The average distance and the diameter of dense random regular graphs

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Abstract

Let \( AD(G_{n,d}) \) be the average distance of \( G_{n,d} \), a random \( n \)-vertex \( d \)-regular graph. For \( d = (\beta + o(1))n^\alpha \) with two arbitrary constants \( \alpha \in (0,1) \) and \( \beta > 0 \), we prove that \(|AD(G_{n,d}) - \mu| < \epsilon \) holds with high probability for any constant \( \epsilon > 0 \), where \( \mu \) is equal to \( \alpha^{-1} + \exp(-\beta^{1/\alpha}) \) if \( \alpha^{-1} \in \mathbb{N} \) and to \( \lceil \alpha^{-1} \rceil + 1 \) otherwise.

Consequently, we show that the diameter of the \( G_{n,d} \) is equal to \( \lceil \alpha^{-1} \rceil + 1 \) with high probability.

Mathematics Subject Classifications: 05C80, 05C12

1 Introduction

The study of the diameter of regular graphs is well motivated in graph theory. A central question is how to construct an \( n \)-vertex \( d \)-regular graph with the minimum possible diameter, which has an application to high-performance computing [12, 17, 26]. Let \( D'(n, d) \) denote the Moore bound, a well-known lower bound of the minimum possible diameter among all \( n \)-vertex \( d \)-regular graphs [26] (we will present the bound in Equation (3)). Let \( \text{diam}(G) \) denote the diameter of a graph \( G \). We define \( \text{diam}(G) = \infty \) if \( G \) is not connected. In this paper, we show that the diameter \( \text{diam}(G_{n,d}) \) of a random \( d \)-regular graph \( G_{n,d} \) of \( d = (\beta + o(1))n^\alpha \) with two arbitrary constants \( \alpha \in (0,1) \) and \( \beta > 0 \) satisfies

\[
\lim_{n \to \infty} (\text{diam}(G_{n,d}) - D'(n, d)) = \begin{cases} 
0 & \text{if either } \alpha^{-1} \notin \mathbb{N} \text{ or } (\alpha^{-1} \in \mathbb{N} \text{ and } \beta < 1), \\
1 & \text{if } \alpha^{-1} \in \mathbb{N} \text{ and } \beta > 1
\end{cases}
\]

with probability \( 1 - o(1) \).
Also, we study the average distance \( AD(G_{n,d}) \) of a random regular graph. The average distance \( AD(G) \) of a connected graph \( G \) is

\[
AD(G) = \left( \frac{n}{2} \right)^{-1} \sum_{\{u,v\} \in \binom{V}{2}} \text{dist}(u,v),
\]

where \( \text{dist}(u,v) \) is the shortest \( uv \)-path length. If \( G \) is not connected, we define \( AD(G) = \infty \).

For a graph property \( P \), we say that an \( n \)-vertex random graph \( G_n \) satisfies \( P \) with high probability (w.h.p.) if \( \lim_{n \to \infty} \Pr[G_n \text{ satisfies } P] = 1 \). In this paper, we prove the following results.

**Theorem 1.** For two constants \( \alpha \in (0,1) \) and \( \beta > 0 \), let \( d = (\beta + o(1))n^{\alpha} \) be an integer. For every constant \( \epsilon > 0 \), it holds w.h.p. that

\[
|AD(G_{n,d}) - \mu| < \epsilon,
\]

where

\[
\mu = \begin{cases} 
\alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}, \\
\lceil \alpha^{-1} \rceil & \text{otherwise.}
\end{cases}
\]

**Theorem 2.** For two constants \( \alpha \in (0,1) \) and \( \beta > 0 \), let \( d = (\beta + o(1))n^{\alpha} \) be an integer. It holds w.h.p. that

\[
diam(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1.
\]

The study of \( G_{n,d} \) originated from the configuration model introduced by Bollobas [3]. Independently, Bender and Canfield [1] considered a similar model. The configuration model usually enables us to study \( G_{n,d} \) for a constant \( d \). The case of \( d = d(n) \gg 1 \) is much less understood, though there is a well-known successful approach called the switching method, introduced by McKay [24]. See [33] for a detailed survey on \( G_{n,d} \). However, results shown by the switching method usually require the condition that \( d \ll n^{\gamma} \) where \( \gamma \ll 1 \) is some reasonable constant. Therefore, \( G_{n,d} \) of \( d = (\beta + o(1))n^{\alpha} \) with arbitrary constant \( \alpha \) seems to be far from these methods.

Another recent remarkable approach for the study of \( G_{n,d} \) is to compare \( G_{n,d} \) with an Erdős-Rényi graph \( G(n,p) \) of \( p = \frac{d}{n} \). Recall that \( G(n,p) \) is an \( n \)-vertex graph where every two distinct vertices \( u \) and \( v \) are joined by an edge with probability \( p \) independent from any other edges. Since each degree of \( G(n,p) \) is concentrated on \( np \), we may expect that \( G(n,p) \) and \( G_{n,d} \) share several structural properties if \( d = (1 + o(1))np \). For \( \log n \ll d \ll n^{1/3}/(\log n)^2 \), Kim and Vu [20] presented a coupling of \( G_{n,d} \) and \( G_n \) of \( p = (1 - o(1))\frac{d}{n} \) such that \( G(n,p) \subseteq G_{n,d} \) holds w.h.p. Dudek et al. [11, 14] improved this result by presenting a coupling having the same property for \( \log n \ll d \ll n \). Their result is

\[1\] In the conference version of this paper [29], we proved Theorem 2.
called the embedding theorem. The embedding theorem enables us to bound \( \text{diam}(G_{n,d}) \) and \( \text{AD}(G_{n,d}) \) from above by \( \text{diam}(G(n,p)) \) and \( \text{AD}(G(n,p)) \), respectively. Very recently, Gao, Isaev, and McKay [15] proved that there is a coupling of \( G(n,p) \) and \( G_{n,d} \) satisfying \( G(n,p) \supseteq G_{n,d} \) if \( p \geq \frac{C \text{dlog} n}{n} \) for some constant \( C \), \( d = \omega(\text{log} n) \) and \( d = o(n) \). We can immediately obtain Theorem 2 by combining the coupling of [15] and known results concerning the diameter of \( G(n,p) \). However, due to the \( O(\text{log} n) \) factor in the condition \( p \geq \frac{C \text{dlog} n}{n} \), Theorem 1 does not follow from [15] immediately.

To study \( \text{diam}(G(n,d)) \) and \( \text{AD}(G(n,d)) \), we shall look at \( \text{diam}(G(n,p)) \) and \( \text{AD}(G(n,p)) \) of \( p = \frac{d}{n} \). It is well known that \( G(n,p) \) of \( p = (\beta + o(1))n^{-1+\alpha} \) has diameter \( \lfloor \alpha^{-1} \rfloor + 1 \) [6, 4, 14].

As for the average distance, we obtain a concentration result of \( \text{AD}(G(n,p)) \), which might be of independent interest.

**Theorem 3.** For two constants \( \alpha \in (0, 1) \) and \( \beta > 0 \), let \( p = \beta n^{-1+\alpha} \) and

\[
\mu = \begin{cases} 
\alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}, \\
\lceil \alpha^{-1} \rceil & \text{otherwise}.
\end{cases}
\]

Then, there exist absolute constants \( C_1, C_2 > 0 \) such that

\[
|\text{AD}(G(n,p)) - \mu| \leq C_1 n^{-C_2}
\]

holds w.h.p.

1.1 Related results and trivial bounds

**Diameter of \( G(n,p) \).** There is a long line of the diameter of \( G(n,p) \) [22, 4, 8, 13, 28]. For dense \( G(n,p) \), Bollobas [4] proved the following result.

