \ll 1$ and m be an positive integer. Suppose that H is a 3-graph of order n with $\delta_2(H) \geqslant (1/2 - \gamma)n$. If H is not 3γ-extremal, then there exists a set $W \subset V(H)$ with $|W| \leqslant \epsilon_1 n$ and $|W| \in 3m\mathbb{N}$, so that for any $U \subset V(H) \setminus W$ with $|U| \leqslant \epsilon_2 n$ and $|U| \in 3m\mathbb{N}$, both $H[W]$ and $H[U \cup W]$ contain $K^3(m)$-factors.

Lemma 2.2 (Almost tiling lemma, Lemma 1.5 in [23]). Let K be a k-partite k-graph. Then there exists a constant $C = C(K)$ such that for any $\alpha > 0$ there exists an integer

$n_0 = n_0(K, \alpha)$ with the property that every k-graph H on $n \geqslant n_0$ vertices with $\delta_{k-1}(H) \geqslant (\sigma(K) + \alpha)n$ admits a K-tiling covering all but at most C vertices of H.

Lemmas 2.3 and 2.4 deal with the extremal case for $K^3(m)$ and $K^3_{m,m}$, respectively.

Lemma 2.3. Let $m \geqslant 2$ be an integer. There exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph on $n \geqslant n_0$ vertices with $\delta_2(H) \geqslant n/2 + m^{\frac{1}{m}} n^{1-\frac{1}{m}}$, $n \in 3m\mathbb{N}$. If H is γ-extremal, then H contains a $K^3(m)$-factor.

Lemma 2.4. There exist $\gamma > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph on $n \geqslant n_0$ vertices with $\delta_2(H)$ satisfying (2), where $n \in 3m\mathbb{N}$. If H is γ-extremal, then H contains a $K^3_{m,m}$-factor.

Proof of Theorems 1.3 and 1.4: Let $0 < \alpha \ll 1$ and $1/n \ll \epsilon_2 \ll \epsilon_1 \ll \gamma \ll 1$ with $n \in 3m\mathbb{N}$. Let H be a 3-graph of order n with $\delta_2(H) \geqslant n/2 + m^{\frac{1}{m}} n^{1-\frac{1}{m}}$ (resp. $\delta_2(H)$ satisfying (2)).

I. H is 3γ-extremal. Then, by Lemma 2.3, H contains a $K^3(m)$-factor (resp. $K^3_{m,m}$-factor by Lemma 2.4).

II. H is not 3γ-extremal. From the definition of $K^3_{m,m}$, one can easily have that $K^3_{m,m}$ is a spanning subgraph of $K^3(m)$. If H has a $K^3(m)$-factor then it also contains a $K^3_{m,m}$-factor. By Lemma 2.1, we can choose an absorbing set $W \subset V(H)$ with $|W| \leqslant \epsilon_1 n$ and $|W| \in 3m\mathbb{N}$ so that for any $U \subset V(H)\backslash W$ with $|U| \leqslant \epsilon_2 n$ and $|U| \in 3m\mathbb{N}$, both $H[W]$ and $H[U \cup W]$ contain $K^3(m)$-factors. Let H' be the 3-graph obtained from H by deleting the vertices of W. Then $|V(H')| = n' \geqslant (1 - \epsilon_1)n$ and $\delta_2(H') \geqslant n/2 - 1 - \epsilon_1 n \geqslant (1/3 + \alpha)n'$. Note that $\sigma(K^3(m)) = 1/3$. The codegree condition in Lemma 2.2 for H' and $K^3(m)$ is satisfied. By Lemma 2.2, H' contains a $K^3(m)$-tiling M_1 covering all but at most C vertices. Let $U = V(H') \setminus V(M_1)$. Then $|U| = n - |W| - |V(M_1)| \in 3m\mathbb{N}$ and $|U| \leqslant C \leqslant \epsilon_2 n$. Hence $H[U \cup W]$ contains a $K^3(m)$-factor M_2. Then $M_1 \cup M_2$ is a $K^3(m)$-factor in H. We are done. □

The rest of the paper is organized as follows. In Section 3, we give the proof of the absorption lemma used in the paper, i.e. Lemma 2.1, and in Section 4, we deal with the extremal case, i.e. we prove Lemmas 2.3 and 2.4.

3 Absorption lemma

To prove the absorption lemma, we need some preliminaries. Let $H = (V, E)$ be a k-graph of order n, and F be a k-graph of order t. Given an integer $i \geqslant 1$, a constant $\eta > 0$, and two vertices $x, y \in V$, a vertex set $S \subset V$ is called an (x, y)-*connector* of length i with respect to F if $S \cap \{x, y\} = \emptyset$, $|S| = ti - 1$ and both $H[S \cup \{x\}]$ and $H[S \cup \{y\}]$ contain F-factors. Two vertices x and y are called (i, η)-*close* with respect to F if there exist at least ηn^{ti-1} (x, y)-connectors of length i with respect to F in H. Let

$$\tilde{N}_{F,i,\eta}(x) = \{y : x \text{ and } y \text{ are } (i, \eta)\text{-close with respect to } F\}.$$

A subset $U \subset V$ is said to be (F, i, η)-closed in H if every pair of vertices in U are (i, η)-close with respect to F. If V is (F, i, η)-closed in H then we simply say that H is (F, i, η)-closed.

The following lemma given by Lo and Markström [21] referred to as absorption lemma provides an absorbing set for any sufficiently small vertex set if H is (F, i, η)-closed.

Lemma 3.1 (Lemma 1.1 in [21]). Let t and i be positive integers and $\eta > 0$. Then there exist η_1, η_2 such that $0 < \eta_2 \ll \eta_1 \ll \eta$ and an integer $n_0 = n_0(i, \eta)$ satisfying the following: Suppose that F is a k-graph of order t and H is an (F, i, η)-closed k-graph of order $n \geqslant n_0$. Then there exists a vertex subset $U \subset V(H)$ of size at most $\eta_1 n$ with $|U| \in t\mathbb{Z}$ such that, for every vertex set $W \subset V \setminus U$ of size at most $\eta_2 n$ with $|W| \in t\mathbb{Z}$, both $H[U]$ and $H[U \cup W]$ contain F-factors.

Lemma 3.2 also given in [21] allows us to find close pairs with respect to a k-partite k-graph F.

Lemma 3.2 (Lemma 4.2 in [21]). Let $k \geqslant 2$ be an integer and $\alpha > 0$. Given a k-partite k-graph F, there exist a constant $\eta_0 = \eta_0(k, F, \alpha) > 0$ and an integer $n_0 = n_0(k, F, \alpha)$ such that the following holds: Let H be a k-graph of order $n \geqslant n_0$ and $x, y \in V(H)$. If

$$|\{S \mid S \in N(x) \cap N(y) \text{ with } |N(S)| \geqslant \alpha n\}| \geqslant \alpha \binom{n}{k-1},$$

then x and y are $(F, 1, \eta)$-close for all $0 < \eta \leqslant \eta_0$.

The following lemma in [12] gives us a partition of $V(H)$ with bounded number of parts such that each of them is closed with respect to F.

Lemma 3.3 (Lemma 6.3 in [12]). Given $\delta > 0$, integers $c, k, t \geqslant 2$ and $0 < \eta \ll 1/c, \delta, 1/t$, there exists a constant $\eta' > 0$ such that the following holds for all sufficiently large n: Let F be a k-graph on t vertices. Assume a k-graph H on n vertices satisfies that $|\tilde{N}_{F,1,\eta}(v)| \geqslant \delta n$ for any $v \in V(H)$ and every set of $c+1$ vertices in $V(H)$ contains two vertices that are $(F, 1, \eta)$-close. Then we can find a partition of $V(H)$ into $V_1, \dots, V_r$ with $r \leqslant \min\{c, 1/\delta\}$ such that for any $j \in [r]$, $|V_j| \geqslant (\delta - \eta)n$ and V_j is $(F, 2^{c-1}, \eta')$-closed in H.

