Chromatic Vertex Folkman Numbers[*]

Xiaodong Xu

Guangxi Academy of Sciences
Nanning, 530007, P.R. China

xxdmaths@sina.com

Meilian Liang

School of Mathematics and Information Science
Guangxi University, Nanning, 530004, P.R. China

gxulml@163.com

Stanisław P. Radziszowski

Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623, U.S.A.

spr@cs.rit.edu

Abstract

For graph G and integers $a_1 \geqslant \cdots \geqslant a_r \geqslant 2$, we write $G \to (a_1, \cdots, a_r)^v$ if and only if for every r-coloring of the vertex set $V(G)$ there exists a monochromatic K_{a_i} in G for some color $i \in \{1, \cdots, r\}$. The vertex Folkman number $F_v(a_1, \cdots, a_r; s)$ is defined as the smallest integer n for which there exists a K_s-free graph G of order n such that $G \to (a_1, \cdots, a_r)^v$. It is well known that if $G \to (a_1, \cdots, a_r)^v$ then $\chi(G) \geqslant m$, where $m = 1 + \sum_{i=1}^{r}(a_i - 1)$. In this paper we study such Folkman graphs G with chromatic number $\chi(G) = m$, which leads to a new concept of chromatic Folkman numbers. We prove constructively some existential results, among others that for all $r, s \geqslant 2$ there exist K_{s+1}-free graphs G such that $G \to (s, \cdots_r, s)^v$ and G has the smallest possible chromatic number $r(s-1) + 1$ with respect to this property. Among others we conjecture that for every $s \geqslant 2$ there exists a K_{s+1}-free graph G on $F_v(s, s; s+1)$ vertices with $\chi(G) = 2s - 1$ and $G \to (s, s)^v$.

Mathematics Subject Classifications: 05C55, 05C35

1 Preliminaries, notation and definitions

Throughout this paper we consider only finite undirected loopless simple graphs. For graph $G = (V, E)$, denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its

edges. A complete graph of order n is denoted by K_n, and a cycle of length n by C_n. The clique number of G is denoted by $cl(G)$, and the chromatic number by $\chi(G)$. An (s,t)-graph is a graph that contains neither an s-clique nor a t-independent set.

The set $\{1, \cdots, n\}$ is denoted by $[n]$. Let $r, s, a_1, \cdots, a_r$ be integers such that $r \geqslant 2$, $s > \max\{a_1, \cdots, a_r\}$ and $\min\{a_1, \cdots, a_r\} \geqslant 2$. We write $G \rightarrow (a_1, \cdots, a_r)^v$ if and only if for every r-coloring of $V(G)$ there exists a monochromatic K_{a_i} in G for some color $i \in [r]$. The sets of vertex Folkman graphs are defined as

$$\mathcal{F}_v(a_1, \cdots, a_r; s) = \{G \mid G \rightarrow (a_1, \cdots, a_r)^v \text{ and } cl(G) < s\},$$

and the vertex Folkman numbers are defined as the smallest orders of graphs in these sets, namely

$$F_v(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v(a_1, \cdots, a_r; s)\}.$$

In 1970, Folkman [8] proved that for every $s > \max\{a_1, \cdots, a_r\}$ the sets $\mathcal{F}_v(a_1, \cdots, a_r; s)$ are nonempty. If $a_1 = \cdots = a_r = a$, then we will use a simpler notation for the corresponding set of vertex Folkman graphs $\mathcal{F}(r, a, s) = \mathcal{F}_v(a_1, \cdots, a_r; s)$, and for numbers $F(r, a, s) = F_v(a_1, \cdots, a_r; s)$. The case of $F(r, s, s+1)$ was studied in particular by Dudek and Rödl [6] and Hàn, Rödl and Szabó [10]. The latter work contains the result stated in the following theorem.