**Theorem 4** (Theorem 6 of [4]). Fix a positive constant \( c \). Let \( D = D(n) \geq 2 \) be a positive integer and \( p = p(n) \in [0, 1] \) be a real number satisfying

\[
p^D n^{D-1} = \log(n^2/c).
\]

Suppose that \( np = \omega(\text{log} n) \). Then, \( G(n,p) \) satisfies

\[
\lim_{n \to \infty} \Pr[\text{diam}(G(n,p)) = k] = \begin{cases} 
\exp(-c/2) & \text{if } k = D, \\
1 - \exp(-c/2) & \text{if } k = D + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

**Corollary 5.** Suppose that \( p = (\beta + o(1))n^{-1+\alpha} \), where \( \alpha \in (0, 1) \) and \( \beta > 0 \) are any constants. Then, \( \text{diam}(G(n,p)) = \lfloor \alpha^{-1} \rfloor + 1 \) holds w.h.p.

It should be noted that Theorem 5 also follows from the main result of Klee and Larman [22].

The diameter of \( G(n,p) \) of small \( p \) has gathered special attention [5, 28, 8]. In this line of work, there is a convention that the diameter of a disconnected graph is the...
maximum among all diameters of its connected components. Bollobás [5] proved that $\text{diam}(G(n, p)) \in A$ holds w.h.p. if $np - \log n = \omega(1)$, where $A = A(n) \subseteq \mathbb{N}$ satisfies $|A| \leq 4$. Chung and Lu [8] studied $\text{diam}(G(n, p))$ with $1 < np \leq c \log n$ where $c$ is some constant. For example, they proved that $\text{diam}(G(n, p)) = (1 + o(1)) \frac{\log n}{\log np}$ holds w.h.p. if $\omega(1) = np < \log n$. Riordan and Wormald [28] strengthened the results of [8], providing the tight estimate for $\text{diam}(G(n, p))$ for $1 + o(1) \leq np = O(1)$. For smaller $p$, Luczak [23] investigated $\text{diam}(G(n, p))$ with $np < 1$.

**Average distance of $G(n, p)$.** The average distance of random graphs with a power law degree sequence has gathered a great deal of attention in network analysis [18, 27, 2, 32, 9, 31]. Focusing on $G(n, p)$ with $np = \omega(\log n)$, one may observe that $\text{AD}(G(n, p)) \approx \text{diam}(G(n, p))$. More precisely, it is easy to see that $\text{AD}(G(n, p)) \leq \text{diam}(G(n, p)) = (1 + o(1)) \frac{\log n}{\log np}$ and $\text{AD}(G(n, p)) \geq (1 - o(1)) \frac{\log n}{\log np}$ hold by considering the maximum degree of $G(n, p)$.

Katzav et al. [18] presented analytical results on $\text{AD}(G(n, p))$ for dense $G(n, p)$ that coincide with Theorem 3. However, to the best of our knowledge, there are no known results with rigorous proofs for $\text{AD}(G(n, p))$ with $np = n^\Omega(1)$.

**Diameter of $G_{n,d}$.** The diameter of regular graphs has gathered special attention in graph theory [12, 17, 26] and has an application in designing efficient network topologies. Note that for every vertex $v$, there are at most $d(d-1)^k$ vertices having distance $k$ from $v$. Thus, for every $n$-vertex $d$-regular graph $G$ of diameter $D$ with $d \geq 3$, we have

$$D \geq \min \left\{ D \in \mathbb{N} : n \leq 1 + \sum_{i=1}^{D} d(d-1)^{i-1} \right\}$$

$$= \left\lfloor \log_{d-1} n + \log_{d-1} \left( 1 - \frac{2}{d} \left( 1 - \frac{1}{n} \right) \right) \right\rfloor$$

$$= \frac{\log n}{\log(d-1)} - O(1).$$

We denote by $D' = D'(n, d)$ this lower bound Equation (3), which is known as the Moore bound [26].

For random regular graphs $G_{n,d}$, Bollobás and de la Vega [7] proved that

$$\text{diam}(G_{n,d}) = D'(n, d) \pm O \left( \frac{\log \log n}{\log(d-1)} \right)$$

holds w.h.p. if the degree $d \geq 3$ is a constant. If $\log n \ll d \ll n^{o(1)}$, the embedding theorem of Dudek et al. [11, 14] and the lower bound Equation (3) together imply that

$$\text{diam}(G_{n,d}) = (1 + o(1)) \frac{\log n}{\log d} = (1 + o(1))D'(n, d)$$

holds w.h.p.
Suppose that \( d = (\beta + o(1))n^{\alpha} \), where \( \alpha \in (0, 1) \) and \( \beta > 0 \) are constants. From the embedding theorem, we have \( \text{diam}(G_{n,d}) \leq \lfloor \alpha^{-1} \rfloor + 1 \) holds w.h.p., as we will confirm in Section 2. On the other hand, by substituting \( d = (\beta + o(1))n^{\alpha} \) to Equation (3), we obtain

\[
\lim_{n \to \infty} D' = \begin{cases} 
[\alpha^{-1}] + 1 & \text{if } \alpha^{-1} \not\in \mathbb{N} \text{ or } (\alpha^{-1} \in \mathbb{N} \land \beta < 1), \\
\alpha^{-1} & \text{if } \alpha^{-1} \in \mathbb{N} \land \beta > 1, \\
\alpha^{-1} - \beta^{1/\alpha} + 1 & \text{if } \alpha^{-1} \in \mathbb{N} \land \beta < 1, \\
\text{depends on the term } o(1) & \text{otherwise.}
\end{cases}
\] (4)

By combining Theorem 2 and eq. (4), we obtain Equation (1). As mentioned earlier, Theorem 1 immediately follows from the result of Gao, Isaev, and McKay [15]. In this paper, we prove Theorem 2 by combining the upper bound from the embedding theorem [11] and Theorem 1 (note that \( \text{diam}(G) \geq \lceil AD(G) \rceil \)).

**Average distance of \( G_{n,d} \).** Let \( N_k \) be the number of vertex pairs of distance \( k \). We use the same argument as for Equation (3) to obtain a lower bound of \( AD(G) \) for any \( d \)-regular graph with \( d \geq 3 \). Suppose \( \text{diam}(G) = D' \) and thus \( N_1 + \cdots + N_{D'} = \binom{n}{2} \). Moreover, for every \( k = 1, \ldots, D' - 1 \), we have \( N_k \leq d(d - 1)^{k-1} \). Therefore, we obtain

\[
AD(G) = \binom{n}{2}^{-1} (N_1 + 2N_2 + \cdots + D'N_{D'}) \\
= D' - \binom{n}{2}^{-1} ((D' - 1)N_1 + (D' - 2)N_2 + \cdots + N_{D'-1}) \\
\geq D' - \binom{n}{2}^{-1} \sum_{k=1}^{D'-1} (D' - k)d(d - 1)^{k-1} \\
= D' - \frac{d(d - 1)^{D'}}{(n-1)(d-2)^2} + \frac{dD'}{(n-1)(d-2)} + \frac{d}{(n-1)(d-2)^2} \\
= \log_{d-1} n - O(1). 
\] (5)

Let \( AD' = AD(n,d) \) denote the lower bound Equation (5). Then, we have

\[
\frac{\log n}{\log(d-1)} - O(1) \leq AD(G_{n,d}) \leq \text{diam}(G_{n,d}).
\]

This implies that

\[
AD(G_{n,d}) = (1 + o(1)) \frac{\log n}{\log(d-1)}
\]

holds w.h.p. if \( d \geq 3 \) is constant or \( \log n \ll d \leq n^{o(1)} \).