Actually here we use a variant absorbing method which is so-called lattice-based absorption developed by Han [9], the notation used were first given by Keevash and Mycroft [14]. Given a k-graph $H = (V, E)$ and a partition $\mathcal{P} = \{V_1, \dots, V_r\}$ of V, the *index vector* $\mathbf{i}_{\mathcal{P}}(S)$ of a subset $S \subset V$ with respect to $\mathcal{P}$ is the vector whose j-th coordinate is the size of the intersection of S and V_j. A vector $\mathbf{v} \in \mathbb{Z}^r$ is called an *s-vector* if all its coordinates are nonnegative and their sum equals to s. Given a k-graph F of order t and $\mu > 0$, a t-vector $\mathbf{v}$ is called a *μ-robust F-vector* if there are at least μn^t copies F' of F in H satisfying $\mathbf{i}_{\mathcal{P}}(V(F'))=\mathbf{v}$. Let $I^{\mu}_{\mathcal{P},F}(H)$ be the set of all μ-robust F-vectors and $L^{\mu}_{\mathcal{P},F}(H)$ be the lattice (i.e. the additive subgroup) generated by $I^{\mu}_{\mathcal{P},F}(H)$. For $j \in [r]$, let $\mathbf{u}_j \in \mathbb{Z}^r$ be the j-th unit vector, namely, $\mathbf{u}_j$ has 1 on the j-th coordinate and 0 on other coordinates. A transferral is a vector of the form $\mathbf{u}_j - \mathbf{u}_\ell$ for some distinct $j, \ell \in [r]$.

The following lemma in [13] states that if $L^{\mu}_{\mathcal{P},F}(H)$ contains all transferrals then H is closed.

Lemma 3.4 (Lemma 3.9 in [13]). Let $i_0, k, r_0 > 0$ be integers and let F be a k-graph on t vertices. Given constants $\epsilon, \eta, \mu > 0$, there exist $\eta' > 0$ and an integer $i_0' \geqslant 0$ such that the following holds for sufficiently large n: Let H be a k-graph on n vertices with a partition $\mathcal{P} = \{V_1, \ldots, V_r\}$ such that $r \leqslant r_0$ and for each $j \in [r]$, $|V_j| \geqslant \epsilon n$ and V_j is (F, i_0, η)-closed in H. If $\mathbf{u}_j - \mathbf{u}_\ell \in L^{\mu}_{\mathcal{P},F}(H)$ for all $1 \leqslant j < \ell \leqslant r$, then H is (F, i_0', η')-closed.

The following lemma helps us to count the number of copies of $K^3(m)$.

Lemma 3.5 (Corollary 2 in [5]). Let F be a k-partite k-graph of order t. For every $\epsilon > 0$, there exists a constant $\mu > 0$ and an integer n_0 such that every k-graph H of order $n \geqslant n_0$ with $e(H) > \epsilon n^k$ contains at least μn^t copies of F.

We also need the following lemma from [10].

Lemma 3.6 (Lemma 3.3 in [10]). Let $0 < 1/n \ll \gamma < 1/100$. Suppose that H is a 3-graph of order n with $\delta_2(H) \geqslant (1/2 - \gamma)n$. Let X, Y be any bipartition of $V(H)$ with $|X|, |Y| \geqslant n/5$. If H is not 3γ-extremal, then H contains at least $\gamma^2 n^3$ XXY-edges and at least $\gamma^2 n^3$ XYY-edges.

Now it is ready to give the proof of our absorption lemma, we restate it here.

Lemma 3.7. Let $0 < \epsilon_2 \ll \epsilon_1 \ll \gamma \ll 1$ and m be an positive integer. Suppose that H is a 3-graph of order n with $\delta_2(H) \geqslant (1/2 - \gamma)n$. If H is not 3γ-extremal, then there exists a set $W \subset V(H)$ with $|W| \leqslant \epsilon_1 n$ and $|W| \in 3m\mathbb{N}$ so that for any $U \subset V(H) \setminus W$ with $|U| \leqslant \epsilon_2 n$ and $|U| \in 3m\mathbb{N}$, both $H[W]$ and $H[U \cup W]$ contain $K^3(m)$-factors.

Proof. Assume γ is sufficiently small and let $\alpha = \gamma/3$. Let $F = K^3(m)$. If we prove that H is (F, i, η)-closed for some $i > 0$ and $0 < \eta \ll \gamma$, then by Lemma 3.1 with $t = 3m$ we obtain the desired absorbing set. So in the following it is sufficient to show that H is (F, i, η)-closed for some parameters $i > 0$ and $0 < \eta \ll \gamma$. The outline of the proof is as follows. The first step is, by applying Lemma 3.3 on H, to obtain a partition $\mathcal{P}$ of $V(H)$ with $|\mathcal{P}| \leqslant 2$ such that each part is $(F, 2, \eta')$-closed and has large enough size. To show that all conditions of Lemma 3.3 are satisfied, we need to verify that for every vertex $v \in V(H)$, $\tilde{N}_{F,1,\eta}(v)$ is large enough (this can be done in Claim 2) and any three vertices contain at least one $(F, 1, \eta)$-close pair (this can be done by using Lemma 3.2). If $|\mathcal{P}| = 1$, then we are done. Otherwise, we show H is closed by applying Lemma 3.4 on H and $\mathcal{P}$, i.e. we prove that $L^{\mu}_{\mathcal{P},F}(H)$ contains all transferrals (this can be done in Claims 3 and 4).

Claim 2. *For each $v \in V(H)$ and some $0 < \eta \ll \gamma$, $\tilde{N}_{F,1,\eta}(v) \geqslant (1/2 - 2\gamma)n$.*

Proof of Claim 2: Fix $v \in V(H)$, we have

$$|N(v)| \geqslant \frac{(1/2 - \gamma)n(n-1)}{2} = (1/2 - \gamma)\binom{n}{2}. \tag{4}$$

Note that $|N(S)| \geqslant (1/2 - \gamma)n \geqslant \alpha n$ for any 2-elements set $S \subseteq V(H)$. By Lemma 3.2, we have $u \in \tilde{N}_{F,1,\eta}(v)$ if $|N(v) \cap N(u)| \geqslant \alpha\binom{n}{2}$ for any $0 < \eta \leqslant \eta_0 = \eta_0(k, F, \alpha)$. Let G be a bipartite graph with partitions $N(v)$ and $V(H) \setminus \{v\}$, and a 2-elements set $S \in N(v)$

and a vertex $w \in V(H) \setminus \{v\}$ are adjacent in G if and only if $S \cup \{w\} \in E(H)$. Then we have

$$e(G) = \sum_{S \in N(v)} d_G(S) = \sum_{S \in N(v)} (|N(S)| - 1) < |\tilde{N}_{F,1,\eta}(v)| \cdot |N(v)| + n \cdot \alpha \binom{n}{2}.$$

Together with $|N(S)| \geqslant (1/2 - \gamma)n$, we have

$$|\tilde{N}_{F,1,\eta}(v)| \geqslant (1/2 - \gamma)n - 1 - \frac{n \cdot \alpha\binom{n}{2}}{(1/2 - \gamma)\binom{n}{2}} \geqslant (1/2 - 2\gamma)n. \qquad \square$$

