The average distance and the diameter of dense random regular graphs

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Abstract

Let $\mathrm{AD}(G_{n,d})$ be the average distance of $G_{n,d}$, a random n-vertex d-regular graph. For $d = (\beta + o(1))n^{\alpha}$ with two arbitrary constants $\alpha \in (0, 1)$ and $\beta > 0$, we prove that $|\mathrm{AD}(G_{n,d}) - \mu| < \epsilon$ holds with high probability for any constant $\epsilon > 0$, where μ is equal to $\alpha^{-1} + \exp(-\beta^{1/\alpha})$ if $\alpha^{-1} \in \mathbb{N}$ and to $\lceil \alpha^{-1} \rceil$ otherwise. Consequently, we show that the diameter of the $G_{n,d}$ is equal to $\lfloor \alpha^{-1} \rfloor + 1$ with high probability.

Mathematics Subject Classifications: 05C80, 05C12

1 Introduction

The study of the diameter of regular graphs is well motivated in graph theory. A central question is how to construct an n-vertex d-regular graph with the minimum possible diameter, which has an application to high-performance computing [12, 17, 26]. Let $D'(n, d)$ denote the Moore bound, a well-known lower bound of the minimum possible diameter among all n-vertex d-regular graphs [26] (we will present the bound in Equation (3)). Let $\mathrm{diam}(G)$ denote the diameter of a graph G. We define $\mathrm{diam}(G) = \infty$ if G is not connected. In this paper, we show that the diameter $\mathrm{diam}(G_{n,d})$ of a random d-regular graph $G_{n,d}$ of $d = (\beta + o(1))n^{\alpha}$ with two arbitrary constants $\alpha \in (0, 1)$ and $\beta > 0$ satisfies

$$\lim_{n\to\infty} (\mathrm{diam}(G_{n,d}) - D'(n, d)) = \begin{cases} 0 & \text{if either } \alpha^{-1} \notin \mathbb{N} \text{ or } (\alpha^{-1} \in \mathbb{N} \text{ and } \beta < 1), \\ 1 & \text{if } \alpha^{-1} \in \mathbb{N} \text{ and } \beta > 1 \end{cases} \tag{1}$$

with probability $1 - o(1)$.

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Also, we study the average distance $\mathrm{AD}(G_{n,d})$ of a random regular graph. The average distance $\mathrm{AD}(G)$ of a connected graph G is

$$\mathrm{AD}(G) = \binom{n}{2}^{-1} \sum_{\{u,v\}\in\binom{V}{2}} \mathrm{dist}(u,v),$$

where $\mathrm{dist}(u,v)$ is the shortest uv-path length. If G is not connected, we define $\mathrm{AD}(G) = \infty$.

For a graph property $\mathcal{P}$, we say that an n-vertex random graph G_n satisfies $\mathcal{P}$ *with high probability* (w.h.p.) if $\lim_{n\to\infty} \mathbf{Pr}[G_n \text{ satisfies } \mathcal{P}] = 1$. In this paper, we prove the following results [1].

Theorem 1. *For two constants $\alpha \in (0,1)$ and $\beta > 0$, let $d = (\beta + o(1))n^\alpha$ be an integer. For every constant $\epsilon > 0$, it holds w.h.p. that*

$$|\mathrm{AD}(G_{n,d}) - \mu| < \epsilon,$$

where

$$\mu = \begin{cases} \alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}, \\ \lceil \alpha^{-1} \rceil & \text{otherwise.} \end{cases} \tag{2}$$

Theorem 2. *For two constants $\alpha \in (0,1)$ and $\beta > 0$, let $d = (\beta + o(1))n^\alpha$ be an integer. It holds w.h.p. that*

$$\mathrm{diam}(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1.$$

The study of $G_{n,d}$ originated from the configuration model introduced by Bollobas [3]. Independently, Bender and Canfield [1] considered a similar model. The configuration model usually enables us to study $G_{n,d}$ for a constant d. The case of $d = d(n) \gg 1$ is much less understood, though there is a well-known successful approach called the *switching method*, introduced by McKay [24]. See [33] for a detailed survey on $G_{n,d}$. However, results shown by the switching method usually require the condition that $d \ll n^\gamma$ where $\gamma \leqslant 1$ is some reasonable constant. Therefore, $G_{n,d}$ of $d = (\beta + o(1))n^\alpha$ with arbitrary constant α seems to be far from these methods.

Another recent remarkable approach for the study of $G_{n,d}$ is to compare $G_{n,d}$ with an Erdős-Rényi graph $G(n,p)$ of $p = \frac{d}{n}$. Recall that $G(n,p)$ is an n-vertex graph where every two distinct vertices u and v are joined by an edge with probability p independent from any other edges. Since each degree of $G(n,p)$ is concentrated on np, we may expect that $G(n,p)$ and $G_{n,d}$ share several structural properties if $d = (1+o(1))np$. For $\log n \ll d \ll n^{1/3}/(\log n)^2$, Kim and Vu [20] presented a coupling of $G_{n,d}$ and $G_{n,d}$ of $p = (1-o(1))\frac{d}{n}$ such that $G(n,p) \subseteq G_{n,d}$ holds w.h.p. Dudek et al. [11, 14] improved this result by presenting a coupling having the same property for $\log n \ll d \ll n$. Their result is

[1] In the conference version of this paper [29], we proved Theorem 2.

called the *embedding theorem*. The embedding theorem enables us to bound $\mathrm{diam}(G_{n,d})$ and $\mathrm{AD}(G_{n,d})$ from above by $\mathrm{diam}(G(n,p))$ and $\mathrm{AD}(G(n,p))$, respectively. Very recently, Gao, Isaev, and McKay [15] proved that there is a coupling of $G(n,p)$ and $G_{n,d}$ satisfying $G(n,p) \supseteq G_{n,d}$ if $p \geqslant \frac{Cd\log n}{n}$ for some constant C, $d = \omega(\log n)$ and $d = o(n)$. We can immediately obtain Theorem 2 by combining the coupling of [15] and known results cencerning the diameter of $G(n,p)$. However, due to the $O(\log n)$ factor in the condition $p \geqslant \frac{Cd\log n}{n}$, Theorem 1 does not follow from [15] immediately.