Theorem 1. (Hàn-Rödl-Szabó, 2018) *For any positive integer r there exists a constant $C = C(r)$ such that for every $s \geqslant 2$ it holds that $F(r, s, s+1) \leqslant Cs^2 \log^2 s$.*

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$. It is well known that if $G \rightarrow (a_1, \cdots, a_r)^v$ then $\chi(G) \geqslant m$ [19]. In this paper we study vertex Folkman graphs G with the corresponding Folkman numbers, when the graphs G satisfy an additional constraint on their chromatic number, namely $\chi(G) = m$. This motivates the concept of the *minimum chromatic Folkman graphs*, $\mathcal{F}_v^\chi(a_1, \cdots, a_r; s)$, and corresponding *minimum chromatic Folkman numbers*, $F_v^\chi(a_1, \cdots, a_r; s)$. Formally:

$$\mathcal{F}_v^\chi(a_1, \cdots, a_r; s) = \{G \mid G \in \mathcal{F}_v(a_1, \cdots, a_r; s) \text{ and } \chi(G) = m\},$$

and

$$F_v^\chi(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_v^\chi(a_1, \cdots, a_r; s)\}.$$

The remainder of this paper focuses on minimum chromatic vertex Folkman graphs and numbers, and we will call them simply chromatic Folkman graphs and numbers, respectively. Similarly as in the classical case, we will use further notational abbreviation for the diagonal chromatic cases, namely, if $a_1 = \cdots = a_r = a$, then we set $\mathcal{F}^\chi(r, a, s) = \mathcal{F}_v^\chi(a_1, \cdots, a_r; s)$ and $F^\chi(r, a, s) = F_v^\chi(a_1, \cdots, a_r; s)$.

The main result of this paper, Theorem 3 in Section 2.1, shows the existence of $F^\chi(r, s, s+1)$ for $r, s \geqslant 2$ by an explicit construction. We must admit that the implied

upper bound is much larger than the bound in Theorem 1, but our construction gives an additional constraint on the chromatic number. We also conjecture that, in some cases, our construction is the best possible one, in particular that for every $s \geqslant 2$ there exists a K_{s+1}-free graph G on $F_v(s, s; s+1)$ vertices with the chromatic number $\chi(G)$ equal to $m = 2s - 1$ such that $G \rightarrow (s, s)^v$, or equivalently, that $F_v^\chi(s, s; s+1) = F_v(s, s; s+1)$. We wish to remark that a technique as in this paper should lead to more general existence results for $F_v^\chi(a_1, \cdots, a_r; s)$, and with some further enhancements also for generalized Folkman numbers, where one avoids monochromatic graphs G_i instead of K_{a_i}. However, we do not study such extensions in this work.

In another direction, one can consider coloring the edges instead of vertices, which leads to concepts analogous to those above for vertices. Let $M = R(a_1, \cdots, a_r) = \min\{n \mid K_n \rightarrow (a_1, \cdots, a_r)^e\}$, i.e. set M to be the classical multicolor Ramsey number for fixed $a_1, \ldots, a_r$ (In this paragraph the subscript/superscript e means that we color edges instead of vertices). A classical argument easily proves that if $G \rightarrow (a_1, \cdots, a_r)^e$ then $\chi(G) \geqslant M$ [15]. Further, we can define $\mathcal{F}_e^\chi(a_1, \cdots, a_r; s) = \{G \mid G \in \mathcal{F}_e(a_1, \cdots, a_r; s)$ and $\chi(G) = M\}$, and $F_e^\chi(a_1, \cdots, a_r; s) = \min\{|V(G)| \mid G \in \mathcal{F}_e^\chi(a_1, \cdots, a_r; s)\}$. From the construction by Nešetřil and Rödl in [21] we can see that there exists a K_{s+1}-free graph G with the chromatic number equal to $R(s, s)$ such that $G \rightarrow (s, s)^e$. For $3 \leqslant k \leqslant l$, essentially the same reasoning as in [21] implies the existence of K_{s+1}-free graphs G with $\chi(G) = R(k, l)$ and $G \rightarrow (k, l)^e$. In a similar way, one could also show that $F_e^\chi(a_1, \cdots, a_r; s)$ exists.

In the sequel we will be coloring only vertices. The rest of this paper is organized as follows. In Section 2, the existence of the chromatic Folkman numbers $F^\chi(r, s, s+1)$ is proved, the corresponding upper bound is discussed, and a conjecture is posed. In Section 3, we study some minimal Folkman graphs with the smallest minimum degree. Though not yet directly connected to chromatic Folkman numbers, we point to a possible connection between them.

2   Chromatic Folkman numbers

Our main motivation to study chromatic Folkman graphs and numbers is to understand how they may differ from the regular Folkman cases. If for a special family of cases we find that the chromatic cases yield the same numbers as the regular ones, then we could see it as a stronger version of the Folkman theorem. In any case, we may be able to see better the structure of extremal Folkman graphs.