Given any three vertices $x_1, x_2, x_3 \in V(H)$, by (4) and the inclusion-exclusion principle, we have

$$\begin{aligned}
\sum_{1 \leqslant i < j \leqslant 3} |N(x_i) \cap N(x_j)| &= \sum_{i=1}^{3} |N(x_i)| - |\cup_{i=1}^{3} N(x_i)| + |\cap_{i=1}^{3} N(x_i)| \\
&\geqslant 3(1/2 - \gamma)\binom{n}{2} - |\cup_{i=1}^{3} N(x_i)| + |\cap_{i=1}^{3} N(x_i)| \\
&\geqslant 3\alpha\binom{n}{2} + \binom{n}{2} - |\cup_{i=1}^{3} N(x_i)| + |\cap_{i=1}^{3} N(x_i)| \\
&\geqslant 3\alpha\binom{n}{2}.
\end{aligned}$$

By the pigeonhole principle, there exists at least one pair x_i, x_j so that $|N(x_i) \cap N(x_j)| \geqslant \alpha\binom{n}{2}$, by Lemma 3.2, such a pair x_i, x_j is $(F, 1, \eta)$-close.

Now apply Lemma 3.3 to F and H with $\delta = (1/2 - 2\gamma)$, $c = 2$ and $\eta \ll \gamma$, we have that there exist a constant $\eta' > 0$ and a partition $\mathcal{P}$ of V with at most 2 parts such that each part has size at least $(1/2 - 3\gamma)n$ and is $(F, 2, \eta')$-closed in H. If $|\mathcal{P}| = 1$, then H is $(F, 2, \eta')$-closed, as desired. So, we assume $|\mathcal{P}| = 2$ and $\mathcal{P} = \{X, Y\}$. Since H is not 3γ-extremal, by Lemma 3.6, both $e(XXY)$ and $e(XYY)$ are at least $\gamma^2 n^3$.

Define

$$E_0 = \{xy : x \in X, y \in Y,\, d_X(xy) \geqslant \gamma^2 n, d_Y(xy) \geqslant \gamma^2 n\},$$
$$E_1 = \{xy : x \in X, y \in Y,\, d_X(xy) \geqslant \gamma^2 n, d_Y(xy) < \gamma^2 n\},$$

and

$$E_2 = \{xy : x \in X, y \in Y,\, d_X(xy) < \gamma^2 n, d_Y(xy) \geqslant \gamma^2 n\}.$$

Then $E(K^2(X,Y)) = E_0 \cup E_1 \cup E_2$. So $|E_i| \leqslant e(K^2(X,Y)) \leqslant \frac{n^2}{4}$ for any $i \in \{0,1,2\}$. Let $V_1 = X$, $V_2 = Y$. By Lemma 3.4, to show that H is closed it suffices to show $\mathbf{u}_1 - \mathbf{u}_2 \in L^{\mu}_{\mathcal{P},F}(H)$ for some μ. Or equivalently, we need to show that there exists an ℓ such that H contains at least μn^{3m} copies of $K^3(m)$ of types $(\ell, 3m - \ell)$ and $(\ell + 1, 3m - \ell - 1)$, respectively. We split the following proof into two cases according to the size of E_0.

Claim 3. *There exists $\mu_1 > 0$ for any given integers $0 \leqslant s, t \leqslant m$ with $s + t = m$ such that the following holds: If $|E_0| \geqslant \gamma^4 n^2$, then H contains at least $\mu_1 n^{3m}$ copies of $K^3(m)$ of type $(m + s, m + t)$.*

Proof of Claim 3: Choose $0 < \gamma_1 \ll \gamma$. Construct an auxiliary 4-partite 4-graph G_1 as follows. Let $V(G_1) = X' \cup X \cup Y \cup Y'$, where X' and Y' are copies of X' and Y', respectively; for $a \in X',\ x \in X, y \in Y, b \in Y'$, $axyb \in E(G_1)$ if and only if $axy, xyb \in H$. Then $|V(G_1)| = 2n$, and

$$|G_1| \geqslant \sum_{xy \in E_0} d_X(xy) d_Y(xy) \geqslant \gamma^4 n^2 \cdot \gamma^2 n \cdot \gamma^2 n = \gamma^8/16 |V(G_1)|^4.$$

Hence, by Lemma 3.5, there exists a positive constant μ_1 such that G_1 contains at least $\mu_1 n^{4m}$ copies of $K^4(m)$. Fix a pair (s,t), a copy of $K^4(s,m,m,t)$ is contained in at most $\binom{|X'|}{t}\binom{|Y'|}{s} \leqslant n^{s+t} = n^m$ copies of $K^4(m,m,m,m)$. Therefore, G_1 contains at least $\mu_1' n^{3m}$ copies of $K^4(s,m,m,t)$ for some $\mu_1' > 0$. Observe that a copy of $K^4(s,m,m,t)$ in G_1 gives us a copy of $K^3(m)$ of type $(m+s, m+t)$. Then H contains at least $\mu_1' n^{3m}$ copies of $K^3(m)$ of type $(m+s, m+t)$. □

Claim 4. *Given integers $0 \leqslant s, t \leqslant m$ with $s + t = m$, there exists $\mu_1' > 0$ such that the following holds: If $|E_0| < \gamma^4 n^2$, then H contains at least $\mu_1' n^{3m}$ copies of $K^3(m)$ of the same type either $(2m+s, t)$ or $(t, 2m+s)$.*

Proof of Claim 4: Without loss of generality, assume that $|E_1| \leqslant |E_2|$. First, we show $3\gamma^2 n^2 \leqslant |E_1| \leqslant \frac{1}{8} n^2$. The upper bound is trivial by the assumption that $|E_1| \leqslant |E_2|$. Now suppose that $|E_1| < 3\gamma^2 n^2$. Then, we have

$$\begin{aligned} e(XXY) \quad &= \frac{1}{2} \sum_{x \in X, y \in Y} d_X(xy) \\ &< \frac{1}{2} \left(|E_0| \cdot |X| + |E_1| \cdot |X| + |E_2| \cdot \gamma^2 n \right) \\ &< \frac{1}{2} \left(\left(\gamma^4 + 3\gamma^2\right) n^2 \cdot \left(\frac{1}{2} + 3\gamma\right) n + \frac{n^2}{4} \cdot \gamma^2 n \right) \\ &< \gamma^2 n^3, \end{aligned}$$

a contradiction to $e(XXY) \geqslant \gamma^2 n^3$. Thus, we have $|E_1| \geqslant 3\gamma^2 n^2$. Note that for $xy \in E_1$, we have $d_X(xy) \geqslant (1/2 - \gamma - \gamma^2)n$ and hence $(1/2 - \gamma - \gamma^2)n \leqslant |X|, |Y| \leqslant (1/2 + \gamma + \gamma^2)n$.

Let $Y' = \{y \in Y : d_{E_0}(y) \leqslant \gamma^2 n\}$. Since $|E_0| \leqslant \gamma^4 n^2$, there are at most $\gamma^2 n$ vertices y in Y such that $d_{E_0}(y) > \gamma^2 n$. Thus we have $|Y'| \geqslant |Y| - \gamma^2 n$.