Suppose that \( d = (\beta + o(1))n^{\alpha} \), where \( \alpha \in (0, 1) \) and \( \beta > 0 \) are constants. From the lower bound Equation (5), we have

\[
\lim_{n \to \infty} AD' = \begin{cases} 
[\alpha^{-1}] + 1 & \text{if } \alpha^{-1} \not\in \mathbb{N}, \\
\alpha^{-1} & \text{if } \alpha^{-1} \in \mathbb{N} \land \beta > 1, \\
\alpha^{-1} - \beta^{1/\alpha} + 1 & \text{if } \alpha^{-1} \in \mathbb{N} \land \beta < 1, \\
\text{depends on the term } o(1) & \text{otherwise.}
\end{cases}
\] (6)
1.2 Definitions and notation

For two positive integers $k$ and $m$ with $k \leq m$, we denote by $(m)_k$ the falling factorial $m(m-1) \cdots (m-k+1)$. For a finite set $X$ and a positive integer $k \leq |X|$, we use

$$\binom{X}{k} := \{\{x_1, \ldots, x_k\} \subseteq X : |\{x_1, \ldots, x_k\}| = k\},$$

$$\binom{X}{k} := \left\{(x_1, \ldots, x_k) : \{x_1, \ldots, x_k\} \in \binom{X}{k}\right\}.$$

For a graph $G$, we denote by $V(G)$ and $E(G)$, respectively, the vertex set and the edge set of $G$. Note that $E(G) \subseteq \binom{V(G)}{2}$ is a set of unordered vertex pairs. Throughout the paper, the number of vertices of a graph is denoted by $n$, and the vertex set is denoted by $V = \{1, \ldots, n\}$.

We simply write $H \subseteq G$ if $H$ is contained in $G$, that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ hold. Two graphs $G \cup H$ and $G \cap H$ are defined by

$$G \cup H = (V(G) \cup V(H), E(G) \cup E(H)),$$

$$G \cap H = (V(G) \cap V(H), E(G) \cap E(H)).$$

It should be noted that $G$ and $H$ are labelled.

A path is defined to be a graph $P = (\{v_0, \ldots, v_\ell\}, \{\{v_0, v_1\}, \ldots, \{v_{\ell-1}, v_\ell\}\})$ for distinct vertices $v_0, \ldots, v_\ell$. The vertices of degree one in a path are called endpoints. We call a path of endpoints $s$ and $t$ an $st$-path. The length of a path is the number of edges. For a graph $G$ and its two distinct vertices $s$ and $t$, the distance $\text{dist}_G(s, t)$ is the minimum length among all $st$-paths contained in $G$. We define $\text{dist}_G(s, t) = \infty$ if $G$ does not contain any $st$-paths. For a graph $G = (V, E)$ of $n$ vertices, the average distance $\text{AD}(G)$ of $G$ is

$$\text{AD}(G) = \left(\frac{n}{2}\right)^{-1} \sum_{\{s,t\} \in \binom{V}{2}} \text{dist}_G(s, t).$$

The diameter $\text{diam}(G)$ of $G$ is

$$\text{diam}(G) = \max_{s \neq t} \text{dist}_G(s, t).$$

Note that $\text{diam}(G) = \text{AD}(G) = \infty$ if $G$ is not connected. We use $\text{dist}(s, t)$ rather than $\text{dist}_G(s, t)$ if the graph $G$ is clear from the context.

For an event $Z$ on a graph $G$ (say, $\text{dist}_G(1, 2) \geq \ell$), we use

$$\mathbb{1}_Z(G) = \begin{cases} 1 & \text{if } G \text{ satisfies an event } Z, \\ 0 & \text{otherwise} \end{cases}$$

as the indicator function.
1.3 Tools

**Lemma 6** (The Chernoff bound; Theorem 10.1 and 10.5 of [10]). Let $X_1, X_2, \ldots, X_n$ be independent binary random variables satisfying that $\Pr[X_i = 1] = p_i$ and let $X = \sum_{i=1}^n X_i$ be the sum. Then, for any $\epsilon \geq 0$,

\[
\Pr[X \geq (1 + \epsilon) \mathbb{E}[X]] \leq \exp \left(-\frac{\min\{\epsilon, \epsilon^2\} \mathbb{E}[X]}{3}\right)
\]

and

\[
\Pr[X \leq (1 - \epsilon) \mathbb{E}[X]] \leq \exp \left(-\frac{\epsilon^2 \mathbb{E}[X]}{2}\right).
\]

**Lemma 7** (Multivariate version of Brun’s sieve; Lemma 2.8 of [33]). Let $S_n^{(1)}, \ldots, S_n^{(k)}$ be random variables defined on the same space $\Omega_n$ such that each $S_n^{(i)}$ can be written as the sum of binary random variables. Suppose that there exist positive constants $\lambda_1, \ldots, \lambda_k$ satisfying

\[
\lim_{n \to \infty} \mathbb{E} \left[ \prod_{i=1}^k (S_n^{(i)})_{r_i} \right] = \prod_{i=1}^k \lambda_i^{r_i}
\]

for every fixed integers $r_1, \ldots, r_k \geq 0$.

Then, for any constants $j_1, \ldots, j_k \geq 0$, it holds that

\[
\lim_{n \to \infty} \Pr \left[ \bigwedge_{i=1}^k [S_n^{(i)} = j_i] \right] = \prod_{i=1}^k \exp(-\lambda_i) \frac{\lambda_i^{j_i}}{j_i!}.
\]

**Lemma 8** (Lemma 2.1 of [19]). Suppose that $1 \ll d \ll n$. For any fixed graph $H$, it holds that

\[
\Pr[H \subseteq G_{n,d}] = (1 + o(1)) \left(\frac{d}{n}\right)^{|E(H)|}.
\]

Let $G[n, m]$ be a graph selected uniformly at random from the set of all graphs of $n$ vertices with exactly $m$ edges.

**Lemma 9** (The embedding theorem; Theorem 10.10 of [14]). There is a constant $C > 0$ that satisfies the following. For any real $\gamma = \gamma(n)$, integer $d = d(n)$ satisfying

\[
C \left( \left( \frac{d}{n} + \frac{\log n}{d} \right)^{1/3} \right) \leq \gamma < 1,
\]

and $m = \lfloor (1 - \gamma)nd/2 \rfloor$, there exists a joint distribution $\pi$ of $G[n, m]$ and $G_{n,d}$ such that

\[
\lim_{n \to \infty} \Pr_{\pi}[G[n, m] \subseteq G_{n,d}] = 1
\]

holds.

In other words, for $\log n \ll d \ll n$, we can choose $m = (1 - o(1))nd/2$ and couple $G[n, m]$ and $G_{n,d}$ such that $G[n, m] \subseteq G_{n,d}$ holds w.h.p.
2 Upper bounds of $\text{AD}(G_{n,d})$ and $\text{diam}(G_{n,d})$

In this section we obtain upper bounds of $\text{AD}(G_{n,d})$ and $\text{diam}(G_{n,d})$ using Theorem 9. As noted in [11], in Theorem 9, one can replace $G[n,m]$ by $G(n,p)$ of $p = (1 - 2\gamma)d/(n - 1)$. This yields the following result.

**Corollary 10.** For $d = d(n)$ satisfying $\log n \ll d \ll n$, there exists $p = (1 - o(1))\frac{\alpha}{n}$ such that $\text{AD}(G_{n,d}) \leq \text{AD}(G(n,p))$ and $\text{diam}(G_{n,d}) \leq \text{diam}(G(n,p))$ hold w.h.p.