To study $\mathrm{diam}(G_{n,d})$ and $\mathrm{AD}(G_{n,d})$, we shall look at $\mathrm{diam}(G(n,p))$ and $\mathrm{AD}(G(n,p))$ of $p = \frac{d}{n}$. It is well known that $G(n,p)$ of $p = (\beta + o(1))n^{-1+\alpha}$ has diameter $\lfloor \alpha^{-1} \rfloor + 1$ [6, 4, 14]. As for the average distance, we obtain a concentration result of $\mathrm{AD}(G(n,p))$, which might be of independent interest.

Theorem 3. *For two constants $\alpha \in (0,1)$ and $\beta > 0$, let $p = \beta n^{-1+\alpha}$ and*

$$\mu = \begin{cases} \alpha^{-1} + \exp(-\beta^{1/\alpha}) & \text{if } \alpha^{-1} \in \mathbb{N}, \\ \lceil \alpha^{-1} \rceil & \text{otherwise.} \end{cases}$$

Then, there exist absolute constants $C_1, C_2 > 0$ such that

$$|\mathrm{AD}(G(n,p)) - \mu| \leqslant C_1 n^{-C_2}$$

holds w.h.p.

1.1 Related results and trivial bounds

Diameter of $G(n,p)$. There is a long line of the diameter of $G(n,p)$ [22, 4, 8, 13, 28]. For dense $G(n,p)$, Bollobas [4] proved the following result.

Theorem 4 (Theorem 6 of [4])**.** *Fix a positive constant c. Let $D = D(n) \geqslant 2$ be a positive integer and $p = p(n) \in [0,1]$ be a real number satisfying*

$$p^D n^{D-1} = \log(n^2/c).$$

Suppose that $np = \omega(\log n)$. Then, $G(n,p)$ satisfies

$$\lim_{n\to\infty} \mathbf{Pr}[\mathrm{diam}(G(n,p)) = k] = \begin{cases} \exp(-c/2) & \text{if } k = D, \\ 1 - \exp(-c/2) & \text{if } k = D+1, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 5. *Suppose that $p = (\beta + o(1))n^{-1+\alpha}$, where $\alpha \in (0,1)$ and $\beta > 0$ are any constants. Then,* $\mathrm{diam}(G(n,p)) = \lfloor \alpha^{-1} \rfloor + 1$ *holds w.h.p.*

It should be noted that Corollary 5 also follows from the main result of Klee and Larman [22].

The diameter of $G(n,p)$ of small p has gathered special attention [5, 28, 8]. In this line of work, there is a convention that the diameter of a disconnected graph is the maximum among all diameters of its connected components. Bollobás [5] proved that $\operatorname{diam}(G(n,p)) \in A$ holds w.h.p. if $np - \log n = \omega(1)$, where $A = A(n) \subseteq \mathbb{N}$ satisfies $|A| \leqslant 4$. Chung and Lu [8] studied $\operatorname{diam}(G(n,p))$ with $1 < np \leqslant c \log n$ where c is some constant. For example, they proved that $\operatorname{diam}(G(n,p)) = (1+o(1))\frac{\log n}{\log np}$ holds w.h.p. if $\omega(1) = np < \log n$. Riordan and Wormald [28] strengthened the results of [8], providing the tight estimate for $\operatorname{diam}(G(n,p))$ for $1+o(1) \leqslant np = O(1)$. For smaller p, Łuczak [23] investigated $\operatorname{diam}(G(n,p))$ with $np < 1$.

Average distance of $G(n,p)$. The average distance of random graphs with a power law degree sequence has gathered a great deal of attention in network analysis [18, 27, 2, 32, 9, 31]. Focusing on $G(n,p)$ with $np = \omega(\log n)$, one may observe that $\mathrm{AD}(G(n,p)) \approx \operatorname{diam}(G(n,p))$. More precisely, it is easy to see that $\mathrm{AD}(G(n,p)) \leqslant \operatorname{diam}(G(n,p)) = (1+o(1))\frac{\log n}{\log np}$ and $\mathrm{AD}(G(n,p)) \geqslant (1-o(1))\frac{\log n}{\log np}$ hold by considering the maximum degree of $G(n,p)$.

Katzav et al. [18] presented analytical results on $\mathrm{AD}(G(n,p))$ for dense $G(n,p)$ that coincide with Theorem 3. However, to the best of our knowledge, there are no known results with rigorous proofs for $\mathrm{AD}(G(n,p))$ with $np = n^{\Omega(1)}$.

Diameter of $G_{n,d}$. The diameter of regular graphs has gathered special attention in graph theory [12, 17, 26] and has an application in designing efficient network topologies. Note that for every vertex v, there are at most $d(d-1)^k$ vertices having distance k from v. Thus, for every n-vertex d-regular graph G of diameter D with $d \geqslant 3$, we have

$$\begin{aligned} D &\geqslant \min\left\{ D \in \mathbb{N} \,:\, n \leqslant 1 + \sum_{i=1}^{D} d(d-1)^{i-1} \right\} \\ &= \left\lceil \log_{d-1} n + \log_{d-1}\left(1 - \frac{2}{d}\left(1 - \frac{1}{n}\right)\right)\right\rceil \\ &= \frac{\log n}{\log(d-1)} - O(1). \end{aligned} \tag{3}$$

We denote by $D' = D'(n,d)$ this lower bound Equation (3), which is known as the Moore bound [26].

For random regular graphs $G_{n,d}$, Bollobás and de la Vega [7] proved that

$$\operatorname{diam}(G_{n,d}) = D'(n,d) \pm O\left(\frac{\log\log n}{\log(d-1)}\right)$$

holds w.h.p. if the degree $d \geqslant 3$ is a constant. If $\log n \ll d \leqslant n^{o(1)}$, the embedding theorem of Dudek et al. [11, 14] and the lower bound Equation (3) together imply that

$$\operatorname{diam}(G_{n,d}) = (1+o(1))\frac{\log n}{\log d} = (1+o(1))D'(n,d)$$

holds w.h.p.