Before we prove the existence of $F^\chi(r, s, s + 1)$, we want to observe that $F_v^\chi(a_1, \cdots, a_r; s)$ and $F_v(a_1, \cdots, a_r; s)$ can be different. It is known that there exists exactly one K_5-free graph Q of order 13 such that $Q \rightarrow (3, 4)^v$ [18]. This graph Q is the complement of the unique $(3, 5)$-Ramsey critical graph, which is cyclic on the set $\mathbb{Z}_{13}$ with arcs of length 1 and 5. One can easily check that $\chi(Q) = 7$. Thus, since in this case $m = 6$, we have $F_v^\chi(3, 4; 5) > 13 = F_v(3, 4; 5)$. Bikov and Nenov [4] provided us with other examples of this type. In particular they observed that $F_v^\chi(4, 4; 6) > F_v(4, 4; 6)$, since the only extremal graph for $F_v(4, 4; 6)$ is $Q + K_1$. More such examples follow from

their recent work [1, 2]. On the other hand we feel that these examples are special in that they exploit larger difference between arrowed and avoided graphs. This is captured in the following problem for the borderline diagonal cases, for two and more colors.

Problem 1. *Is it true that $F^\chi(r, s, s+1) = F(r, s, s+1)$ for all $r, s \geqslant 2$?*

Lemma 2 in the next section gives a positive answer to Problem 1 for all $r \geqslant 2$ with $s = 2$. A graph presented in Figure 1 in [22] and the computations described therein implicitly give a positive answer in one more special 2-color case. Namely, it holds that $F_v^\chi(3, 3; 4) = F_v(3, 3; 4) = 14$. One can ask similar questions related to minimum chromatic edge Folkman numbers (when coloring edges instead of vertices), but these seem much more difficult to answer.

While several general cases of vertex Folkman numbers have been studied, the problem of finding their exact values for small cases remains elusive. This holds even for just two colors when s is smaller than $m - 1$. For instance, the case of $F_v(4, 4; 5)$ seems to be difficult, for which only the bounds $19 \leqslant F_v(4, 4; 5) \leqslant 23$ are known [3, 23]. The computational approach is often too expensive. Just testing a single instance of arrowing for an upper bound witness graph is not easy in most cases, and improving lower bounds is much harder since it may involve a very large number of arrowing instances. Both upper and lower bounds for edge Folkman numbers tend to be still harder computationally.

2.1 Main theorem

The upper bound on $F^\chi(r, s, s+1)$, which can be obtained by the construction of Theorem 3 below, is rather large. This and other bounds we could derive for $F^\chi(r, a, s)$ also seem less tight than those known for $F(r, a, s)$, except the special case for $a = 2$ and $s = 3$ captured by Lemma 2. We follow this lemma by a theorem describing the quite special but more general and more difficult case of $F^\chi(r, s, s+1)$.

Lemma 2. *For all $r \geqslant 2$, $F^\chi(r, 2, 3)$ exists and it is equal to $F(r, 2, 3)$.*

Proof. The sets $\mathcal{F}^\chi(r, 2, 3)$ consist of triangle-free graphs with chromatic number equal to $r + 1$, which in the case of arrowing K_2 clearly coincides with the smallest order graphs in $\mathcal{F}_v(r, 2, 3)$. The latter sets are known to be nonempty for all $r \geqslant 2$. Thus we also have $F^\chi(r, 2, 3) = F(r, 2, 3)$. $\qquad\square$

In the basic case of two colors, one can easily see that $F^\chi(2, 2, 3) = 5$, because $C_5 \in \mathcal{F}_v(2, 2; 3)$ and $\chi(C_5) = m = 3$. Clearly, we also have $F_v(2, 2; 3) = 5$. We know that $F_v(2, 2, 2; 3) = 11$, or the smallest 4-chromatic triangle-free graph has 11 vertices, witnessed by the Grötzsch graph. Also, it is known that $F^\chi(4, 2, 3) = F_v(2, 2, 2, 2; 3) = 22$, or the smallest 5-chromatic triangle-free graph has 22 vertices [11]. The best known bounds in the first open case, $32 \leqslant F^\chi(5, 2, 3) \leqslant 40$, are due to Goedgebeur [9].