We claim that either $d_{E_1}(y) \leqslant 3\gamma^2 n$ or $d_{E_1}(y) \geqslant |X| - 3\gamma n$ for all $y \in Y'$. Fix $y \in Y'$. Let e_y be the number of edges $x_1 x_2 y$ of the form XXY such that exactly one of $\{x_1 y, x_2 y\}$ belongs to E_1. On one hand, we have

$$e_y \geqslant ((1/2 - \gamma - \gamma^2)n - d_{E_1}(y)) \cdot d_{E_1}(y) \geqslant (|X| - 2\gamma n - 2\gamma^2 n - d_{E_1}(y)) \cdot d_{E_1}(y),$$

since for each $x \in N_{E_1}(y)$, there are at least $(1/2 - \gamma - \gamma^2)n - d_{E_1}(y)$ edges $xx'y$ of the form XXY with $x' \in N_{E_0}(y) \cup N_{E_2}(y)$ and $|X| \leqslant (1/2 + \gamma + \gamma^2)n$. On the other hand, we have

$$e_y \leqslant |X| \cdot d_{E_0}(y) + \gamma^2 n \cdot d_{E_2}(y) \leqslant 2\gamma^2 n |X|,$$

since $d_X(x'y) < \gamma n^2$ for $x'y \in E_2$, and the last inequality holds since $d_{E_0}(y) \leqslant \gamma^2 n$ and $d_{E_2}(y) \leqslant |X|$. Therefore, we have

$$(|X| - 2\gamma n - 2\gamma^2 n - d_{E_1}(y)) \cdot d_{E_1}(y) \leqslant 2\gamma^2 n|X|.$$

Solve the inequality we have either $d_{E_1}(y) \leqslant 3\gamma^2 n$ or $d_{E_1}(y) \geqslant |X| - 3\gamma n$ for all $y \in Y'$.

Let $Y_0 = \{y : d_{E_1}(y) \geqslant |X| - 3\gamma n, y \in Y'\}$. Clearly,

$$|Y_0| \geqslant \frac{|E_1| - (|Y| - |Y'|)|X| - |Y| \cdot 3\gamma^2 n}{|X|} \geqslant \frac{3\gamma^2 n^2 - \gamma^2 n|X| - |Y| \cdot 3\gamma^2 n}{|X|} \geqslant \gamma^2 n.$$

Now we claim that there are at least $(1 - 14\gamma)\binom{|X|}{2}$ pairs $x_1x_2 \in \binom{X}{2}$ such that $d_Y(x_1x_2) \geqslant \frac{1}{10}\gamma^2 n$. Clearly,

$$\begin{aligned} e(XXY_0) \quad &= \frac{1}{2}\sum_{x\in X, y\in Y_0} d_X(xy) \\ &\geqslant \frac{|Y_0|(1/2 - \gamma - \gamma^2)n(|X| - 3\gamma n)}{2} \\ &\geqslant \frac{|Y_0||X|(1-5\gamma)|X|(1-7\gamma)}{2} \\ &\geqslant (1-12\gamma)\binom{|X|}{2}|Y_0|. \end{aligned}$$

On the other hand, if the number of pairs $x_1x_2 \in \binom{X}{2}$ with $d_{Y_0}(x_1x_2) \geqslant \frac{1}{10}\gamma^2 n$ is less than $(1-14\gamma)\binom{|X|}{2}$, we have

$$\begin{aligned} e(XXY_0) \quad &= \quad \sum_{x_1x_2 \in \binom{X}{2}} d_{Y_0}(x_1x_2) \\ &< \quad (1-14\gamma)\binom{|X|}{2}|Y_0| + 14\gamma\binom{|X|}{2}\frac{1}{10}\gamma^2 n \\ &\leqslant \quad (1-12\gamma)\binom{|X|}{2}|Y_0| - 2\gamma\binom{|X|}{2}|Y_0| + 14\gamma\binom{|X|}{2}\frac{1}{10}|Y_0| \\ &= \quad (1-12\gamma)\binom{|X|}{2}|Y_0| - \frac{3}{5}\gamma\binom{|X|}{2}|Y_0| \\ &< \quad (1-12\gamma)\binom{|X|}{2}|Y_0|, \end{aligned}$$

a contradiction.

Next, we claim that there are at least $(\frac{1}{2} - 11\gamma)\binom{|X|}{2}$ pairs $x_1x_2 \in \binom{X}{2}$ such that

$d_X(x_1x_2) \geqslant \gamma n$. In fact,

$$\begin{aligned}
\sum_{x_1x_2 \in \binom{X}{2}} d_X(x_1x_2) &= \sum_{x_1x_2 \in \binom{X}{2}} d_H(x_1x_2) - \sum_{x_1x_2 \in \binom{X}{2}} d_Y(x_1x_2) \\
&\geqslant \left(\frac{1}{2} - \gamma\right) n \cdot \binom{|X|}{2} - \frac{1}{2} \sum_{x \in X, y \in Y} d_X(xy) \\
&> \left(\frac{1}{2} - \gamma\right) n \cdot \binom{|X|}{2} - \frac{1}{2} \left(\gamma^4 n^2 \cdot |X| + \frac{n^2}{8} \cdot |X| + \frac{n^2}{4} \cdot \gamma^2 n\right) \\
&\geqslant \left(\frac{1}{2} - \gamma\right) n \cdot \binom{|X|}{2} - \frac{n}{2} \left(\gamma^4 n \cdot |X| + \frac{n}{8} \cdot |X| + \frac{n}{4} \cdot \gamma^2 n\right) \\
&\geqslant \left(\frac{1}{4} - 3\gamma\right) n \binom{|X|}{2},
\end{aligned}$$

the third inequality holds since $d_X(xy) \leqslant |X|$ for any $xy \in E_0 \cup E_1$, $d_X(xy) < \gamma^2 n$ for $xy \in E_2$ and $|E_0| < \gamma^4 n^2$, $|E_1| \leqslant \frac{n^2}{8}$ and $|E_2| \leqslant \frac{n^2}{4}$; the last inequality holds since $(1/2 - 3\gamma)n \leqslant |X| \leqslant (1/2 + 3\gamma)n$. Since

$$\frac{(\frac{1}{4} - 3\gamma)n\binom{|X|}{2} - \gamma n \binom{|X|}{2}}{|X|} \geqslant \frac{\frac{1}{4} - 4\gamma}{\frac{1}{2} + 3\gamma} \binom{|X|}{2} \geqslant \left(\frac{1}{2} - 11\gamma\right) \binom{|X|}{2},$$

there are at least $(\frac{1}{2} - 11\gamma)\binom{|X|}{2}$ pairs $x_1x_2 \in \binom{X}{2}$ such that $d_X(x_1x_2) \geqslant \gamma n$.