Suppose that $d = (\beta + o(1))n^{\alpha}$, where $\alpha \in (0,1)$ and $\beta > 0$ are constants. From the embedding theorem, we have $\mathrm{diam}(G_{n,d}) \leqslant \lfloor \alpha^{-1} \rfloor + 1$ holds w.h.p., as we will confirm in Section 2. On the other hand, by substituting $d = (\beta + o(1))n^{\alpha}$ to Equation (3), we obtain

$$\lim_{n\to\infty} D' = \begin{cases} \lfloor \alpha^{-1} \rfloor + 1 & \text{if } \alpha^{-1} \notin \mathbb{N} \text{ or } (\alpha^{-1} \in \mathbb{N} \wedge \beta < 1), \\ \alpha^{-1} & \text{if } \alpha^{-1} \in \mathbb{N} \wedge \beta > 1, \\ \text{depends on the term } o(1) & \text{if } \alpha^{-1} \in \mathbb{N} \wedge \beta = 1. \end{cases} \tag{4}$$

By combining Theorem 2 and eq. (4), we obtain Equation (1). As mentioned earlier, Theorem 1 immediately follows from the result of Gao, Isaev, and McKay [15]. In this paper, we prove Theorem 2 by combining the upper bound from the embedding theorem [11] and Theorem 1 (note that $\mathrm{diam}(G) \geqslant \lceil \mathrm{AD}(G) \rceil$).

Average distance of $G_{n,d}$. Let N_k be the number of vertex pairs of distance k. We use the same argument as for Equation (3) to obtain a lower bound of $\mathrm{AD}(G)$ for any d-regular graph with $d \geqslant 3$. Suppose $\mathrm{diam}(G) = D'$ and thus $N_1 + \cdots + N_{D'} = \binom{n}{2}$. Moreover, for every $k = 1, \ldots, D'-1$, we have $N_k \leqslant d(d-1)^{k-1}$. Therefore, we obtain

$$\begin{aligned} \mathrm{AD}(G) &= \binom{n}{2}^{-1} (N_1 + 2N_2 + \cdots + D'N_{D'}) \\ &= D' - \binom{n}{2}^{-1} ((D'-1)N_1 + (D'-2)N_2 + \cdots + N_{D'-1}) \\ &\geqslant D' - \binom{n}{2}^{-1} \sum_{k=1}^{D'-1} (D'-k)d(d-1)^{k-1} \\ &= D' - \frac{d(d-1)^{D'}}{(n-1)(d-2)^2} + \frac{dD'}{(n-1)(d-2)} + \frac{d}{(n-1)(d-2)^2} \qquad (5) \\ &= \log_{d-1} n - O(1). \end{aligned}$$

Let $\mathrm{AD}' = \mathrm{AD}(n,d)$ denote the lower bound Equation (5). Then, we have

$$\frac{\log n}{\log(d-1)} - O(1) \leqslant \mathrm{AD}(G_{n,d}) \leqslant \mathrm{diam}(G_{n,d}).$$

This implies that

$$\mathrm{AD}(G_{n,d}) = (1 + o(1)) \frac{\log n}{\log(d-1)}$$

holds w.h.p. if $d \geqslant 3$ is constant or $\log n \ll d \leqslant n^{o(1)}$.

Suppose that $d = (\beta + o(1))n^{\alpha}$, where $\alpha \in (0,1)$ and $\beta > 0$ are constants. From the lower bound Equation (5), we have

$$\lim_{n\to\infty} \mathrm{AD}' = \begin{cases} \lfloor \alpha^{-1} \rfloor + 1 & \text{if } \alpha^{-1} \notin \mathbb{N}, \\ \alpha^{-1} & \text{if } \alpha^{-1} \in \mathbb{N} \text{ and } \beta > 1, \\ \alpha^{-1} - \beta^{1/\alpha} + 1 & \text{if } \alpha^{-1} \in \mathbb{N} \text{ and } \beta < 1, \\ \text{depends on the term } o(1) & \text{otherwise.} \end{cases} \tag{6}$$

1.2 Definitions and notation

For two positive integers k and m with $k \leqslant m$, we denote by $(m)_k$ the falling factorial $m(m-1)\cdots(m-k+1)$. For a finite set X and a positive integer $k \leqslant |X|$, we use

$$\binom{X}{k} := \{\{x_1, \dots, x_k\} \subseteq X : |\{x_1, \dots, x_k\}| = k\},$$
$$(X)_k := \left\{(x_1, \dots, x_k) : \{x_1, \dots, x_k\} \in \binom{X}{k}\right\}.$$

For a graph G, we denote by $V(G)$ and $E(G)$, respectively, the vertex set and the edge set of G. Note that $E(G) \subseteq \binom{V(G)}{2}$ is a set of unordered vertex pairs. Throughout the paper, the number of vertices of a graph is denoted by n, and the vertex set is denoted by $V = \{1, \dots, n\}$.

We simply write $H \subseteq G$ if H is contained in G, that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ hold. Two graphs $G \cup H$ and $G \cap H$ are defined by

$$G \cup H = (V(G) \cup V(H), E(G) \cup E(H)),$$
$$G \cap H = (V(G) \cap V(H), E(G) \cap E(H)).$$

It should be noted that G and H are labelled.

A *path* is defined to be a graph $P = (\{v_0, \dots, v_\ell\}, \{\{v_0, v_1\}, \dots, \{v_{\ell-1}, v_\ell\}\})$ for distinct vertices $v_0, \dots, v_\ell$. The vertices of degree one in a path are called *endpoints*. We call a path of endpoints s and t an *st-path*. The *length* of a path is the number of edges. For a graph G and its two distinct vertices s and t, the *distance* $\mathrm{dist}_G(s,t)$ is the minimum length among all st-paths contained in G. We define $\mathrm{dist}_G(s,t) = \infty$ if G does not contain any st-paths. For a graph $G = (V, E)$ of n vertices, the *average distance* $\mathrm{AD}(G)$ of G is

$$\mathrm{AD}(G) = \binom{n}{2}^{-1} \sum_{\{s,t\} \in \binom{V}{2}} \mathrm{dist}_G(s,t).$$

The *diameter* $\mathrm{diam}(G)$ of G is

$$\mathrm{diam}(G) = \max_{s \neq t} \mathrm{dist}_G(s,t).$$

Note that $\mathrm{diam}(G) = \mathrm{AD}(G) = \infty$ if G is not connected. We use $\mathrm{dist}(s,t)$ rather than $\mathrm{dist}_G(s,t)$ if the graph G is clear from the context.

For an event Z on a graph G (say, $\mathrm{dist}_G(1,2) \geqslant \ell$), we use

$$\mathbb{1}_{[Z]}(G) = \begin{cases} 1 & \text{if } G \text{ satisfies an event } Z, \\ 0 & \text{otherwise} \end{cases}$$

as the indicator function.