Theorem 3. *For given integers $r \geqslant 2$ and $s \geqslant 3$, let $b_i = i(s-1)+1$ for $i \in [r-1]$, and $B = \prod_{i=1}^{r-1} b_i$. Then $F^\chi(r, s, s+1)$ exists and*

$$F^\chi(r, s, s+1) \leqslant 1 + s + \sum_{i=2}^{r-1} F^\chi(i, s, s+1) + B \cdot F^\chi(r, s-1, s). \tag{1}$$

In particular, for all $s \geqslant 3$, the chromatic Folkman number $F^\chi(2, s, s+1)$ exists and we have $F^\chi(2, s, s+1) \leqslant 1 + s + sF^\chi(2, s-1, s)$.

Proof. For $r \geqslant 3$, we will construct a graph $G(r, s) \in \mathcal{F}^\chi(r, s, s+1)$ given any graphs in each of $\mathcal{F}^\chi(i, s, s+1)$ for $2 \leqslant i \leqslant r-1$ and any graph in $\mathcal{F}^\chi(r, s-1, s)$. The vertices of the graph $G(r, s)$ will be formed by vertices of given graphs corresponding to the terms of the right-hand-side of (1). The proof is using simultaneous induction on r and s, and it has two main parts: construction of $G(r, s)$, and the proof that $G(r, s)$ has required properties.

Note that the second part of the theorem is just an instantiation of the first part for two colors, $r = 2$, in which case the main summation of (1) is empty. Thus, the basis of our induction is formed by the sets $\mathcal{F}^\chi(i, 2, 3)$, which are nonempty by Lemma 2, and where the corresponding Folkman numbers satisfy $F^\chi(i, 2, 3) = F(i, 2, 3)$ for all $i \geqslant 2$.

Construction of the graph $G(r, s)$ by induction for $r, s \geqslant 3$.

Let G_0 be the graph of order 1, $G_1 = K_s$, and set $V_0 = V(G_0)$, $V_1 = V(G_1)$. We may assume that the graphs $G_i = (V_i, E_i)$ such that $|V_i| = F^\chi(i, s, s+1)$ and $G_i \in \mathcal{F}^\chi(i, s, s+1)$ have been already constructed, for $2 \leqslant i < r$. Therefore, we know that $\chi(G_i) = b_i = i(s-1)+1$ for $i \in \{0, \cdots, r-1\}$, and hence we can partition each of the sets of vertices V_i into $\chi(G_i)$ nonempty independent sets $V_i(j)$ in G_i, so that

$$V_i = \bigcup_{j=1}^{\chi(G_i)} V_i(j).$$

Let H be any graph in the set $\mathcal{F}^\chi(r, s-1, s)$ with $|V(H)| = F^\chi(r, s-1, s)$ vertices. For the part of $G(r, s)$ corresponding to the last term of (1), we take $B = \prod_{i=1}^{r-1} b_i$ isomorphic copies $H(j_0, \cdots, j_{r-1})$ of H indexed by r-tuples $(j_0, \cdots, j_{r-1})$, where $1 \leqslant j_k \leqslant \chi(G_k)$ for $k \in \{0, \cdots, r-1\}$. Note that the order of $G(r, s)$ is equal to the right-hand-side of (1) because the sizes of parts described above match exactly its terms,

$$V = V(G(r, s)) = V_0 \cup V_1 \cup \bigcup_{i=2}^{r-1} V_i \cup \bigcup_{(j_0, \cdots, j_{r-1})} V(H(j_0, \cdots, j_{r-1})).$$

Finally, we complete the construction of $G(r, s)$ by adding the edges with one end in any of the sets of vertices $V(H(j_0, \cdots, j_{r-1}))$ and the other end in the sets $V_i(j)$, as follows: For each fixed r-tuple $(j_0, \cdots, j_{r-1})$, where $1 \leqslant j_i \leqslant b_i$ for $i \in \{0, \cdots, r-1\}$, we add all possible edges with one end in $V(H(j_0, \cdots, j_{r-1}))$ and the other end in $V_i(j_i)$.

Proof that $G(r, s) \in \mathcal{F}^{\chi}(r, s, s + 1)$.

We need to show that for all $r, s \geqslant 3$ we have: (i) $cl(G(r, s)) < s + 1$, (ii) in every r-coloring of the vertices V we have a monochromatic K_s, and (iii) $\chi(G(r, s)) = m = r(s - 1) + 1$.