Therefore, there are at least $(1 - 14\gamma + \frac{1}{2} - 11\gamma - 1)\binom{|X|}{2} \geqslant \frac{n^2}{100}$ pairs $x_1x_2 \in \binom{X}{2}$ such that $d_X(x_1x_2) \geqslant \gamma n$ and $d_Y(x_1x_2) \geqslant \frac{1}{10}\gamma^2 n$. As what we have done in the proof of Claim 3, define an auxiliary 4-graph G_2 as follows. Let $V(G_2) = X' \cup X \cup Y$, where X' is a copy of X; for $x' \in X', x_1, x_2 \in X, y \in Y$, $x'x_1x_2y \in E(G_2)$ if and only if $x'x_1x_2, x_1x_2y \in H$. Hence, $n < |V(G_2)| = n + |X| < 2n$, and

$$|G_2| \geqslant \gamma n \cdot \frac{\gamma^2}{10} n \cdot \frac{n^2}{100} > \gamma' |V(G_2)|^4.$$

By Lemma 3.5, there exists a positive constant μ_2 such that G_2 contains at least $\mu_2 n^{4m}$ copies of $K^4(m)$. As the same argument shown in the proof of Claim 3, H contains at least $\mu_2' n^{3m}$ copies of $K^3(m)$ of type $(2m + s, t)$ for some positive μ_2'. ☐

This completes the proof of Lemma 2.1. ☐

4 Extremal case

In this section, we prove Lemmas 2.3 and 2.4. Let G and H be two k-graphs on the same vertex set V and let $G \setminus H$ be the graph $(V, E(G) \setminus E(H))$. Suppose that $|V| = n$ and $0 \leqslant \alpha \leqslant 1$, we say a vertex $v \in H$ *α-good* with respect to G if $d_{G \setminus H}(v) \leqslant \alpha n^{k-1}$, otherwise call it *$\alpha$-bad*. We call H α-good with respect to G if all of vertices in H are α-good with respect to G. First we deal with a special case when H is α-good with respect to the extremal graph. We need a lemma from [13] which follow with some extra work from a perfect packing theorem of Lu and Székely [22]. Given $V = A \cup B$, let $\mathcal{D}[A, B]$ be the k-graph on V consisting of all edges of type AB^{k-1}.

Lemma 4.1 (Lemma 6.1 in [13]). Let K be a complete k-partite k-graph of order t with the first part of size a_1. Given $0 < \rho \ll 1/m$ and a sufficiently large integer n, suppose H is a k-graph on $n \in t\mathbb{Z}$ vertices with a partition of $V(H) = X \cup Y$ such that $a_1|Y| = (t-a_1)|X|$. Furthermore, assume that H is ρ-good with respect to $\mathcal{D}[X, Y]$. Then H contains a K-factor.

Lemma 4.2. Let α, ϵ be any given constants with $0 < \epsilon \ll \alpha$ and m be an integer. Suppose that H is a 3-graph with large enough order n and $V(H)$ has a partition $A \cup B$ with $||A| - |B|| < \epsilon n$ such that H is α-good with respect to $\mathcal{B}[A, B]$. Then H contains a $K^3(m)$-tiling covering all but at most $2\epsilon n$ vertices. Furthermore, if $n \in 12m\mathbb{Z}$ and $|A| = |B|$. Then H contains a $K^3(m)$-factor.

Proof. Without loss of generality, assume $|A| \leqslant |B|$. Let $|A| = 6mn' + s$ and $|B| = 6mn' + t$, where $0 \leqslant s < 6m$ and $t = |B| - |A| + s < \epsilon n + s$. Let A_0 and B_0 be the sets obtained from A and B by deleting s and t vertices from A and B, respectively. Then $|A_0 \cup B_0| = 12mn' \in 12m\mathbb{N}$. Let $H_0 = H[A_0 \cup B_0]$ and $n_0 := V(H_0)$. Then H_0 must be α'-good with respect to $\mathcal{B}[A_0, B_0]$ for some constant $\alpha' > 0$. Partition A_0 into three subsets $A_1,\ A_2,\ A_2'$ with $|A_1| = 3mn', |A_2| = mn'$ and $|A_2'| = 2mn'$. Let $H_1 = H_0[A_1 \cup B_0]$ and $H_2 = H_0[A_2 \cup A_2']$. Then we have $|V(H_1)| = \frac{3}{4}n_0$ and $|V(H_2)| = \frac{1}{4}n_0$. One can examine that H_1 is $\frac{16}{9}\alpha'$-good with respect to $\mathcal{D}[A_1, B_0]$ and H_2 is $16\alpha'$-good with respect to $\mathcal{D}[A_2, A_2']$. Set $K = K^3(m)$. Applying Lemma 4.1 to H_1 and H_2 with parameters $\frac{16}{9}\alpha'$ and $16\alpha'$, we obtain $K^3(m)$-factors $\mathcal{M}_1$ in H_1 and $\mathcal{M}_2$ in H_2, respectively. Therefore, $\mathcal{M}_1 \cup \mathcal{M}_2$ is a desired $K^3(m)$-factor of H_0.

If $n \in 12m\mathbb{Z}$ and $|A| = |B|$ then $H_0 = H$. Hence $\mathcal{M}_1 \cup \mathcal{M}_2$ is a $K^3(m)$-factor of H. □

Remark: Note that, in the above proof, the $K^3(m)$-factors $\mathcal{M}_1$ and $\mathcal{M}_2$ have the following property:
(1) Each member in $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) has type $(m, 2m)$ (resp. $(3m, 0)$) with respect to the partition $A \cup B$, and
(2) both $|\mathcal{M}_1|(\sim \frac{n}{4})$ and $|\mathcal{M}_2|(\sim \frac{n}{12})$ are large enough.

The following classical result [16] also will be used.

Lemma 4.3 (Kövári-Sós-Turán, 1954). For all $t \geqslant s \geqslant 2$, the Turán function of the complete bipartite graph $K^2(s, t)$ is

$$ex_2(n, K^2(s, t)) \leqslant \frac{1}{2}((t-1)^{1/s} n^{2-1/s} + (s+1)n).$$

4.1 Proofs of Lemmas 2.3 and 2.4

Since H is γ-extremal, there is a partition $V = A \cup B$ such that $|A| \leqslant |B| \leqslant \lceil n/2 \rceil$ and H is γ-extremal with respect to $\mathcal{B}[A, B]$. Set $\gamma_1 = \sqrt{\gamma}$. By the definition of γ-extremal, all but at most $\gamma_1 n$ vertices in V are γ_1-good with respect to $\mathcal{B}[A, B]$. Let A_0 and B_0 be the sets of γ_1-bad vertices in A and B, respectively. Then $|A_0 \cup B_0| \leqslant \gamma_1 n$. For a vertex $x \in A_0 \cup B_0$, we call it B-*acceptable* if $|E(H_x) \cap E(K^2(A,B))| \geqslant \frac{n^2}{40}$; otherwise we call it A-*acceptable*. Note that $|E(H_x)| \geqslant \delta_1(H) \geqslant (n-1)(\lfloor n/2 \rfloor - 1)/2$. If x is A-acceptable then $|E(H_x) \cap \binom{A}{2}| \geqslant \frac{3}{4}\binom{|A|}{2}$ and $|E(H_x) \cap \binom{B}{2}| \geqslant \frac{3}{4}\binom{|B|}{2}$. Now move all A-acceptable vertices into A and B-acceptable vertices into B, we get a new partition $V = A' \cup B'$ with the property that

1) $n/2 - \gamma_1 n \leqslant |A'|, |B'| \leqslant n/2 + \gamma_1 n$ (since $|A_0 \cup B_0| \leqslant \gamma_1 n$);

2) H γ_2-contains $\mathcal{B}[A', B']$ for some constant $\gamma_2 \gg \gamma_1$.

Moreover, we can partition A' into A_1, A_2 so that:

A1) Every vertex in A_1 is γ_2-good with respect to $\mathcal{B}[A', B']$;

A2) $|A_2| \leqslant \gamma_1 n$;

A3) for every $x \in A_2$, $|E(H_x) \cap \binom{A'}{2}| \geqslant \frac{2}{3}\binom{|A'|}{2}$ and $|E(H_x) \cap \binom{B'}{2}| \geqslant \frac{2}{3}\binom{|B'|}{2}$.

Similarly, there is a partition B_1, B_2 of B' so that:

B1) Every vertex in B_1 is γ_2-good with respect to $\mathcal{B}[A', B']$;

B2) $|B_2| \leqslant \gamma_1 n$;

B3) for every $x \in B_2$, $|E(H_x) \cap E(K^2(A', B'))| \geqslant \frac{n^2}{50}$.