1.3 Tools

Lemma 6 (The Chernoff bound; Theorem 10.1 and 10.5 of [10])**.** *Let $X_1, X_2, \ldots, X_n$ be independent binary random variables satisfying that $\mathbf{Pr}[X_i = 1] = p_i$ and let $X = \sum_{i=1}^{n} X_i$ be the sum. Then, for any $\epsilon \geqslant 0$,*

$$\mathbf{Pr}[X \geqslant (1+\epsilon)\,\mathbf{E}[X]] \leqslant \exp\left(-\frac{\min\{\epsilon, \epsilon^2\}\,\mathbf{E}[X]}{3}\right)$$

and

$$\mathbf{Pr}[X \leqslant (1-\epsilon)\,\mathbf{E}[X]] \leqslant \exp\left(-\frac{\epsilon^2\,\mathbf{E}[X]}{2}\right).$$

Lemma 7 (Multivariate version of Brun's sieve; Lemma 2.8 of [33])**.** *Let $S_n^{(1)}, \ldots, S_n^{(k)}$ be random variables defined on the same space Ω_n such that each $S_n^{(i)}$ can be written as the sum of binary random variables. Suppose that there exist positive constants $\lambda_1, \ldots, \lambda_k$ satisfying*

$$\lim_{n\to\infty} \mathbf{E}\left[\prod_{i=1}^{k} (S_n^{(i)})_{r_i}\right] = \prod_{i=1}^{k} \lambda_i^{r_i}$$

for every fixed integers $r_1, \ldots, r_k \geqslant 0$.

Then, for any constants $j_1, \ldots, j_k \geqslant 0$, it holds that

$$\lim_{n\to\infty} \mathbf{Pr}\left[\bigwedge_{i=1}^{k} [S_n^{(i)} = j_i]\right] = \prod_{i=1}^{k} \exp(-\lambda_i)\frac{\lambda^{j_i}}{j_i!}.$$

Lemma 8 (Lemma 2.1 of [19])**.** *Suppose that $1 \ll d \ll n$. For any fixed graph H, it holds that*

$$\mathbf{Pr}[H \subseteq G_{n,d}] = (1+o(1))\left(\frac{d}{n}\right)^{|E(H)|}.$$

Let $G[n,m]$ be a graph selected uniformly at random from the set of all graphs of n vertices with exactly m edges.

Lemma 9 (The embedding theorem; Theorem 10.10 of [14])**.** *There is a constant $C > 0$ that satisfies the following. For any real $\gamma = \gamma(n)$, integer $d = d(n)$ satisfying*

$$C\left(\left(\frac{d}{n} + \frac{\log n}{d}\right)^{1/3}\right) \leqslant \gamma < 1, \tag{7}$$

and $m = \lfloor (1-\gamma)nd/2 \rfloor$, there exists a joint distribution π of $G[n,m]$ and $G_{n,d}$ such that

$$\lim_{n\to\infty} \mathbf{Pr}_{\pi}[G[n,m] \subseteq G_{n,d}] = 1$$

holds.

In other words, for $\log n \ll d \ll n$, we can choose $m = (1-o(1))nd/2$ and couple $G[n,m]$ and $G_{n,d}$ such that $G[n,m] \subseteq G_{n,d}$ holds w.h.p.

2 Upper bounds of $\mathrm{AD}(G_{n,d})$ and $\mathrm{diam}(G_{n,d})$

In this section we obtain upper bounds of $\mathrm{AD}(G_{n,d})$ and $\mathrm{diam}(G_{n,d})$ using Lemma 9. As noted in [11], in Lemma 9, one can replace $G[n,m]$ by $G(n,p)$ of $p=(1-2\gamma)d/(n-1)$. This yields the following result.

Corollary 10. *For $d=d(n)$ satisfying $\log n \ll d \ll n$, there exists $p=(1-o(1))\frac{d}{n}$ such that $\mathrm{AD}(G_{n,d}) \leqslant \mathrm{AD}(G(n,p))$ and $\mathrm{diam}(G_{n,d}) \leqslant \mathrm{diam}(G(n,p))$ hold w.h.p.*

For $d=(\beta+o(1))n^{\alpha}$, take γ of Lemma 9 satisfying $\gamma=o(1)$, and let $p=(1-2\gamma)\frac{d}{n-1}=(\beta+o(1))n^{-1+\alpha}$. Then, from Theorem 3 and corollary 10, it holds w.h.p. that

$$\mathrm{AD}(G_{n,d}) \leqslant \mathrm{AD}(G(n,p)) \leqslant \mu + o(1). \tag{8}$$

Similarly, from Corollaries 5 and 10, a random regular graph $G_{n,d}$ w.h.p. satisfies

$$\mathrm{diam}(G_{n,d}) \leqslant \mathrm{diam}(G(n,p)) \leqslant \lfloor \alpha^{-1} \rfloor + 1. \tag{9}$$

3 Lower bounds of $\mathrm{AD}(G_{n,d})$ and $\mathrm{diam}(G_{n,d})$

If $\alpha^{-1} \notin \mathbb{N}$, the lower bound Equation (6) and the upper bound Equation (8) yield that

$$\mathrm{AD}(G_{n,d}) = \lfloor \alpha^{-1} \rfloor + 1 - o(1)$$

holds w.h.p. Now we focus on the case where $\alpha^{-1} \in \mathbb{N}$. This section is devoted to prove the following.

Lemma 11. *Let $d=(\beta+o(1))n^{\alpha}$, where $\alpha \in (0,1)$ and $\beta>0$ are any constants satisfying $\alpha^{-1} \in \mathbb{N}$. For any constant $\epsilon>0$,*

$$\lim_{n\to\infty} \mathbf{Pr}[\mathrm{AD}(G_{n,d}) \leqslant \mu - \epsilon] = 0,$$

where $\mu = \alpha^{-1} + \exp(-\beta^{1/\alpha})$.

Remark. By combining Equation (8) and lemma 11, we complete the proof of Theorem 1. Moreover, Lemma 11 implies

$$\mathrm{diam}(G_{n,d}) \geqslant \lceil \mathrm{AD}(G_{n,d}) \rceil = \lfloor \alpha^{-1} \rfloor + 1$$

holds w.h.p., which completes the proof of Theorem 2.