(i) Assume the contrary, and suppose that some set $S \subset V$ of order $s + 1$ induces K_{s+1}. Let $k_i = |S \cap V_i|$ for $0 \leqslant i < r$. From the construction we see that $k_i \leqslant s$ and there exists exactly one t for which $k_t > 0$. Similarly, there exists exactly one r-tuple $(j_0, \cdots, j_{r-1})$ such that $h = |S \cap V(H(j_0, \cdots, j_{r-1}))| > 0$. Note that since $h \leqslant s - 1$ and $k_t + h = s + 1$, then $k_t \geqslant 2$. However, each vertex of $H(j_0, \cdots, j_{r-1})$ can be adjacent only to independent sets $V_i(j) \subset V_i$, hence we have a contradiction for $i = t$, and thus $cl(G(r, s)) \leqslant s$. We actually observe that $cl(G(r, s)) = s$ since the graph $G(r, s)$ contains $G_1 = K_s$.

(ii) Assume the contrary, and suppose that some r-coloring C of V does not contain any monochromatic K_s. Let c_0 be the color of V_0, and c_1 be a different color of one of the vertices in V_1. Using the assumptions that $G_i \in \mathcal{F}^{\chi}(i, s, s + 1)$ and that there is no monochromatic K_s in C restricted to V_i, we can see that the $(i + 1)$-st color c_i must be used for some vertex in V_i. Thus, we can find a vertex $v_i \in V_i$ in a new color $c_i = C(v_i)$, for each $2 \leqslant i < r$. Let j_i' be such that $v_i \in V_i(j_i')$. The graph $H(j_0', \cdots, j_{r-1}')$ is isomorphic to $H \in \mathcal{F}^{\chi}(r, s - 1, s)$, and therefore it contains a monochromatic $(s - 1)$-clique S in color c_i for some $0 \leqslant i < r$. Now in $G(r, s)$, this S can be extended to a monochromatic K_s in color c_i by adding vertex v_i, which is a contradiction.

(iii) Part (ii) implies that $\chi(G(r, s)) \geqslant r(s - 1) + 1$, hence we only need to prove that $\chi(G(r, s)) \leqslant r(s - 1) + 1$. We will show how to appropriately color V with $r(s - 1) + 1$ colors. First, for each $i \in \{0, \cdots, r - 1\}$, we color the vertices in independent sets $V_i(j)$ with color j, for $j \in [\chi(G_i)]$. This step, by the inductive assumption, can use exactly $(r - 1)(s - 1) + 1$ colors. Next, also by the inductive assumption, we color properly the graphs $H(j_0, \cdots, j_{r-1})$ just themselves using $r(s - 2) + 1$ colors. Note that this is r less than the total number of allowed colors. Let k_i, for $0 \leqslant i < r$, denote these extra colors. We use colors k_i to recolor some vertices of the H-graphs, and thus obtain a proper coloring of the entire $G(r, s)$ with $r(s - 1) + 1$ colors, as follows: For each r-tuple $(j_0, \cdots, j_{r-1})$, for every vertex $v \in V(H(j_0, \cdots, j_{r-1}))$ which has color j_i, recolor v with a new color k_i, for each $i \in \{0, \cdots, r - 1\}$. $\qquad\square$

It is interesting to see that if the vertex of V_0 is removed from $G(r, s)$, then in the part (iii) of the proof of Theorem 3 the chromatic number $\chi(G(r, s))$ drops to $m - 1$, and thus the arrowing of part (ii) would not hold.

We could improve a little the upper bound in Theorem 3 using an approach as in [24], but at the cost of a significantly more complex construction, and hence we decided to not include it.

Recall that the chromatic Folkman graphs and numbers we study here have just one additional constraint on top of regular Folkman graphs and numbers, namely the minimal possible chromatic number. In general, in Theorem 3 and other places of this paper, if

χ is removed from an upper bound, then it also holds for regular Folkman numbers. In particular, the special case of our Theorem 3 for two colors but without considering chromatic numbers gives the bound $F(2, s, s+1) \leqslant 1+s+sF(2, s-1, s)$. The same follows from a construction by Nenov [17] using the corona product of graphs. Bikov and Nenov [4] pointed out that the constructions of [17, 14] could be used as building blocks in our proofs. Unfortunately, at the time of writing the first version of this paper we were not aware of them. Another upper bound construction of a similar type for vertex Folkman graphs was presented by Xu et al. in [24]. Some special multicolor cases of Theorem 3 for regular Folkman numbers (without considering chromatic numbers), appear in Theorem 6 in [16] by Łuczak, Ruciński and Urbański (2001).