For $d = (\beta+o(1))n^\alpha$, take $\gamma$ of Theorem 9 satisfying $\gamma = o(1)$, and let $p = (1-2\gamma)\frac{\alpha}{n-1} = (\beta + o(1))n^{1+\alpha}$. Then, from Theorems 3 and 10, it holds w.h.p. that

$$ \text{AD}(G_{n,d}) \leq \text{AD}(G(n,p)) \leq \mu + o(1). \quad (8) $$

Similarly, from Theorems 5 and 10, a random regular graph $G_{n,d}$ w.h.p. satisfies

$$ \text{diam}(G_{n,d}) \leq \text{diam}(G(n,p)) \leq \lfloor \alpha^{-1} \rfloor + 1. \quad (9) $$

3 Lower bounds of $\text{AD}(G_{n,d})$ and $\text{diam}(G_{n,d})$

If $\alpha^{-1} \notin \mathbb{N}$, the lower bound Equation (6) and the upper bound Equation (8) yield that

$$ \text{AD}(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1 - o(1) $$

holds w.h.p. Now we focus on the case where $\alpha^{-1} \in \mathbb{N}$. This section is devoted to prove the following.

**Lemma 11.** Let $d = (\beta+o(1))n^\alpha$, where $\alpha \in (0,1)$ and $\beta > 0$ are any constants satisfying $\alpha^{-1} \in \mathbb{N}$. For any constant $\epsilon > 0$,

$$ \lim_{n \to \infty} \Pr[\text{AD}(G_{n,d}) \leq \mu - \epsilon] = 0, $$

where $\mu = \alpha^{-1} + \exp(-\beta^{1/\alpha})$.

**Remark.** By combining Equation (8) and theorem 11, we complete the proof of Theorem 1. Moreover, Theorem 11 implies

$$ \text{diam}(G_{n,d}) \geq \lfloor \text{AD}(G_{n,d}) \rfloor = \lfloor \alpha^{-1} \rfloor + 1 $$

holds w.h.p., which completes the proof of Theorem 2.

**Proof of Theorem 11.** Note that

$$ \text{AD}(G_{n,d}) = \left(\frac{n}{2}\right)^{-1} \sum_{\{s,t\} \in V_n^2} \text{dist}(s,t) $$

where $V_n = \{1, 2, \ldots, n\}$.
\[
\sum_{\ell=1}^{\infty} \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} 1_{\text{dist}(s,t) \geq \ell} \\
\geq \sum_{\ell=1}^{\alpha - 1 + 1} \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} 1_{\text{dist}(s,t) \geq \ell}.
\]

For \( \ell \in \{1, \ldots, \alpha - 1 + 1\} \), let \( p_\ell = p_\ell(G_{n,d}) = \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} 1_{\text{dist}(s,t) \geq \ell} \). We evaluate \( p_\ell \) using the following result.

**Lemma 12.** Consider \( G_{n,d} \) of \( d = (\beta + o(1))n^\alpha \). Fix two constants \( \alpha \in (0, 1) \) and \( \beta > 0 \) satisfying \( \alpha^{-1} \in \mathbb{N} \). For any constant \( k \in \mathbb{N} \), fix \( 2k \) distinct vertices \( s_1, \ldots, s_k, t_1, \ldots, t_k \). For any fixed \( \ell_1, \ldots, \ell_k \in \{1, \ldots, \alpha - 1 + 1\} \), it holds that

\[
\lim_{n \to \infty} \Pr \left[ \bigwedge_{i=1}^{k} \left[ \text{dist}(s_i, t_i) \geq \ell_i \right] \right] = \exp(-M \beta^{1/\alpha})
\]

where \( M = |\{i \in \{1, \ldots, k\} : \ell_i = \alpha^{-1} + 1\}| \).

We will prove Theorem 12 in Section 3.1. For \( \ell \in \{1, \ldots, \alpha - 1 + 1\} \), let

\[
\mu_\ell = \begin{cases} 
1 & \text{if } 1 \leq \ell \leq \alpha^{-1}, \\
\exp(-\beta^{1/\alpha}) & \text{if } \ell = \alpha^{-1} + 1.
\end{cases}
\]

From Theorem 12, we have

\[
E[p_\ell] = \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} \Pr[\text{dist}(s,t) \geq \ell] \\
= \Pr[\text{dist}(1,2) \geq \ell] = \mu + o(1)
\]

and

\[
E[p_\ell^2] = \binom{n}{2}^{-2} \sum_{\{s,t\},\{s',t'\} \in \binom{V}{2}} \Pr[\text{dist}(s,t) \geq \ell \land \text{dist}(s',t') \geq \ell] \\
= \binom{n}{2}^{-2} \left( O(n^3) + \sum_{\{s,t\},\{s',t'\} \in \binom{V}{2} : \{s,t\} \cap \{s',t'\} = \emptyset} \Pr[\text{dist}(s,t) \geq \ell \land \text{dist}(s',t') \geq \ell] \right) \\
= \Pr[\text{dist}(1,2) \geq \ell \land \text{dist}(3,4) \geq \ell] + o(1) = \mu^2 + o(1).
\]

From the Chebyshev inequality, for every constant \( \epsilon > 0 \), we have

\[
\Pr[|p_\ell - E[p_\ell]| \geq \epsilon] \leq \frac{\Var[p_\ell]}{\epsilon^2} = o(1).
\]
Thus we obtain

\[ \Pr \left[ \left| \left( \sum_{\ell=1}^{\alpha^{-1}+1} p_{\ell} \right) - \mu \right| > \epsilon \right] \leq \sum_{\ell=1}^{\alpha^{-1}+1} \Pr \left[ |p_{\ell} - \mu_{\ell}| > \epsilon/((\alpha^{-1} + 1) \right] = o(1). \]

Therefore, it holds w.h.p. that

\[ \text{AD}(G_{n,d}) \geq \sum_{\ell=1}^{\alpha^{-1}+1} p_{\ell} \geq \mu - o(1), \]

which completes the proof of Theorem 11.

3.1 Distances of fixed vertex pairs of $G_{n,d}$

This part is devoted to prove Theorem 12. We start with establishing the following result.

**Lemma 13.** Consider $G_{n,d}$ of $d = (\beta + o(1))n^\alpha$ for constants $\alpha \in (0,1)$ and $\beta > 0$. For two fixed distinct vertices $s$ and $t$, it holds w.h.p. that $\text{dist}(s, t) \in \{\lceil \alpha^{-1} \rceil, \lceil \alpha^{-1} \rceil + 1\}$.

**Proof.** For two fixed vertices $s, t$ of $G_{n,d}$ and an integer $\ell$, we denote by $P$ the set of paths of length $\ell$ connecting $s$ and $t$ in a complete graph. Let $X_\ell = X_\ell(G_{n,d})$ be the number of paths $P \in P$ contained in $G_{n,d}$, that is,

\[ X_\ell = |\{P \in P : P \subseteq G_{n,d}\}|. \tag{10} \]

Fix an integer $\ell$ satisfying $\ell \alpha < 1$ (or equivalently, $\ell \leq \lfloor \alpha^{-1} \rfloor - 1$). Then, from Theorem 8, we have

\[ \mathbb{E}(X_\ell) = \sum_{P \in P} \Pr[P \subseteq G_{n,d}] = (1 + o(1))n^{\ell-1} \left( \frac{d}{n} \right)^\ell = o(1). \]

From the Markov’s inequality, we obtain

\[ \Pr[\text{dist}(s, t) \leq \ell] \leq \Pr[X_1 + \cdots + X_\ell > 0] \leq \sum_{i=1}^\ell \mathbb{E}(X_i) = o(1). \]

In other words, $\text{dist}(s, t) \geq \ell + 1 \geq \lfloor \alpha^{-1} \rfloor$ holds w.h.p.