Proof of Lemma 11. Note that

$$\mathrm{AD}(G_{n,d}) = \binom{n}{2}^{-1} \sum_{\{s,t\}\in\binom{V}{2}} \mathrm{dist}(s,t)$$

$$= \sum_{\ell=1}^{\infty} \binom{n}{2}^{-1} \sum_{\{s,t\}\in\binom{V}{2}} \mathbb{1}_{[\mathrm{dist}(s,t)\geqslant \ell]}$$

$$\geqslant \sum_{\ell=1}^{\alpha^{-1}+1} \binom{n}{2}^{-1} \sum_{\{s,t\}\in\binom{V}{2}} \mathbb{1}_{[\mathrm{dist}(s,t)\geqslant \ell]}.$$

For $\ell \in \{1,\dots,\alpha^{-1}+1\}$, let $p_\ell = p_\ell(G_{n,d}) = \binom{n}{2}^{-1}\sum_{\{s,t\}\in\binom{V}{2}} \mathbb{1}_{[\mathrm{dist}(s,t)\geqslant \ell]}$. We evaluate p_ℓ using the following result.

Lemma 12. *Consider $G_{n,d}$ of $d = (\beta + o(1))n^\alpha$. Fix two constants $\alpha \in (0,1)$ and $\beta > 0$ satisfying $\alpha^{-1} \in \mathbb{N}$. For any constant $k \in \mathbb{N}$, fix $2k$ distinct vertices $s_1,\dots,s_k,t_1,\dots,t_k$. For any fixed $\ell_1,\dots,\ell_k \in \{1,\dots,\alpha^{-1}+1\}$, it holds that*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\bigwedge_{i=1}^{k} [\mathrm{dist}(s_i,t_i) \geqslant \ell_i]\right] = \exp(-M\beta^{1/\alpha})$$

where $M = |\{i \in \{1,\dots,k\} : \ell_i = \alpha^{-1}+1\}|$.

We will prove Lemma 12 in Section 3.1. For $\ell \in \{1,\dots,\alpha^{-1}+1\}$, let

$$\mu_\ell = \begin{cases} 1 & \text{if } 1 \leqslant \ell \leqslant \alpha^{-1}, \\ \exp(-\beta^{1/\alpha}) & \text{if } \ell = \alpha^{-1}+1. \end{cases}$$

From Lemma 12, we have

$$\begin{aligned} \mathbf{E}[p_\ell] &= \binom{n}{2}^{-1} \sum_{\{s,t\}\in\binom{V}{2}} \mathbf{Pr}[\mathrm{dist}(s,t) \geqslant \ell] \\ &= \mathbf{Pr}[\mathrm{dist}(1,2) \geqslant \ell] = \mu + o(1) \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}[p_\ell^2] &= \binom{n}{2}^{-2} \sum_{\{s,t\},\{s',t'\}\in\binom{V}{2}} \mathbf{Pr}[\mathrm{dist}(s,t) \geqslant \ell \wedge \mathrm{dist}(s',t') \geqslant \ell] \\ &= \binom{n}{2}^{-2} \left(O(n^3) + \sum_{\substack{\{s,t\},\{s',t'\}\in\binom{V}{2}: \\ \{s,t\}\cap\{s',t'\}=\varnothing}} \mathbf{Pr}[\mathrm{dist}(s,t) \geqslant \ell \wedge \mathrm{dist}(s',t') \geqslant \ell] \right) \\ &= \mathbf{Pr}[\mathrm{dist}(1,2) \geqslant \ell \wedge \mathrm{dist}(3,4) \geqslant \ell] + o(1) = \mu^2 + o(1). \end{aligned}$$

From the Chebyshev inequality, for every constant $\epsilon > 0$, we have

$$\mathbf{Pr}[|p_\ell - \mathbf{E}[p_\ell]| \geqslant \epsilon] \leqslant \frac{\mathbf{Var}[p_\ell]}{\epsilon^2} = o(1).$$

Thus we obtain

$$\mathbf{Pr}\left[\left|\left(\sum_{\ell=1}^{\alpha^{-1}+1} p_\ell\right) - \mu\right| > \epsilon\right] \leqslant \sum_{\ell=1}^{\alpha^{-1}+1} \mathbf{Pr}\left[|p_\ell - \mu_\ell| > \epsilon/(\alpha^{-1}+1)\right] = o(1).$$

Therefore, it holds w.h.p. that

$$\mathrm{AD}(G_{n,d}) \geqslant \sum_{\ell=1}^{\alpha^{-1}+1} p_\ell \geqslant \mu - o(1),$$

which completes the proof of Lemma 11. □

3.1 Distances of fixed vertex pairs of $G_{n,d}$

This part is devoted to prove Lemma 12. We start with establishing the following result.

Lemma 13. *Consider $G_{n,d}$ of $d = (\beta + o(1))n^\alpha$ for constants $\alpha \in (0, 1)$ and $\beta > 0$. For two fixed distinct vertices s and t, it holds w.h.p. that* $\mathrm{dist}(s, t) \in \{\lceil\alpha^{-1}\rceil, \lfloor\alpha^{-1}\rfloor + 1\}$.

Proof. For two fixed vertices s, t of $G_{n,d}$ and an integer ℓ, we denote by $\mathcal{P}$ the set of paths of length ℓ connecting s and t in a complete graph. Let $X_\ell = X_\ell(G_{n,d})$ be the number of paths $P \in \mathcal{P}$ contained in $G_{n,d}$, that is,

$$X_\ell = |\{P \in \mathcal{P} : P \subseteq G_{n,d}\}|. \tag{10}$$

Fix an integer ℓ satisfying $\ell\alpha < 1$ (or equivalently, $\ell \leqslant \lceil\alpha^{-1}\rceil - 1$). Then, from Lemma 8, we have

$$\begin{aligned}\mathrm{E}(X_\ell) &= \sum_{P\in\mathcal{P}} \mathbf{Pr}[P \subseteq G_{n,d}] \\ &= (1 + o(1))n^{\ell-1}\left(\frac{d}{n}\right)^\ell \\ &= o(1).\end{aligned}$$

From the Markov's inequality, we obtain

$$\begin{aligned}\mathbf{Pr}[\mathrm{dist}(s, t) \leqslant \ell] &\leqslant \mathbf{Pr}[X_1 + \cdots + X_\ell > 0] \\ &\leqslant \sum_{i=1}^{\ell} \mathrm{E}(X_i) \\ &= o(1).\end{aligned}$$

In other words, $\mathrm{dist}(s, t) \geqslant \ell + 1 \geqslant \lceil\alpha^{-1}\rceil$ holds w.h.p.

On the other hand, from Equation (9), we have $\mathrm{dist}(s, t) \leqslant \mathrm{diam}(G_{n,d}) \leqslant \lfloor\alpha^{-1}\rfloor + 1$. This completes the proof of Lemma 13. □

Proof of Lemma 12. Fix an integer $k > 0$ and $2k$ distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$ of $G_{n,d}$, where $d = (\beta + o(1))n^\alpha$. From Lemma 13, it holds w.h.p. that $\mathrm{dist}(s,t) \in \{\alpha^{-1}, \alpha^{-1}+1\}$.