2.2 Bounds on F_v^χ for 2 colors

A simple upper bound on $F^\chi(r, s, s+1)$ for $r = 2$ based on Theorem 3 is $cs!$, for some positive constant c, which is much larger than the upper bound on $F(2, s, s+1)$ in Theorem 1. In this section we look at some other 2-color cases: off-diagonal $F_v^\chi(a, b; s+1)$ in Theorem 4 and some special subcases of $F^\chi(s, s; s+1)$ in the sequel.

Theorem 4. *For any integers a, b and s such that $2 \leqslant a, b \leqslant s$, $F_v^\chi(a, b; s+1)$ exists and we have*

$$F_v^\chi(a, b; s+1) \leqslant \frac{a+b-1}{2s-1} F_v^\chi(s, s; s+1). \tag{2}$$

Proof. Suppose that $G \in \mathcal{F}_v^\chi(s, s; s+1)$, and the order of G is $F_v^\chi(s, s; s+1)$. We clearly have $\chi(G) = 2s - 1$. Write the set of vertices of G as a partition

$$V(G) = \bigcup_{j=1}^{2s-1} I_j,$$

where I_j's are independent sets for $j \in [2s-1]$, and $|I_{j_1}| \leqslant |I_{j_2}|$ for $j_1 < j_2$ and $j_1, j_2 \in [2s-1]$. Let G_i be the subgraph of G induced by $\bigcup_{j=1}^{i} I_j$. Note that this implies $\chi(G_i) = i$ for each i, since otherwise $\chi(G) < 2s - 1$.

We claim that $G_{a+b-1} \in \mathcal{F}_v^\chi(a, b; s+1)$. By the comments above we see that $\chi(G_{a+b-1}) = a + b - 1 = m$ as required, and $cl(G_i) \leqslant s$ holds by construction. It remains to be shown that $G_{a+b-1} \to (a, b)^v$. For a contradiction suppose that we have a red-blue coloring of $V(G_{a+b-1})$ without any red K_a and without any blue K_b. We can extend this coloring to a full red-blue coloring of $V(G)$ by coloring red all the vertices in I_j's for $a+b \leqslant j \leqslant s+b-1$, and coloring blue all the vertices in I_j's for $s+b \leqslant j \leqslant 2s-1$. This coloring does not contain any monochromatic K_s, which contradicts the assumption that $G \in \mathcal{F}_v^\chi(s, s; s+1)$. Considering the non-decreasing orders of the sets I_j, we have that

$$|V(G_i)| \leqslant \frac{a+b-1}{2s-1} F_v^\chi(s, s; s+1),$$

and thus the bound (2) and the theorem follow. $\square$

Bollobás and Thomason [5] studied the set-coloring of graphs, where an r-set-coloring of a graph G is defined as an assignment of r distinct colors to each vertex of G so that the sets of colors assigned to adjacent vertices are disjoint. The set-coloring variants of the vertex- and edge Folkman numbers were previously introduced and studied by the first two authors of this paper jointly with Wenfei Zhao and Zehui Shao [25].

Let us denote the minimum number of colors required to r-set-color any given graph G by $\chi^{(r)}(G)$. In 1979, Bollobás and Thomason proved that $\min\{\chi^{(r)}(G) \mid \chi(G) = t\} = t + 2r - 2$ [5]. We need a simple lemma using this result as follows.

Lemma 5. $\chi^{(2s-1)}(C_{4s-1}) = 4s - 1$.

Proof. Using the above result by Bollobás and Thomason on $\chi^{(r)}(G)$, since $\chi(C_{4s-1}) = 3$, we clearly have $\chi^{(2s-1)}(C_{4s-1}) \geqslant 4s - 1$. On the other hand, it is easy to give a proper $(2s-1)$-set-coloring witnessing $\chi^{(2s-1)}(C_{4s-1}) \leqslant 4s - 1$. We take both the vertices of the cycle and colors to be in the set $\mathbb{Z}_{4s-1}$, and assume that the edges of the cycle are $\{i, i+1\}$ for $i \in \mathbb{Z}_{4s-1}$, all modulo $4s - 1$. We assign the colors $\{i(2s-1) + j \mid 0 \leqslant j \leqslant 2s - 2\}$ to the vertex i of C_{4s-1}, for each $i \in \mathbb{Z}_{4s-1}$. One can easily see that the sets of colors assigned to adjacent vertices are disjoint. $\qquad\square$