On the other hand, from Equation (9), we have $\text{dist}(s, t) \leq \text{diam}(G_{n,d}) \leq \lfloor \alpha^{-1} \rfloor + 1$. This completes the proof of Theorem 13.

\[ \square \]
Proof of Theorem 12. Fix an integer \( k > 0 \) and \( 2k \) distinct vertices \( s_1, \ldots, s_k, t_1, \ldots, t_k \) of \( G_{n,d} \), where \( d = (\beta + o(1))n^{\alpha} \). From Theorem 13, it holds w.h.p. that \( \text{dist}(s,t) \in \{\alpha^{-1}, \alpha^{-1} + 1\} \).

Suppose that \( \ell_1 \leq \alpha^{-1} \) and thus \( \text{dist}(s_1, t_1) \geq \ell_1 \) holds w.h.p. Then we have

\[
\Pr \left[ \bigwedge_{i=2}^{k} [\text{dist}(s_i, t_i) \geq \ell_i] \right] - \Pr[\text{dist}(s_1, t_1) < \ell_1] \leq \Pr \left[ \bigwedge_{i=1}^{k} [\text{dist}(s_i, t_i) \geq \ell_i] \right]
\]

\[
\leq \Pr \left[ \bigwedge_{i=2}^{k} [\text{dist}(s_i, t_i) \geq \ell_i] \right]
\]

and thus

\[
\Pr \left[ \bigwedge_{i=1}^{k} [\text{dist}(s_i, t_i) \geq \ell_i] \right] = \Pr \left[ \bigwedge_{i=2}^{k} [\text{dist}(s_i, t_i) \geq \ell_i] \right] - o(1).
\]

Hence, we may assume that \( \ell_i = \alpha^{-1} + 1 \) for all \( i = 1, \ldots, k \) (i.e., \( M = k \) in Theorem 12).

Let \( P^{(i)} \) denote the set of \( s_it_i \)-paths of length \( \alpha^{-1} \) contained in the complete graph \( K_n \). Define \( X^{(i)} \) as the number of paths of \( P^{(i)} \) contained in \( G_{n,d} \), that is,

\[
X^{(i)} = |\{P \in P^{(i)} : P \subseteq G(n,p)\}|.
\]

Then, we have

\[
\Pr \left[ \bigwedge_{i=1}^{k} [\text{dist}(s_i, t_i) \geq \alpha^{-1} + 1] \right] = \Pr \left[ \bigwedge_{i=1}^{k} [\text{dist}(s_i, t_i) \geq \alpha^{-1} + 1] \land \bigwedge_{i=1}^{k} [X^{(i)} = 0] \right]
\]

\[
= \Pr \left[ \bigwedge_{i=1}^{k} [X^{(i)} = 0] \right] - o(1).
\]

We evaluate Equation (11) using the following result, which will be shown in Section 3.2.

Lemma 14. Consider \( G_{n,d} \) of \( d = (\beta + o(1))n^{\alpha} \), where \( \alpha \in (0,1) \) and \( \beta > 0 \) are any constants satisfying \( \alpha^{-1} \in \mathbb{N} \). Fix \( 2k \) distinct vertices \( s_1, \ldots, s_k, t_1, \ldots, t_k \), where \( k \) is any constant. For \( i = 1, \ldots, k \), let \( X^{(i)} \) denote the number of \( s_it_i \)-paths of length \( \alpha^{-1} \in \mathbb{N} \) contained in \( G(n,p) \). Fix arbitrary nonnegative integers \( r_1, \ldots, r_k \). Then, it holds that

\[
\mathbb{E} \left[ \prod_{i=1}^{k} (X^{(i)})^{r_i} \right] = (\beta^{1/\alpha})^R + o(1),
\]

where \( R = r_1 + \cdots + r_k \).

From Theorem 14 and the Poisson approximation theorem (Theorem 7), we have

\[
\Pr \left[ \bigwedge_{i=1}^{k} [X^{(i)} = 0] \right] = \exp(-k\beta^{1/\alpha}) + o(1).
\]

(12)
By combining Equations (11) and (12), we have

$$\Pr \left[ \bigwedge_{i=1}^{k} \{ \text{dist}(s_i, t_i) \geq \alpha^{-1} + 1 \} \right] = \exp(-k\beta^{1/\alpha}) - o(1).$$

This completes the proof of Theorem 12 and thus Theorem 11.

### 3.2 Proof of Theorem 14

We first prove the following result and then show Theorem 14.

**Lemma 15.** Fix an integer $\ell \geq 1$ and consider $G(n, p)$ satisfying $(np)^\ell = \Omega(n)$. Fix $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, where $k$ is arbitrary constant. For $i = 1, \ldots, k$, let $X^{(i)}$ denote the number of $s_i t_i$-paths of length $\ell \in \mathbb{N}$ contained in $G(n, p)$.

Then, for any fixed nonnegative integers $r_1, \ldots, r_k$,

$$\mathbb{E} \left[ \prod_{i=1}^{k} \left( X^{(i)} \right)^{r_i} \right] = n^{R_{(1)}} p^{R_{(2)}} \left( 1 + O \left( \frac{1}{np} \right) \right),$$

where $R = r_1 + \cdots + r_k$.

**Corollary 16.** Consider $G(n, p)$ of $p = (\beta + o(1))n^{-1+\alpha}$, where $\alpha \in (0, 1)$ and $\beta > 0$ are any constants satisfying $\alpha^{-1} \in \mathbb{N}$. Fix arbitrary nonnegative integers $r_1, \ldots, r_k$. Then, it holds that

$$\mathbb{E} \left[ \prod_{i=1}^{k} \left( X^{(i)} \right)^{r_i} \right] = (\beta^{1/\alpha})^R + o(1),$$

where $R = r_1 + \cdots + r_k$.

**Proof of Theorem 15.** For a positive constant $k$, fix $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. For every $i \in \{1, \ldots, k\}$, let $\mathcal{P}^{(i)}$ denote the set of all $s_i t_i$-paths of length $\ell$ contained in the complete graph. We denote by $X^{(i)}$ the number of paths of $\mathcal{P}^{(i)}$ contained in $G(n, p)$.

Fix nonnegative integers $k, r_1, \ldots, r_k$. We may assume that $r_i > 0$ for every $i = 1, \ldots, k$. Let $\mathcal{A} = (\mathcal{P}^{(1)})_{r_1} \times \cdots \times (\mathcal{P}^{(k)})_{r_k}$. Each element $A \in \mathcal{A}$ is a tuple

$$A = (P_1^{(1)}, \ldots, P_{r_1}^{(1)}, \ldots, (P_1^{(k)}, \ldots, P_{r_k}^{(k)})),$$

where each $P_j^{(i)} \in \mathcal{P}_i$ is an $s_i t_i$-path of length $\ell$ and $P_j^{(i)} \neq P_j^{(i)}$ holds for every $i$ and $j \neq j'$. For notational convention, we write $A = (P_1, \ldots, P_R) \in \mathcal{A}$. Since $r_k > 0$, it holds that $P_R \in \mathcal{P}^{(k)}$.

For a tuple $A = (P_1, \ldots, P_t)$ of $t$ paths, let $E(A) = \bigcup_{i=1}^{t} E(P_i)$ and $V(A) = \bigcup_{i=1}^{t} V(P_i)$ (we will use induction on $R$ and hence we assume $t \leq R$ here). For $\mathcal{S} \subseteq \mathcal{A}$, we consider

$$\Gamma_\mathcal{S} = \sum_{A \in \mathcal{S}} p_{E(A)}^{t^E(A)}.$$

**Proof of Theorem 14.** Fix an integer $\ell \geq 1$ and consider $G(n, p)$ satisfying $(np)^\ell = \Omega(n)$. Fix $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, where $k$ is arbitrary constant. For $i = 1, \ldots, k$, let $X^{(i)}$ denote the number of $s_i t_i$-paths of length $\ell \in \mathbb{N}$ contained in $G(n, p)$.