Suppose that $\ell_1 \leqslant \alpha^{-1}$ and thus $\mathrm{dist}(s_1,t_1) \geqslant \ell_1$ holds w.h.p. Then we have

$$\begin{aligned}\mathbf{Pr}\left[\bigwedge_{i=2}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \ell_i]\right] - \mathbf{Pr}[\mathrm{dist}(s_1,t_1) < \ell_1] &\leqslant \mathbf{Pr}\left[\bigwedge_{i=1}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \ell_i]\right] \\ &\leqslant \mathbf{Pr}\left[\bigwedge_{i=2}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \ell_i]\right]\end{aligned}$$

and thus

$$\mathbf{Pr}\left[\bigwedge_{i=1}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \ell_i]\right] = \mathbf{Pr}\left[\bigwedge_{i=2}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \ell_i]\right] - o(1).$$

Hence, we may assume that $\ell_i = \alpha^{-1}+1$ for all $i = 1, \dots, k$ (i.e., $M = k$ in Lemma 12).

Let $\mathcal{P}^{(i)}$ denote the set of s_it_i-paths of length α^{-1} contained in the complete graph K_n. Define $X^{(i)}$ as the number of paths of $\mathcal{P}^{(i)}$ contained in $G_{n,d}$, that is,

$$X^{(i)} = |\{P \in \mathcal{P}^{(i)} : P \subseteq G(n,p)|.$$

Then, we have

$$\begin{aligned}\mathbf{Pr}\left[\bigwedge_{i=1}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \alpha^{-1}+1]\right] &= \mathbf{Pr}\left[\bigwedge_{i=1}^{k}[\mathrm{dist}(s_i,t_i) \geqslant \alpha^{-1}] \wedge \bigwedge_{i=1}^{k}[X^{(i)}=0]\right] \\ &= \mathbf{Pr}\left[\bigwedge_{i=1}^{k}[X^{(i)}=0]\right] - o(1). \qquad (11)\end{aligned}$$

We evaluate Equation (11) using the following result, which will be shown in Section 3.2.

Lemma 14. *Consider $G_{n,d}$ of $d = (\beta + o(1))n^\alpha$, where $\alpha \in (0,1)$ and $\beta > 0$ are any constants satisfying $\alpha^{-1} \in \mathbb{N}$. Fix $2k$ distinct vertices $s_1, \dots, s_k, t_1, \dots, t_k$, where k is any constant. For $i = 1, \dots, k$, let $X^{(i)}$ denote the number of s_it_i-paths of length $\alpha^{-1} \in \mathbb{N}$ contained in $G(n,p)$. Fix arbitrary nonnegative integers $r_1, \dots, r_k$. Then, it holds that*

$$\mathbf{E}\left[\prod_{i=1}^{k}(X^{(i)})_{r_i}\right] = (\beta^{1/\alpha})^R + o(1),$$

where $R = r_1 + \cdots + r_k$.

From Lemma 14 and the Poisson approximation theorem (Lemma 7), we have

$$\mathbf{Pr}\left[\bigwedge_{i=1}^{k}[X^{(i)}=0]\right] = \exp(-k\beta^{1/\alpha}) + o(1). \qquad (12)$$

By combining Equations (11) and (12), we have

$$\mathbf{Pr}\left[\bigwedge_{i=1}^{k}[\mathrm{dist}(s_i,t_i)\geqslant \alpha^{-1}+1]\right]=\exp(-k\beta^{1/\alpha})-o(1).$$

This completes the proof of Lemma 12 and thus Lemma 11.

3.2 Proof of Lemma 14

We first prove the following result and then show Lemma 14.

Lemma 15. *Fix an integer $\ell \geqslant 1$ and consider $G(n,p)$ satisfying $(np)^\ell=\Omega(n)$. Fix $2k$ distinct vertices $s_1,\dots,s_k,t_1,\dots,t_k$, where k is arbitrary constant. For $i=1,\dots,k$, let $X^{(i)}$ denote the number of s_it_i-paths of length $\ell\in\mathbb{N}$ contained in $G(n,p)$.*

Then, for any fixed nonnegative integers $r_1,\dots,r_k$,

$$\mathbf{E}\left[\prod_{i=1}^{k}\left(X^{(i)}\right)_{r_i}\right]=n^{R(\ell-1)}p^{R\ell}\left(1\pm O\left(\frac{1}{np}\right)\right),$$

where $R=r_1+\dots+r_k$.

Corollary 16. *Consider $G(n,p)$ of $p=(\beta+o(1))n^{-1+\alpha}$, where $\alpha\in(0,1)$ and $\beta>0$ are any constants satisfying $\alpha^{-1}\in\mathbb{N}$. Fix arbitrary nonnegative integers $r_1,\dots,r_k$. Then, it holds that*

$$\mathbf{E}\left[\prod_{i=1}^{k}(X^{(i)})_{r_i}\right]=(\beta^{1/\alpha})^R+o(1),$$

where $R=r_1+\dots+r_k$.

Proof of Lemma 15. For a positive constant k, fix $2k$ distinct vertices $s_1,\dots,s_k,t_1,\dots,t_k$. For every $i\in\{1,\dots,k\}$, let $\mathcal{P}^{(i)}$ denote the set of all s_it_i-paths of length ℓ contained in the complete graph. We denote by $X^{(i)}$ the number of paths of $\mathcal{P}^{(i)}$ contained in $G(n,p)$.

Fix nonnegative integers $k,r_1,\dots,r_k$. We may assume that $r_i>0$ for every $i=1,\dots,k$. Let $\mathcal{A}=(\mathcal{P}^{(1)})_{r_1}\times\dots\times(\mathcal{P}^{(k)})_{r_k}$. Each element $A\in\mathcal{A}$ is a tuple

$$A=((P^{(1)}_1,\dots,P^{(1)}_{r_1}),\dots,(P^{(k)}_1,\dots,P^{(k)}_{r_k})),$$

where each $P^{(i)}_j\in\mathcal{P}_i$ is an s_it_i-path of length ℓ and $P^{(i)}_j\neq P^{(i)}_{j'}$ holds for every i and $j\neq j'$. For notational convention, we write $A=(P_1,\dots,P_R)\in\mathcal{A}$. Since $r_k>0$, it holds that $P_R\in\mathcal{P}^{(k)}$.