The composition of simple graphs G and H is denoted by $G[H]$, and it is defined as the graph with vertex set $V(G) \times V(H)$, in which vertex (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. We will need another simple lemma involving $G[H]$ and $\chi^{(r)}(G)$, namely:

Lemma 6. *If G and H are graphs and $\chi(H) = r$, then $\chi^{(r)}(G) = \chi(G[H])$.*

Proof. Klavžar proved that if $\chi(H) = r$, then $\chi(G[H]) = \chi(G[K_r])$ [12]. We can also easily see that $\chi^{(r)}(G) = \chi(G[K_r])$. The lemma follows. $\qquad\square$

The bound $F_v(2s, 2s; 2s + 1) \leqslant 5F_v(s, s; s + 1)$ was obtained constructively by Kolev who used the composition of graphs $G[H]$ [13]. We will use a similar approach to obtain an upper bound on the chromatic Folkman numbers of the form $F_v^\chi(2s, 2s; 2s + 1)$.

Theorem 7. *For any integer $s \geqslant 2$, we have*

$$F_v^\chi(2s, 2s; 2s + 1) \leqslant (4s - 1)F_v^\chi(s, s; s + 1).$$

Proof. Let H be any graph in $\mathcal{F}_v^\chi(s, s; s + 1)$ of order $F_v^\chi(s, s; s + 1)$, and thus $\chi(H) = 2s - 1$. Observe that $C_{4s-1}[H] \rightarrow (2s, 2s)^v$. By Lemmas 5 and 6, we have $\chi(C_{4s-1}[H]) = \chi^{(2s-1)}(C_{4s-1}) = 4s - 1$. This in turn implies that $C_{4s-1}[H] \in \mathcal{F}_v^\chi(2s, 2s; 2s + 1)$. Finally, since the order of $C_{4s-1}[H]$ is equal to $(4s-1)F_v^\chi(s, s; s+1)$ and clearly $F_v^\chi(2s, 2s; 2s+1) \leqslant |V(C_{4s-1}[H])|$, this completes the proof. $\qquad\square$

2.3 A conjecture

The classical Turán graph $T_{n,r}$ is a complete multipartite graph on n vertices whose r partite sets have sizes as equal as possible. We say that a graph G is *equitably r-colorable*

if its vertices can be properly colored with r-colors so that the sizes of any two color classes differ by at most 1. Clearly, any n-vertex graph G is equitably r-colorable if and only if G is a subgraph of $T_{n,r}$. This motivates the formulation of the following conjecture.

Conjecture. *For any integer $s \geqslant 2$, let $n = F_v(s, s; s+1)$. Then there exists an n-vertex K_{s+1}-free subgraph G of the Turán graph $T_{n,2s-1}$, such that $G \to (s, s)^v$.*

Any subgraph of $T_{n,2s-1}$ has the chromatic number upper bounded by $2s-1$. Therefore, we can easily see that the Conjecture implies the equality $F_v^\chi(s, s; s+1) = F_v(s, s; s+1)$, in particular it would give a positive answer to Problem 1 for two colors ($r = 2$). The Conjecture holds easily for $s = 2$ and $n = 5$ by considering C_5. For $s = 3$, we note that the graph in Figure 1 of [22] witnessing $F_v^\chi(3, 3; 4) = F_v(3, 3; 4) = 14$ is equitably 5-colorable, and so it gives the second positive case. The Conjecture, if proven true, would make the search for the upper bound witnesses for $F_v(s, s; s+1)$ much easier, including an approach using computer constructions.

Nenov studied several problems related to $F_v(r, 2, s + 1)$, for instance in [20]. Observe that using essentially only the definitions, we can easily see that $F_v(2s - 2, 2, s + 1) \leqslant F_v^\chi(s, s; s+1)$, though we suspect that a much better upper bound on $F_v(2s - 2, 2, s + 1)$ is true. Independently, $F_v(2s - 2, 2, s + 1)$ may be much smaller than $F_v(s, s; s+1)$. In another direction, one could use C_5, similarly as we used C_{4k-1} in Theorem 7, to study the cases of $F_v^\chi(2s, 2s, 2s; 2s + 1)$ and more general diagonal and non-diagonal cases. All of these problems seem interesting but difficult.

3 Minimum degree of graphs in $\mathcal{F}_v(s, s; s + 1)$

In this section we prove a theorem and then pose a problem concerning lower bound on the minimum degree in some minimal Folkman graphs. We consider only the case of $F_v(s, s; s + 1)$.