Then, for any fixed nonnegative integers $r_1, \ldots, r_k$,

$$\mathbb{E} \left[ \prod_{i=1}^{k} \left( X^{(i)} \right)^{r_i} \right] = n^{R_{(1)}} p^{R_{(2)}} \left( 1 + O \left( \frac{1}{np} \right) \right),$$

where $R = r_1 + \cdots + r_k$.

**Corollary 16.** Consider $G(n, p)$ of $p = (\beta + o(1))n^{-1+\alpha}$, where $\alpha \in (0, 1)$ and $\beta > 0$ are any constants satisfying $\alpha^{-1} \in \mathbb{N}$. Fix arbitrary nonnegative integers $r_1, \ldots, r_k$. Then, it holds that

$$\mathbb{E} \left[ \prod_{i=1}^{k} \left( X^{(i)} \right)^{r_i} \right] = (\beta^{1/\alpha})^R + o(1),$$

where $R = r_1 + \cdots + r_k$. 

**Proof of Theorem 15.** For a positive constant $k$, fix $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. For every $i \in \{1, \ldots, k\}$, let $\mathcal{P}^{(i)}$ denote the set of all $s_i t_i$-paths of length $\ell$ contained in the complete graph. We denote by $X^{(i)}$ the number of paths of $\mathcal{P}^{(i)}$ contained in $G(n, p)$.

Fix nonnegative integers $k, r_1, \ldots, r_k$. We may assume that $r_i > 0$ for every $i = 1, \ldots, k$. Let $\mathcal{A} = (\mathcal{P}^{(1)})_{r_1} \times \cdots \times (\mathcal{P}^{(k)})_{r_k}$. Each element $A \in \mathcal{A}$ is a tuple

$$A = (P^{(1)}_1, \ldots, P^{(1)}_{r_1}, \ldots, (P^{(k)}_1, \ldots, P^{(k)}_{r_k})),$$

where each $P_j^{(i)} \in \mathcal{P}_i$ is an $s_i t_i$-path of length $\ell$ and $P_j^{(i)} \neq P_j^{(i)}$ holds for every $i$ and $j \neq j'$. For notational convention, we write $A = (P_1, \ldots, P_R) \in \mathcal{A}$. Since $r_k > 0$, it holds that $P_R \in \mathcal{P}^{(k)}$.

For a tuple $A = (P_1, \ldots, P_t)$ of $t$ paths, let $E(A) = \bigcup_{i=1}^{t} E(P_i)$ and $V(A) = \bigcup_{i=1}^{t} V(P_i)$ (we will use induction on $R$ and hence we assume $t \leq R$ here). For $\mathcal{S} \subseteq \mathcal{A}$, we consider

$$\Gamma_\mathcal{S} = \sum_{A \in \mathcal{S}} p_{E(A)}^{t^E(A)}.$$
Note that $\mathbf{E}\left[\prod_{i=1}^{k}\left(X^{(i)}_{r_i}\right)\right] = \sum_{A \in \mathcal{A}} \mathbf{Pr}[E(A) \subseteq E(G(n,p))] = \Gamma_A$. We claim
\begin{equation}
n^{R(\ell-1)}p^{R\ell}\left(1 - O\left(\frac{1}{n}\right)\right) \leq \Gamma_A \leq n^{R(\ell-1)}p^{R\ell}\left(1 + O\left(\frac{1}{np}\right)\right),
\end{equation}
which completes the proof of Theorem 15.

Figure 1: A tuple $A \in \mathcal{A} \setminus \mathcal{F}$. Figure 2: A tuple $A \in \mathcal{F}$.

For any $A \in \mathcal{A}$, it holds that $|E(A)| \leq R\ell$ and the equality holds if and only if any two distinct paths $P_i, P_j$ of $A$ shares no edges (see Figure 1). Let
\begin{align*}
\mathcal{F} &= \{A \in \mathcal{A} : |E(A)| < R\ell\} \\
&= \{(P_1, \ldots, P_{R\ell}) \in \mathcal{A} : \exists i \neq j, E(P_i) \cap E(P_j) \neq \emptyset\}.
\end{align*}
Figure 2 illustrates an example. Then, $\Gamma_A$ can be decomposed into
\begin{equation}
\Gamma_A = \Gamma_{\mathcal{F}} + \Gamma_{\mathcal{A} \setminus \mathcal{F}}.
\end{equation}
The second term $\Gamma_{\mathcal{A} \setminus \mathcal{F}}$ satisfies
\begin{align*}
\Gamma_{\mathcal{A} \setminus \mathcal{F}} &= p^{R\ell} |\{A \in \mathcal{A} : |E(A)| = R\ell\}| \\
&\geq p^{R\ell} |\{A \in \mathcal{A} : |E(A)| = R\ell \text{ and } |V(A)| = R(\ell - 1) + 2k\}| \\
&= (n - 2k)R(\ell - 1)p^{R\ell} \\
&\geq n^{R(\ell-1)}p^{R\ell}\left(1 - O\left(\frac{1}{n}\right)\right).
\end{align*}
This implies the lower bound $\Gamma_A \geq \Gamma_{\mathcal{A} \setminus \mathcal{F}} \geq n^{R(\ell-1)}p^{R\ell}\left(1 - O\left(\frac{1}{n}\right)\right)$.

Now it suffices to bound $\Gamma_A$ from above. Observe that $\Gamma_{\mathcal{A} \setminus \mathcal{F}}$ satisfies
\begin{equation}
\Gamma_{\mathcal{A} \setminus \mathcal{F}} = p^{R\ell} |\{A \in \mathcal{A} : |E(A)| = R\ell\}| \leq n^{R(\ell-1)}p^{R\ell}.
\end{equation}
We show that this term is dominating in $\Gamma_A$. Theorem 15 immediately follows from Equations (15) and (16) and the following result:

\begin{thebibliography}{10}
\end{thebibliography}
Lemma 17. Suppose that \((np)^\ell = \Omega(n)\). Define \(\mathcal{F}\) as Equation (14). It holds that

\[
\Gamma_{\mathcal{F}} = O\left(\frac{n^{R(\ell-1)}p^{R\ell}}{np}\right).
\]

Proof. We use induction on \(R\). For the base case of \(R = 1\), we have \(\mathcal{F} = \emptyset\) and thus

\[
\Gamma_{\mathcal{A}} \leq n^{\ell-1}p^{\ell},
\]

\[
\Gamma_{\mathcal{F}} = 0.
\]

Suppose that \(R \geq 2\) and that Theorem 17 holds for \(R - 1\). Note that Theorem 15 also holds for \(R - 1\) since Theorem 17 implies Theorem 15. Let

\[
\mathcal{A}' = (\mathcal{P}^{(1)})_{r_1} \times \cdots \times (\mathcal{P}^{(k-1)})_{r_{k-1}}.
\]

Then, each element \(A = (P_1, \ldots, P_R) \in \mathcal{A}\) can be decomposed into \(A' = (P_1, \ldots, P_{R-1}) \in \mathcal{A}'\) and \(P_R \in \mathcal{P}^{(k)}\). Note that the edge set \(E(A')\) for \(A' \in \mathcal{A}'\) are defined in the same way as \(E(A)\) and it holds that \(|E(A')| \leq (R - 1)\ell\). Let

\[
\mathcal{F}' = \{A' \in \mathcal{A}' : |E(A')| < (R - 1)\ell\}.
\]