Our strategy is to find vertex-disjoint $K^3(m)$-tiling $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ in H so that the union of them is a $K^3(m)$-factor of H, in which $\mathcal{K}_1$ is so-called 'parity breaking' copies dealing with the case $|B'| \not\equiv 0 \pmod{2m}$, $\mathcal{K}_2$ covers all vertices in $A_2 \cup B_2$, and $\mathcal{K}_3$ is used to guarantee the divisibility condition required by Lemma 4.2 after removing the vertices covered by $\mathcal{K}_1$ and $\mathcal{K}_2$. Furthermore, $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are all small enough such that the graph obtained by deleting $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ is γ_3-good for some constant γ_3. Finally, we apply Lemma 4.2 to obtain $\mathcal{K}_4$.

In Claims 5 and 6, we show that such 'parity breaking' copies of $K^3(m)$ (resp. $K^3_{m,m}$) do exist.

Claim 5. *If $\delta_2(H) \geqslant n/2 + m^{1/m} n^{1-1/m}$, then H contains either $2m-1$ disjoint copies of $K^3(m)$ of type $(m+1, 2m-1)$ or $2m-1$ disjoint copies of $K^3(m)$ of type $(3m-1, 1)$.*

Proof of Claim 5: If we can find a copy of $K^3(m)$ of type $(m+1, 2m-1)$ or $(3m-1, 1)$ avoiding any given vertex set $W \subset V$ with $|W| \leqslant C$ for some constant $C \geqslant 6m^2$, then we can greedily find $2m-1$ disjoint copies of $K^3(m)$ of desired type because we always can find a new copy of $K^3(m)$ avoiding the vertices of copies of $K^3(m)$ we have found (since $C \geqslant 6m^2$). So the rest of the proof is to show the statement is true. Choose any vertex set $W \subset V$ with $|W| \leqslant C$ for some constant $C \geqslant 6m^2$. We split the proof into two cases according to the size of B'.

First assume that $|B'| \leqslant n/2$. For any $a \in A', b \in B'$, we have $|N_H(ab) \cap A'| \geqslant m^{1/m} n^{1-1/m}$ since $\delta_2(H) \geqslant n/2 + m^{1/m} n^{1-1/m}$. Construct an auxiliary bipartite graph G as follows: set $V(G) = A' \cup B'$ and $E(G)$ consists of all pairs ab with $a \in A', b \in B'$ and $|N_H(ab) \cap B'| \geqslant (1 - \sqrt{\gamma_2})|B'|$. Since H γ_2-contains $\mathcal{B}[A', B']$, there are at most $\gamma_2 n^3$ $A'B'B'$-edges missing in H. Clearly, we have that at most $2\gamma_2 n^3/(\sqrt{\gamma_2}|B'|) \leqslant 8\sqrt{\gamma_2} n^2$ pairs ab missing in G. By double-counting the number of ordered pairs (v, e) with $v \in$

$A' \setminus W$ and $e \in N_H(v) \cap E(G-W)$, we have

$$\sum_{v\in A'\setminus W} |N_H(v)\cap E(G-W)| \geqslant (|G|-Cn)\cdot(m^{1/m}n^{1-1/m}-|A'\cap W|).$$

Note that $(|G|-Cn)(m^{1/m}n^{1-1/m}-|A'\cap W|)/|A'\setminus W| \geqslant \frac{1}{2}(m-\frac{1}{2})^{1/m}n^{2-1/m}$. We can choose a vertex $v\in A'\setminus W$ such that $|N_H(v)\cap E(G-W)| \geqslant \frac{1}{2}(m-\frac{1}{2})^{1/m}n^{2-1/m}$. Lemma 4.3 implies that there exists a copy of $K^2(m)$, denoted by M, in $N_H(v)\cap E(G-W)$. By the definition of $E(G)$,

$$\left|\left(\bigcap_{e\in M} N_H(e)\right)\cap(B'\setminus W)\right| \geqslant |B'|-m^2\sqrt{\gamma_2}|B'|-C \geqslant m-1$$

for sufficiently large n and small γ_2. Pick such any $m-1$ vertices together with v and $V(M)$, we obtain a copy of $K^3(m)$ of type $(m+1,2m-1)$ avoiding W.

Now assume $|B'|>n/2$. For any pair $aa'\in\binom{A'}{2}$, we have $|N_H(aa')\cap B'| \geqslant m^{1/m}n^{1-1/m}$. Construct another auxiliary graph G' as follows: set $V(G')=A'$ and $E(G')$ consists of all pairs $aa'\in\binom{A'}{2}$ with $|N_H(aa')\cap A'| \geqslant (1-\sqrt{\gamma_2})|A'|$. Similarly, since there are at most $\gamma_2 n^3$ $A'A'A'$-edges missing in H, there are at most $3\gamma_2 n^3/(\sqrt{\gamma_2}|A'|) \leqslant 8\sqrt{\gamma_2}n^2$ edges aa' missing in G'. By double-counting the number of ordered pairs (v,e) with $v\in B'\setminus W$ and $e\in N_H(v)\cap E(G'-W)$, we have

$$\sum_{v\in B'\setminus W} |N_H(v)\cap E(G'-W)| \geqslant (|G'|-C|A'|)\cdot(m^{1/m}n^{1-1/m}-|B'\cap W|).$$

Note that $(|G'|-C|A'|)(m^{1/m}n^{1-1/m}-|B'\cap W|)/|B'\setminus W| > \frac{1}{2}m^{1/m}|A'|^{2-1/m}$. We can choose a vertex $v\in B'\setminus W$ such that $|N_H(v)\cap E(G'-W)| > \frac{1}{2}m^{1/m}|A'|^{2-1/m}$. Lemma 4.3 implies that there is a copy of $K^2(m)$, denoted by M', in $N(v)\cap E(G'-W)$. By the definition of $E(G')$,

$$\left|\bigcap_{e\in M'} N(e)\cap(A'\setminus W)\right| \geqslant |A'|-m^2\sqrt{\gamma_2}|A'|-C \geqslant m-1.$$

Pick any such $m-1$ vertices together with v and $V(M')$, we obtain a copy of $K^3(m)$ of type $(3m-1,1)$ avoiding W. This completes the proof of claim 5. □

Claim 6. *If $\delta_2(H)$ satisfies (2) in Theorem 1.4, then H contains a copy K of $K^3_{m,m}$ of type $(m+1,2m-1)$ or $(3m-1,1)$, unless $|B'|=\lfloor n/2\rfloor$ when $n\equiv 1 \pmod 4$. Furthermore, for any $0\leqslant t\leqslant m$, H contains a copy K' of $K^3_{m,m}$ of type $(m+2t,2m-2t)$ disjoint from K.*

Proof of Claim 6: If there exists a pair $a_1a_1'\in\binom{A'}{2}$ such that $|N_H(a_1a_1')\cap B'| \geqslant 2\gamma_1 n$, then we can choose m distinct vertices $b_1,\ldots,b_m\in N_H(a_1a_1')\cap B_1$ since $2\gamma_1 n-|B_2|>m$. Note that for a γ_2-good vertex $b\in B'$,

$$|E(H_b)\cap E(K^2(A',B'))| \geqslant |A'||B'|-\gamma_2 n^2 > \frac{m}{m+1}|A'||B'|,$$

we have

$$\left|\bigcap_{i=1}^{m} E(H_{b_i}) \cap E(K^2(A', B'))\right| \geqslant \frac{1}{m+1}|A'||B'|.$$

Thus $(\bigcap_{i=1}^{m} E(H_{b_i})) \cap E(K^2(A', B'))$ contains a matching of order $m-1$, choose such a matching $a_2b_2', \ldots, a_mb_m'$. So the subgraph induced by $\{a_1', a_1, a_2, \ldots, a_m\} \cup \{b_1, \ldots, b_m\} \cup \{b_2', \ldots, b_m'\}$ of H contains a copy of $K^3_{m,m}$ of type $(m+1, 2m-1)$.