For a tuple $A=(P_1,\dots,P_t)$ of t paths, let $E(A)=\bigcup_{i=1}^t E(P_i)$ and $V(A)=\bigcup_{i=1}^t V(P_i)$ (we will use induction on R and hence we assume $t\leqslant R$ here). For $\mathcal{S}\subseteq\mathcal{A}$, we consider

$$\Gamma_{\mathcal{S}}=\sum_{A\in\mathcal{S}}p^{|E(A)|}.$$

Note that $\mathbf{E}\left[\prod_{i=1}^{k}\left(X^{(i)}\right)_{r_i}\right] = \sum_{A\in\mathcal{A}} \mathbf{Pr}[E(A)\subseteq E(G(n,p))] = \Gamma_{\mathcal{A}}$. We claim

$$n^{R(\ell-1)}p^{R\ell}\left(1-O\left(\frac{1}{n}\right)\right) \leqslant \Gamma_{\mathcal{A}} \leqslant n^{R(\ell-1)}p^{R\ell}\left(1+O\left(\frac{1}{np}\right)\right), \tag{13}$$

which completes the proof of Lemma 15.

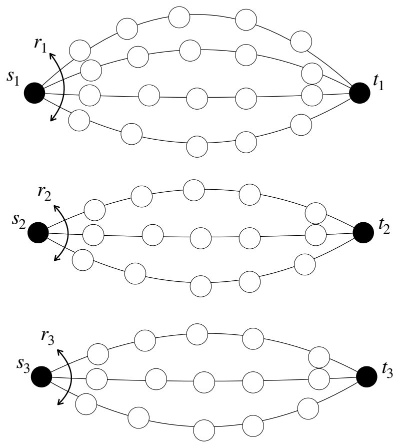

Figure 1: A tuple $A \in \mathcal{A}\setminus\mathcal{F}$.

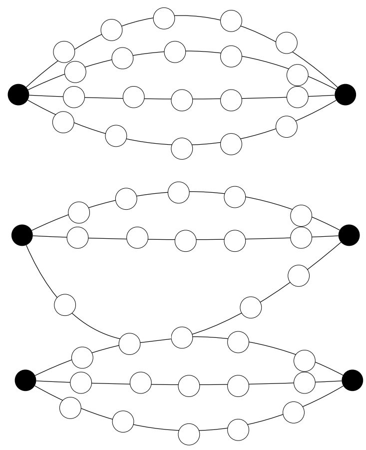

Figure 2: A tuple $A \in \mathcal{F}$.

For any $A \in \mathcal{A}$, it holds that $|E(A)| \leqslant R\ell$ and the equality holds if and only if any two distinct paths P_i, P_j of A shares no edges (see Figure 1). Let

$$\begin{aligned}\mathcal{F} &= \{A\in\mathcal{A} : |E(A)| < R\ell\}\\ &= \{(P_1,\dots,P_R)\in\mathcal{A} : \exists i\neq j,\ E(P_i)\cap E(P_j)\neq\varnothing\}.\end{aligned} \tag{14}$$

Figure 2 illustrates an example. Then, $\Gamma_{\mathcal{A}}$ can be decomposed into

$$\Gamma_{\mathcal{A}} = \Gamma_{\mathcal{F}} + \Gamma_{\mathcal{A}\setminus\mathcal{F}}. \tag{15}$$

The second term $\Gamma_{\mathcal{A}\setminus\mathcal{F}}$ satisfies

$$\begin{aligned}\Gamma_{\mathcal{A}\setminus\mathcal{F}} &= p^{R\ell}\left|\{A\in\mathcal{A} : |E(A)| = R\ell\}\right|\\ &\geqslant p^{R\ell}\left|\{A\in\mathcal{A} : |E(A)| = R\ell \text{ and } |V(A)| = R(\ell-1)+2k\}\right|\\ &= (n-2k)_{R(\ell-1)}p^{R\ell}\\ &\geqslant n^{R(\ell-1)}p^{R\ell}\left(1-O\left(\frac{1}{n}\right)\right).\end{aligned}$$

This implies the lower bound $\Gamma_{\mathcal{A}} \geqslant \Gamma_{\mathcal{A}\setminus\mathcal{F}} \geqslant n^{R(\ell-1)}p^{R\ell}\left(1-O\left(\frac{1}{n}\right)\right)$.

Now it suffices to bound $\Gamma_{\mathcal{A}}$ from above. Observe that $\Gamma_{\mathcal{A}\setminus\mathcal{F}}$ satisfies

$$\Gamma_{\mathcal{A}\setminus\mathcal{F}} = p^{R\ell}\left|\{A\in\mathcal{A} : |E(A)| = R\ell\}\right| \leqslant n^{R(\ell-1)}p^{R\ell}. \tag{16}$$

We show that this term is dominating in $\Gamma_{\mathcal{A}}$. Lemma 15 immediately follows from Equations (15) and (16) and the following result:

Lemma 17. *Suppose that* $(np)^\ell = \Omega(n)$*. Define* $\mathcal{F}$ *as Equation* (14)*. It holds that*

$$\Gamma_{\mathcal{F}} = O\left(\frac{n^{R(\ell-1)}p^{R\ell}}{np}\right).$$

Proof. We use induction on R. For the base case of $R=1$, we have $\mathcal{F}=\varnothing$ and thus

$$\begin{aligned}\Gamma_{\mathcal{A}} &\leqslant n^{\ell-1}p^\ell,\\ \Gamma_{\mathcal{F}} &= 0.\end{aligned}$$

Suppose that $R \geqslant 2$ and that Lemma 17 holds for $R-1$. Note that Lemma 15 also holds for $R-1$ since Lemma 17 implies Lemma 15. Let

$$\mathcal{A}' = (\mathcal{P}^{(1)})_{r_1} \times \cdots \times (\mathcal{P}^{(k-1)})_{r_k-1}.$$