By the induction assumption on \(\mathcal{F}'\) and \(\mathcal{A}'\), we have

\[
\Gamma_{\mathcal{A}'} \leq n^{(R-1)(\ell-1)p^{(R-1)}\ell} \left(1 + C_1 \frac{1}{np}\right),
\]

\[
\Gamma_{\mathcal{F}'} \leq C_2 \left(1 + \frac{1}{np}\right).
\]

for some constants \(C_1, C_2 > 0\). For \(A = (P_1, \ldots, P_R) \in \mathcal{F}\), let \(A' = (P_1, \ldots, P_{R-1}) \in \mathcal{A}'\). Since \(A \in \mathcal{F}\), either

(i) \(E(P_R) \cap E(P_i) \neq \emptyset\) for some \(1 \leq i < R\), or

(ii) \(E(P_R) \cap E(A') = \emptyset\) and \(E(P_i) \cap E(P_j) \neq \emptyset\) for some \(1 \leq i < j < R\) (thus \(A' \in \mathcal{F}'\)) holds. Therefore, we have

\[
\Gamma_{\mathcal{F}} = \sum_{A \in \mathcal{F}} p^{\{|E(A)|\}}
\]

\[
\leq \sum_{A' \in \mathcal{A}'} \sum_{P_R \in \mathcal{P}^{(k)} : E(A) \cap E(P_R) \neq \emptyset} p^{\{|E(A')\cup E(P_R)|\}} + \sum_{A' \in \mathcal{A}'} \sum_{P_R \in \mathcal{P}^{(k)} : E(P_R) \cap E(A') = \emptyset} p^{\{|E(A')\cup E(P_R)|\}}.
\]

From the induction assumption, the second term satisfies

\[
\sum_{A' \in \mathcal{A}'} \sum_{P_R \in \mathcal{P}^{(k)} : E(P_R) \cap E(A') = \emptyset} p^{\{|E(A')\cup E(P_R)|\}} = \sum_{A' \in \mathcal{A}'} p^{\{|E(A')|\}} \sum_{P_R \in \mathcal{P}^{(k)} : E(P_R) \cap E(A') = \emptyset} p^{\{|E(P_R)|\}}
\]

\[
\leq \Gamma_{\mathcal{F}'} \cdot n^{\ell-1}p^{\ell}.
\]

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The first term can be rewritten as
\[
\sum_{A' \in A'} \sum_{P_R \in \mathcal{P}^{(k)}: E(A') \cap E(P_R) \neq \emptyset} p^{E(A') \cup E(P_R)} = \sum_{A' \in A'} \sum_{P_R \in \mathcal{P}^{(k)}: E(A') \cap E(P_R) \neq \emptyset} p^{E(A')} \sum_{P_R \in \mathcal{P}^{(k)}: E(A) \cap E(P_R) \neq \emptyset} p^{E(P_R) \setminus E(A')}.
\]

Fix \(A' = (P_1, \ldots, P_{R-1}) \in A'\). Let \(S = \{s_1, \ldots, s_k, t_1, \ldots, t_k\}\) be the endpoints of the paths and let \(V_i = S \cup V(P_1) \cup \cdots \cup V(P_{R-1})\). To bound the number of \(P_R\) satisfying the condition (ii), we consider two cases: \(E(P_R) \not\subseteq E(A')\) and \(E(P_R) \subseteq E(A')\).

**Case I.** \(E(P_R) \not\subseteq E(A')\). The edge set \(E(P_R) \cap E(A')\) forms a forest. Since \(E(P_R) \not\subseteq E(A')\), this forest is not connected and thus we have \(|V(P_R) \cap V_1| - |E(P_R) \cap E(A')| \geq 2\). This yields
\[
|V(P_R) \setminus V_1| = |V(P_R)| - |V(P_R) \cap V_1| \
\leq \ell - |E(P_R) \cap E(A')| - 1.
\]

Let \(|E(P_R) \cap E(A')| = t < \ell\). Then, \(P_R\) consists of two type of vertices: at most \(\ell - t - 1\) from \(V \setminus V_1\) and the others from \(V_1\). Therefore, there are at most \(n^{\ell-t-1} |V_1|^t \leq C' n^{\ell-t-1}\) candidates for the path \(P_R\) satisfying \(|E(P_R) \cap E(A')| = t < \ell\), where \(C = (R - 1)(\ell + 1)\) (recall that two endpoints of \(P_R\) are fixed and thus they are not taken into account).

**Case II.** \(E(P_R) \subseteq E(A')\). We claim \(A' \in \mathcal{F}'\). If not, it holds that \(E(P_i) \cap E(P_j) = \emptyset\) for any \(i < j < R\). Hence, \(E(P_R) \subseteq E(A')\) implies \(P_R = P_i\) for some \(i < R\). This contradicts to the definition of \(\mathcal{A}\) (\(P_i \neq P_j\) for any \(i < j \leq R\)). Moreover, the number of \(P_R \in \mathcal{P}^{(k)}\) satisfying \(E(P_R) \subseteq E(A')\) is at most \(|V_1|^{\ell-1} \leq C_R^{\ell-1}\). Therefore, we have
\[
\sum_{A' \in A'} \sum_{P_R \in \mathcal{P}^{(k)}: E(A') \cap E(P_R) \neq \emptyset} p^{E(A') \cup E(P_R)} \
\leq \sum_{A' \in A'} \sum_{t=1}^{\ell-1} \sum_{P_R \in \mathcal{P}^{(k)}: |E(A') \cap E(P_R)| = t} p^{E(P_R) \setminus E(A')} + \sum_{A' \in \mathcal{F}'} p^{E(A')} C_R^{\ell-1} \\
\leq \sum_{A' \in A'} \sum_{t=1}^{\ell-1} C'^{t} n^{\ell-t-1} p^{t} + C_R^{\ell-1} \Gamma_{\mathcal{F}'} \\
\leq \Gamma_{\mathcal{A}'} \cdot \frac{C'n^{\ell-1}p^{\ell}}{np} \left(1 + \frac{1.01C'}{np}\right) + C_R^{\ell-1} \Gamma_{\mathcal{F}'}.
\]

From Equations (17) to (19) and the induction assumption, we have
\[
\Gamma_{\mathcal{F}} \leq \Gamma_{\mathcal{F}'} \cdot n^{\ell-1} p^{\ell} + \Gamma_{\mathcal{A}'} \cdot \frac{C'n^{\ell-1}p^{\ell}}{np} \left(1 + \frac{1.01C'}{np}\right) + C_R^{\ell-1} \Gamma_{\mathcal{F}'} \\
\leq O\left(\frac{n^{R(\ell-1)} p^{R\ell}}{np}\right).
\]
This completes the proof of Theorem 17 and thus Theorem 15 (Here, we have used the assumption that \((np)^{\ell} = \Omega(n)\)).