Theorem 8. *For all integers $s \geqslant 3$, we have:*
(a) For every graph G, if $G \to (s, s)^v$ and $G - u \not\to (s, s)^v$ for every vertex $u \in V(G)$, then the minimum degree $\delta(G)$ satisfies $\delta(G) \geqslant 2s - 2$, and
(b) There exists a K_{s+1}-free graph G with minimum degree $\delta(G) = 2s - 2$, such that $G \to (s, s)^v$ and $G - u \not\to (s, s)^v$ for every vertex $u \in V(G)$.

Proof. (a) For a contradiction, let $u \in V(G)$ be any vertex of degree at most $2s - 3$. Assuming that $G - u \not\to (s, s)^v$, consider any K_s-free bipartition $V_1 \cup V_2$ of the remaining vertices, so that $V(G) = V_1 \cup V_2 \cup \{u\}$. Without loss of generality we can also assume that $|V_1 \cap N_G(u)| \leqslant s - 2$. Color the vertices in $V_1 \cup \{u\}$ red and those in V_2 blue, and note that this coloring has no monochromatic K_s. This contradicts $G \to (s, s)^v$, and thus the part (a) follows.

(b) Let H be any K_{s+1}-free graph such that $H \to (s, s)^v$. Assume furthermore that H is both vertex- and edge-minimal with respect to arrowing, that is to say, $H - u \not\to (s, s)^v$

for all vertices $u \in V(H)$ and $H - e \not\to (s,s)^v$ for all edges $e \in E(H)$. Fix some vertex $u_0 \in V(H)$, and let A be the set of all $(2s-2)$-element subsets of $N_H(u_0)$ that contain two vertex-disjoint K_{s-1}'s. Suppose that A consists of m sets, i.e. $A = \{V_i \mid 1 \leqslant i \leqslant m\}$. Note that vertex-minimality of H implies that A is nonempty, so $m \geqslant 1$. First, we extend graph H to H' by adding new vertices $\{u_i \mid 1 \leqslant i \leqslant m\}$ and edges $\{u_i v \mid v \in V_i\}$ for all $i \in [m]$. Next, we delete vertex u_0 from H'. Observe that for all $i \in [m]$ the degree of vertex u_i in H' is equal to $2s-2$.

We claim that $H' \to (s,s)^v$. Suppose the contrary, namely that there exists a red-blue coloring C' of $V(H')$ without any monochromatic K_s. Thus, the restriction of C' to a coloring of the vertices of $H - u_0$, say C, is a witness of $H - u_0 \not\to (s,s)^v$. Since $H \to (s,s)^v$, C must contain two vertex-disjoint monochromatic K_{s-1}'s, furthermore they must be in different colors, and both are contained in $N_H(u_0)$. This however contradicts the properties of C' following from the construction of H'. Hence $H' \to (s,s)^v$.

We will define the final graph G satisfying (b) to be an induced subgraph of H' on the vertex set of the form $V(H') \setminus B$, where $B \subset \{u_i \mid 1 \leqslant i \leqslant m\}$. We choose B so that its vertex indices form a maximal subset of $[m]$ still giving $G \to (s,s)^v$. The properties of H' stated above guarantee that such B must be a proper subset of $[m]$. This can be seen, since if we delete $\{u_i \mid 1 \leqslant i \leqslant m\}$, we obtain a graph isomorphic to $H - u_0$, which does not arrow $(s,s)^v$. This way, we can obtain a minimal graph G which arrows $(s,s)^v$ with at least one vertex u_i of degree $2s-2$. Together with part (a), this completes the proof of (b). $\qquad \square$

Bikov and Nenov suggested [4] that in our proof of Theorem 8(b) we could use graphs M_k, for odd k, defined and studied by Nenov ([17], page 351).

Finally, we pose the following question.

Problem 2. *For which integers $n, s \geqslant 3$ does there exist a $(2s-2)$-regular K_{s+1}-free graph G on n vertices such that $G \to (s,s)^v$?*

It seems that even the case of $s = 3$ is not obvious. If the answer to this problem for each s is YES for at least some n, then the chromatic number of such a graph must be equal to $2s - 1$. Thus, it could give another proof of the existence of $F_v^\chi(s,s;s+1)$. Of course, the order n of such a graph G may be much larger than $F_v^\chi(s,s;s+1)$.

Acknowledgements

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