Now assume $|N_H(a_1a_2) \cap B'| < 2\gamma_1 n$ for any $a_1a_2 \in \binom{A'}{2}$. Then $|N_H(a_1a_2) \cap A'| > n/2 - 2 - 2\gamma_1 n$. Let F be the spanning subgraph consisting of all the edges of type $A'A'B'$ of H. We claim that if there is some $b \in B'$ such that $|F_b| > 2m\gamma_1 n$, then H contains a copy of $K^3_{m,m}$ of type $(3m-1, 1)$. In fact, assume that there is some $b \in B'$ with $|F_b| > 2m\gamma_1 n$. First, suppose that F_b contains a matching of size m. Let $a_1a_1', \ldots, a_ma_m'$ be a matching of F_b. Since

$$\left|\bigcap_{i=1}^{m} N_H(a_ia_i') \cap A'\right| > m(n/2 - 2 - 2\gamma_1 n) - (m-1)|A'| > |A'|/2,$$

one can choose $m-1$ distinct vertices $a_1'', \ldots, a_{m-1}'' \in \bigcap_{i=1}^{m} N_H(a_ia_i') \cap A'$. And then the edges $a_ia_i'a_j'', a_ia_i'b \in E(H)$ $(i \in [m], j \in [m-1])$ form a copy of $K^3_{m,m}$ of type $(3m-1, 1)$. Now suppose that M is a maximum matching in F_b of size at most $m-1$. Clearly, $V(M)$ is a vertex cover of F_b and thus there exists a vertex a in $V(M)$ of degree at least $\frac{2m\gamma_1 n}{2(m-1)} \geqslant |A_2| + m$. That is to say, there are m distinct γ_2-good vertices $a_1'', \ldots, a_m''$ in $N_{A'}(ab)$. Note that for a γ_2-good vertex $a_i'' \in A'$, $|E(H_{a_i''}) \cap \binom{A'}{2}| \geqslant \binom{|A'|}{2} - \gamma_2 n^2 > \frac{m}{m+1}\binom{|A'|}{2}$, we have

$$\left|\bigcap_{i=1}^{m} E(H_{a_i''}) \cap \binom{A'}{2}\right| \geqslant \frac{1}{m+1}\binom{|A'|}{2}.$$

Thus $\bigcap_{i=1}^{m} E(H_{a_i''}) \cap \binom{A'}{2}$ contains a matching of order $m-1$, choose such a matching $a_2a_2', \ldots, a_ma_m'$. Therefore, the subgraph of H induced by $\{a_1'', \ldots, a_m''\} \cup \{a_2, a_2', \ldots, a_m, a_m'\} \cup \{a, b\}$ contains a copy of $K^3_{m,m}$ of type $(3m-1, 1)$, as desired. So the rest of the case is to show that such a vertex $b \in B'$ with $|F_b| \geqslant 2m\gamma_1 n$ does exist.

If $n \equiv 1 \pmod 4$ and $|B'| \leqslant \lfloor n/2 \rfloor - 1$ or $n \not\equiv 1 \pmod 4$ and $|B'| \leqslant \lceil n/2 \rceil - 1$, then for every pair ab with $a \in A'$, $b \in B'$, we have $|N_H(ab) \cap A'| \geqslant 1$. Hence for any $b \in B'$, we have $\delta(F_b[A']) \geqslant 1$ and so $|F_b| \geqslant |A'|/2 \geqslant 2m\gamma_1 n$, we are done. Now assume $|B'| \geqslant \lceil n/2 \rceil$. Then for any pair $aa' \in \binom{A'}{2}$, we have $|N_H(aa') \cap B'| \geqslant 1$. Since

$$\binom{|A'|}{2} \Big/ |B'| \geqslant \binom{n/2 - \gamma_1 n}{2} \Big/ (n/2 + \gamma_1 n) > 2m\gamma_1 n,$$

there exist at least one vertex $b \in B'$ such that $|F_b| > 2m\gamma_1 n$.

Next, we show that H contains a copy K' of $K^3_{m,m}$ of type $(m+2t, 2m-2t)$ disjoint from K, $0 \leqslant t \leqslant m$. Choose any m distinct γ_2-good vertices $a_1, \ldots, a_t \in A' \setminus V(K)$ and $b_{t+1}, \ldots, b_m \in B' \setminus V(K)$. Since $|E(H_{a_i}) \cap \binom{A'}{2}| \geqslant \binom{|A'|}{2} - \gamma_2 n^2$, there exists at least $6m+1$ vertices $a' \in A'$ with $|N_{H_{a_i}}(a') \cap A'| \geqslant |A'| - \sqrt{\gamma_2}n$, that is we can choose t distinct vertices $a'_1, \ldots, a'_t \in A' \setminus V(K)$ such that $|N_{A'}(a_i a'_i)| \geqslant |A'| - \sqrt{\gamma_2} n$ for $1 \leqslant i \leqslant t$. Similarly, since $|E(H_b) \cap E(K^2(A', B'))| \geqslant |A'||B'| - \gamma_2 n^2$, we can choose $m - t$ distinct vertices $b'_{t+1}, \ldots, b'_m \in B' \setminus V(K)$ such that $|N_{A'}(b_j b'_j)| \geqslant |A'| - \sqrt{\gamma_2} n$ for $t+1 \leqslant j \leqslant n$. Therefore, we have $|\,(\cap_{i=1}^{t} N_{A'}(a_i a'_i)) \cap \big(\cap_{i=t+1}^{m} N_A(b_i b_i)\big) \cap A'| \geqslant |A'|/2$. So we can pick m vertices $a''_1, \ldots, a''_m \in (\cap_{i=1}^{t} N_{A'}(a_i a'_i)) \cap \big(\cap_{i=t+1}^{m} N_A(b_i b_i)\big) \cap A'$ different from a_i, b_i, a'_i, b'_i, $i \in [m]$. Clearly, the subgraph of H induced by $\{a_i, a'_i : i \in [t]\} \cup \{b_i, b'_i : t+1 \leqslant i \leqslant m\} \cup \{a''_i : i \in [m]\}$ contains a copy of $K^3_{m,m}$ of type $(m + 2t, 2m - 2t)$. This completes the proof. □

The next claim shows that we can find a small $K^3(m)$-tiling to cover the vertices in $A_2 \cup B_2$.

Claim 7. *Suppose that* $\delta_2(H) \geqslant \lfloor \frac{n}{2} \rfloor - 1$. *Let* $W \subset V(H)$ *with* $|W| \leqslant \gamma_2 n$. *Every vertex* $x \in (A_2 \cup B_2) \setminus W$ *can be covered by a copy of* $K^3(m)$ *of type* $(m, 2m)$ *avoiding* W.

Proof of Claim 7: Recall that every vertex in $A_1 \cup B_1$ is γ_2-good with respect to $\mathcal{B}[A, B]$. Let G be the graph on vertex set V and edge set consisting of all pairs $xy \in \binom{V}{2}$ satisfying $d_{\mathcal{B} \setminus H}(xy) \leqslant \sqrt{\gamma_2} n$. By the definition of γ_2-good, for each vertex $x \in A_1 \cup B_1$, we have $d_G(x) \geqslant n - \sqrt{\gamma_2} n$.