**Proof of Theorem 14.** Let \(d = (1 + o(1))np = (\beta + o(1))n^\alpha\). From Theorem 8, we have \(\Pr[H \subseteq G(n, p)] = (1 + o(1)) \Pr[H \subseteq G(n, d)]\) for any fixed graph \(H\). Let \(R = r_1 + \cdots + r_k\) and \(A = (\mathcal{P}^{(1)})_{r_1} \times \cdots \times (\mathcal{P}^{(k)})_{r_k}\). We write each element \(A \in A\) as a tuple \(A = (P_1, \ldots, P_R)\) of \(R\) paths. Then, from Theorem 16, we have

\[
\mathbb{E}_{G_{n,d}} \left[ \prod_{i=1}^{k} \left( X^{(i)} \right)_{r_i} \right] = \sum_{(P_1, \ldots, P_R) \in A} \Pr[E(P_1 \cup \cdots \cup P_R) \subseteq G_{n,d}]
= (1 + o(1)) \sum_{(P_1, \ldots, P_R)} \Pr[E(P_1 \cup \cdots \cup P_R) \subseteq G(n, p)]
= (1 + o(1)) \mathbb{E}_{G(n,p)} \left[ \prod_{i=1}^{k} \left( X^{(i)} \right)_{r_i} \right]
= (\beta + o(1))^{1/\alpha}.
\]

**4 Concentration of \(\text{AD}(G(n, p))\)**

We prove Theorem 3. We use \(\text{AD} = \text{AD}(G(n, p))\) and \(\text{diam} = \text{diam}(G(n, p))\) as random variables. Let \(D = \lceil \mu \rceil = \lfloor \alpha^2 - 1 \rfloor + 1\). From Theorem 5, we have

\[
\Pr[|\text{AD} - \mu| > \epsilon] \leq \Pr[|\text{AD} - \mu| > \epsilon | \text{diam} = D] \Pr[\text{diam} = D] + \Pr[\text{diam} \neq D] \\
\leq \Pr[|\text{AD} - \mu| > \epsilon | \text{diam} = D] + o(1)
\]

for any \(\epsilon = \epsilon(n) > 0\). Therefore, we may put the condition that \(\text{diam} = D\).

For \(i = 1, \ldots, D\), let

\[
N_i = \left\{ \{s, t\} \in \left( \frac{V}{2} \right) : \text{dist}(s, t) = i \right\}.
\]

We will prove the following result in Section 4.1:

**Lemma 18.** Let \(C > 0\) be a sufficiently large constant and \(\epsilon = \epsilon(n) := \sqrt{\frac{\log n}{np}}\). Then, \(|N_i - M_i| \leq C\epsilon M_i\) holds w.h.p. for all \(i = 1, \ldots, D - 1\), where

\[
M_i = \begin{cases} \frac{(np)^i}{n} \binom{\frac{n}{2}}{i} & \text{if } i < \alpha^2, \\ \left(1 - \exp(-\beta^{1/\alpha})\right)^{\frac{n}{2}} & \text{if } i = \alpha^2 \in \mathbb{N}. \end{cases}
\]
An upper bound of $AD$. Conditioned on $diam = D$, it immediately holds that $AD \leq diam \leq D$. Thus, if $\alpha^{-1} \not\in \mathbb{N}$, we have

$$AD \leq D = \mu$$

with probability $1 - \exp(-n^{\Omega(n)})$.

Now we focus on the case where $\alpha^{-1} \in \mathbb{N}$. Let $\epsilon = C\sqrt{\frac{\log n}{np}}$ for sufficiently large constant $C > 0$. Conditioned on $diam = D$, Theorem 18 implies

$$N_D = \binom{n}{2} - N_1 - \cdots - N_{D-1} \leq (1 + O(\epsilon)) \exp(-\beta^{1/\alpha})\binom{n}{2}.$$ 

Therefore, conditioned on $diam = D$, we have

$$\binom{n}{2} \cdot AD = \sum_{i=1}^{D} iN_i \leq DN_D + (D - 1)\left(\binom{n}{2} - N_D\right) = N_D + (D - 1)\binom{n}{2} \leq (1 + O(\epsilon))\mu\binom{n}{2}.$$ 

In other words, $AD \leq \mu + O(\epsilon)$ holds w.h.p.

A lower bound of $AD$. Conditioned on $diam = D$, we have $N_1 + \cdots + N_D = \binom{n}{2}$ and thus

$$\binom{n}{2} \cdot AD = \sum_{i=1}^{D} iN_i = N_1 + 2N_2 + \cdots + (D - 1)N_{D-1} + D\left(\binom{n}{2} - N_1 - \cdots - N_{D-1}\right) \geq (1 - O(\epsilon))\mu\binom{n}{2}.$$ 

In the last inequality, we used Theorem 18. This completes the proof of Theorem 3.
4.1 Proof of Theorem 18

The proof of Theorem 18 is a slight modification of the proof of Theorem 7.1 of [14].

Consider $G(n, p)$ of $p = (\beta + o(1))n^{-1+\alpha}$. Let $D = [\alpha^{-1}] + 1$. We consider the breadth first search process on $G(n, p)$ from a fixed vertex. Fix a vertex $v$. For $k \geq 0$, let $N_k(v) = \{w \in V : \text{dist}(v, w) = k\}$.

Note that $N_0(v) = \{v\}$. For sufficiently large constant $C > 0$ and $\epsilon := \sqrt{\frac{\log n}{np}}$, let $\mathcal{F}_k$ be the event of $G(n, p)$ that

$$\left| N_i(v) \right| - \frac{2M_i}{n} \leq \frac{CeM_i}{n} \text{ for all } i = 1, \ldots, k,$$

where $M_i$ is given in Theorem 18. Note that, if we are given $N_0(v), \ldots, N_{k-1}(v)$, the random variable $|N_k(v)|$ is distributed as a binomial random variable, that is,

$$|N_k(v)| \sim \text{Bin} \left( n - \sum_{i=0}^{k-1} |N_i(v)|, 1 - (1 - p)^{\left| N_{k-1}(v) \right|} \right).$$

Consider $\mathbb{E}[|N_k(v)| | \mathcal{F}_{k-1}]$. For every $k = 1, \ldots, D - 1$, conditioned on $\mathcal{F}_{k-1}$, we have

$$n \geq n - \sum_{i=0}^{k-1} |N_i(v)| \geq (1 - O(\epsilon))n.$$

Here, recall that $(np)^{D-1} = O(n)$. Using the inequality $e^{-\frac{x^2}{2}} \leq 1 - x \leq e^{-x}$ for every $x \in [0, 1)$ (c.f., Lemma 21.1 of [14]), we obtain

$$1 - (1 - p)^{\left| N_{k-1}(v) \right|} = \begin{cases} (1 \pm O(\epsilon))p(np)^{k-1} & \text{if } k = 1, \ldots, D - 2, \\ (1 \pm O(\epsilon))\exp(-\beta^{1/\alpha}) & \text{if } k = D - 1. \end{cases}$$

Therefore, we have

$$\mathbb{E}[|N_k(v)| | \mathcal{F}_{k-1}] = \begin{cases} (1 \pm O(\epsilon))(np)^k & \text{if } k = 1, \ldots, D - 2, \\ (1 \pm O(\epsilon))\exp(-\beta^{1/\alpha})n & \text{if } k = D - 1. \end{cases}$$

$$= (1 \pm O(\epsilon))\frac{2M_k}{n}.$$

From the Chernoff bound (Theorem 6), we have

$$\Pr[\mathcal{F}_k | \mathcal{F}_{k-1}] \geq 1 - \exp \left( -\Theta(\epsilon^2(np)^k) \right) \geq 1 - O(n^{-2})$$

if the constant $C$ is sufficiently large (recall that $C$ is the constant in the definition of $\mathcal{F}_k$). Therefore, $\mathcal{F}_{D-1}$ holds with probability $1 - O(n^{-2})$ for sufficiently large $C$. Taking the union bound, it holds w.h.p. that $|N_i(v)| = (1 \pm O(\epsilon))\frac{2M_i}{n}$ for all $v$. Consequently, we have $N_i = \frac{1}{2} \sum_{v \in V} |N_i(v)| = (1 \pm O(\epsilon))M_i$, which completes the proof of Theorem 18.
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References