If $x \in A_2 \setminus W$, by A3), we have $|H_x[B_1 \setminus W]| \geqslant \frac{2}{3}\binom{|B'|}{2} - \gamma_1 n^2 - 2\gamma_n^2 \geqslant \frac{1}{2}\binom{|B'|}{2}$. Hence $|E(H_x[B_1 \setminus W]) \cap E(G)| \geqslant \frac{1}{3}\binom{|B'|}{2}$. Thus, by Lemma 4.3, $H_x[B_1 \setminus W] \cap G$ contains a copy of $K^2(m)$, denoted by M. Since $d_H(e) \geqslant |A'| - \sqrt{\gamma_2} n$ for any $e \in M$, we have $|\bigcap_{e \in M} N_H(e) \cap A'| \geqslant |A'| - m^2 \sqrt{\gamma_2} n$. Hence we can choose $\{a_1, \ldots, a_{m-1}\} \subset \big(\bigcap_{e \in M} N_H(e) \cap A'\big) \setminus W$. Therefore, the subgraph of H induced by $\{x, a_1, \ldots, a_{m-1}\} \cup V(M)$ contains a copy of $K^3(m)$ of type $(m, 2m)$ covering x.

Now suppose $x \in B_2 \setminus W$. B3) together with A2), B2) imply that

$$|E(H_x) \cap E(G[A_1 \setminus W, B_1 \setminus W])| \geqslant \frac{n^2}{50} - 2\gamma_1 n^2 - \gamma_2 n^2 - \sqrt{\gamma}_2 n^2 \geqslant \frac{1}{100}|A'||B'|.$$

By Lemma 4.3, $H_x \cap G[A_1 \setminus W, B_1 \setminus W]$ contains a copy of $K^2(m)$ avoiding W, denoted by M'. Since $d_H(e) \geqslant |B'| - \sqrt{\gamma_2} n$ for any $e \in M'$, we have $|\bigcap_{e \in M} N_H(e) \cap B'| \geqslant |B'| - m^2 \sqrt{\gamma_2} n$. Hence we can choose $m-1$ distinct vertices $b_1, \ldots, b_{m-1} \in \big(\bigcap_{e \in M'} N_H(e) \cap B'\big) \setminus W$. Therefore, the subgraph of H induced by $\{x, b_1, \ldots, b_{m-1}\} \cup V(M)$ contains a copy of $K^3(m)$ of type $(m, 2m)$ covering x, as desired. □

Proof of Lemma 2.3: Let $t \equiv |B'| \pmod{2m}$ such that $0 \leqslant t \leqslant 2m - 1$. Let $\mathcal{K}_1$ be $2m - t$ disjoint copies of $K^3(m)$ of type $(m + 1, 2m - 1)$ or t disjoint copies of $K^3(m)$ of type $(3m - 1, 1)$ in H guaranteed by Claim 5. Note that $|V(\mathcal{K}_1)| \leqslant 6m^2$ is small enough. We can apply Claim 7 recursively to H to obtain a $K^3(m)$-tiling $\mathcal{K}_2$ covering all vertices of $(A_2 \cup B_2) \setminus V(\mathcal{K}_1)$. Moreover, every copy of $K^3(m)$ in $\mathcal{K}_2$ is of type $(m, 2m)$. Let $A'' := A' \setminus V(\mathcal{K}_1 \cup \mathcal{K}_2)$ and $B'' := B' \setminus V(\mathcal{K}_1 \cup \mathcal{K}_2)$. Clearly, $|B''| \equiv 0 \pmod{2m}$. Since $n \in 3m\mathbb{N}$, we have $|A'' \cup B''| \equiv 0 \pmod{3m}$ and $|A''| \equiv 0 \pmod{m}$. Since $|\mathcal{K}_1| < 2m$ and $|\mathcal{K}_2| \leqslant 2\gamma_1 n$, we have $n/2 - 5m\gamma_1 n \leqslant |A''|, |B''| \leqslant n/2 + \gamma_1 n$. Let $|A''| = (6a + s)m, |B''| = (6b' + 2t')m$. Then it is easy to check that $s \equiv t' \pmod 3$. So we can set $|A''| = (6a + s)m$ and $|B''| = (6b + 2s)m$ for some $0 \leqslant s \leqslant 5$. Now, each vertex in $A'' \cup B''$

is γ_2'-good with respect to $\mathcal{B}(A'', B'')$ for some constant $\gamma_2' \gg \gamma_2$. By Lemma 4.2, we can find $6(b-a)+s$ disjoint copies of $K^3(m)$ of type $(m, 2m)$ if $b - a \geqslant 0$, or $2(a-b)$ disjoint copies of $K^3(m)$ of type $(3m, 0)$ and s disjoint copies of $K^3(m)$ of type $(m, 2m)$ if $b - a < 0$. Let $\mathcal{K}_3$ be these copies of $K^3(m)$. Thus, $|\mathcal{K}_3| \leqslant 6|b-a| + s \leqslant 6\gamma_1 n$. Let $A^* = A'' \setminus V(\mathcal{K}_3)$ and $B^* = B'' \setminus V(\mathcal{K}_3)$. Then we have $|A^*| = |B^*| \equiv 0 \pmod{6m}$ and $|A^*| = |B^*| \geqslant n/2 - 100\gamma_1 mn$. Clearly, $A^* \subset A_1$ and $B^* \subset B_1$. Let $H^* = H[A^* \cup B^*]$. Since both $|A_2|$ and $|B_2|$ are small, it can be checked that there is some constant $\gamma_3 \gg \gamma_2'$ such that every vertex in H^* is γ_3-good with respect to $\mathcal{B}[A^*, B^*]$. By Lemma 4.2, H^* contains a $K^3(m)$-factor, say $\mathcal{K}_4$. Therefore, $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4$ is a $K^3(m)$-factor of H. □

Proof of Lemma 2.4: The proof is similar to the one of Lemma 2.3. Note that $n \in 3m\mathbb{N}$ and $\delta_2(H)$ satisfies condition (2). Let $t \equiv |B'| \pmod{2m}$ with $0 \leqslant t \leqslant 2m-1$. If t is even (note that $|B'| = \lfloor n/2 \rfloor$ and $n \equiv 1 \pmod 4$ belongs to this case), by Claim 6 for $m - t/2$, we can find a copy K' of $K^3_{m,m}$ of type $(3m-t, t)$ in H. Set $\mathcal{K}_1 = \{K'\}$. Now assume t is odd. Then we can find two disjoint copies K, K' of $K^3_{m,m}$ of types $(m+1, 2m-1)$ and $(3m-t-1, t+1)$ (by Claim 6 for $m-(t+1)/2$), respectively, or of types $(3m-1, 1)$ and $(3m-t+1, t-1)$ (by Claim 6 for $m-(t-1)/2$), respectively. In this case, set $\mathcal{K}_1 = \{K, K'\}$. For each case, we have $|B' \setminus V(\mathcal{K}_1)| \equiv 0 \pmod{2m}$ and $|A' \setminus V(\mathcal{K}_1)| \equiv 0 \pmod{m}$. Since $K^3_{m,m}$ is a spanning subgraph of $K^3(m)$, the existence of $K^3_{m,m}$-tiling $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ follows from the existence of $K^3(m)$-tilings $\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$ in H with the same argument as in Lemma 2.3. Finally we have $\mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_4$ is a $K^3_{m,m}$-factor of H. □

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