Then, each element $A = (P_1, \dots, P_R) \in \mathcal{A}$ can be decomposed into $A' = (P_1, \dots, P_{R-1}) \in \mathcal{A}'$ and $P_R \in \mathcal{P}^{(k)}$. Note that the edge set $E(A')$ for $A' \in \mathcal{A}'$ are defined in the same way as $E(A)$ and it holds that $|E(A')| \leqslant (R-1)\ell$. Let

$$\mathcal{F}' = \{A' \in \mathcal{A}' : |E(A')| < (R-1)\ell\}.$$

By the induction assumption on $\mathcal{F}'$ and $\mathcal{A}'$, we have

$$\Gamma_{\mathcal{A}'} \leqslant n^{(R-1)(\ell-1)}p^{(R-1)\ell}\left(1 + \frac{C_1}{np}\right), \quad \Gamma_{\mathcal{F}'} \leqslant C_2\left(\frac{n^{(R-1)(\ell-1)}p^{(R-1)\ell}}{np}\right)$$

for some constants $C_1, C_2 > 0$. For $A = (P_1, \dots, P_R) \in \mathcal{F}$, let $A' = (P_1, \dots, P_{R-1}) \in \mathcal{A}'$. Since $A \in \mathcal{F}$, either

(i) $E(P_R) \cap E(P_i) \neq \varnothing$ for some $1 \leqslant i < R$, or

(ii) $E(P_R) \cap E(A') = \varnothing$ and $E(P_i) \cap E(P_j) \neq \varnothing$ for some $1 \leqslant i < j < R$ (thus $A' \in \mathcal{F}'$)

holds. Therefore, we have

$$\begin{aligned}\Gamma_{\mathcal{F}} &= \sum_{A \in \mathcal{F}} p^{|E(A)|}\\ &\leqslant \sum_{A' \in \mathcal{A}'} \sum_{\substack{P_R \in \mathcal{P}^{(k)}:\\ E(A) \cap E(P_R) \neq \varnothing}} p^{|E(A') \cup E(P_R)|} + \sum_{A' \in \mathcal{F}'} \sum_{\substack{P_R \in \mathcal{P}^{(k)}:\\ E(P_R) \cap E(A') = \varnothing}} p^{|E(A') \cup E(P_R)|}. \end{aligned} \tag{17}$$

From the induction assumption, the second term satisfies

$$\begin{aligned}\sum_{A' \in \mathcal{F}'} \sum_{\substack{P_R \in \mathcal{P}^{(k)}:\\ E(P_R) \cap E(A') = \varnothing}} p^{|E(A') \cup E(P_R)|} &= \sum_{A' \in \mathcal{F}'} p^{|E(A')|} \sum_{\substack{P_R \in \mathcal{P}^{(k)}:\\ E(P_R) \cap E(A') = \varnothing}} p^{|E(P_R)|}\\ &\leqslant \Gamma_{\mathcal{F}'} \cdot n^{\ell-1}p^\ell. \end{aligned} \tag{18}$$

The first term can be rewritten as

$$\sum_{A'\in\mathcal{A}'}\sum_{\substack{P_R\in\mathcal{P}^{(k)}:\\ E(A')\cap E(P_R)\neq\varnothing}} p^{|E(A')\cup E(P_R)|}=\sum_{A'\in\mathcal{A}'}p^{|E(A')|}\sum_{\substack{P_R\in\mathcal{P}^{(k)}:\\ E(A)\cap E(P_R)\neq\varnothing}} p^{|E(P_R)\setminus E(A')|}.$$

Fix $A'=(P_1,\dots,P_{R-1})\in\mathcal{A}'$. Let $S=\{s_1,\dots,s_k,t_1,\dots,t_k\}$ be the endpoints of the paths and let $V_1=S\cup V(P_1)\cup\cdots\cup V(P_{R-1})$. To bound the number of P_R satisfying the condition (ii), we consider two cases: $E(P_R)\not\subseteq E(A')$ and $E(P_R)\subseteq E(A')$.

Case I. $\boldsymbol{E(P_R)\not\subseteq E(A')}$. The edge set $E(P_R)\cap E(A')$ forms a forest. Since $E(P_R)\not\subseteq E(A')$, this forest is not connected and thus we have $|V(P_R)\cap V_1|-|E(P_R)\cap E(A')|\geqslant 2$. This yields

$$\begin{aligned}|V(P_R)\setminus V_1|&=|V(P_R)|-|V(P_R)\cap V_1|\\&\leqslant \ell-|E(P_R)\cap E(A')|-1.\end{aligned}$$

Let $|E(P_R)\cap E(A')|=t<\ell$. Then, P_R consists of two type of vertices: at most $\ell-t-1$ from $V\setminus V_1$ and the others from V_1. Therefore, there are at most $n^{\ell-t-1}|V_1|^t\leqslant C^t n^{\ell-t-1}$ candidates for the path P_R satisfying $|E(P_R)\cap E(A')|=t<\ell$, where $C=(R-1)(\ell+1)$ (recall that two endpoints of P_R are fixed and thus they are not taken into account).

Case II. $\boldsymbol{E(P_R)\subseteq E(A')}$. We claim $A'\in\mathcal{F}'$. If not, it holds that $E(P_i)\cap E(P_j)=\varnothing$ for any $i<j<R$. Hence, $E(P_R)\subseteq E(A')$ implies $P_R=P_i$ for some $i<R$. This contradicts to the definition of $\mathcal{A}$ ($P_i\neq P_j$ for any $i<j\leqslant R$). Moreover, the number of $P_R\in\mathcal{P}^{(k)}$ satisfying $E(P_R)\subseteq E(A')$ is at most $|V_1|^{\ell-1}\leqslant C^{R(\ell-1)}$. Therefore, we have

$$\begin{aligned}
&\sum_{A'\in\mathcal{A}'}\sum_{\substack{P_R\in\mathcal{P}^{(k)}:\\ E(A')\cap E(P_R)\neq\varnothing}} p^{|E(A')\cup E(P_R)|}\\
&\leqslant \sum_{A'\in\mathcal{A}'}p^{|E(A')|}\left(\sum_{t=1}^{\ell-1}\sum_{\substack{P_R\in\mathcal{P}^{(k)}:\\ |E(A)\cap E(P_R)|=t}} p^{|E(P_R)\setminus E(A')|}\right)+\sum_{A'\in\mathcal{F}'}p^{|E(A')|}C^{R(\ell-1)}\\
&\leqslant \sum_{A'\in\mathcal{A}'}p^{|E(A')|}\cdot\sum_{t=1}^{\ell-1}C^t n^{\ell-t-1}p^{\ell-t}+C^{R(\ell-1)}\Gamma_{\mathcal{F}'}\\
&\leqslant \Gamma_{\mathcal{A}'}\cdot\frac{Cn^{\ell-1}p^{\ell}}{np}\left(1+\frac{1.01C}{np}\right)+C^{R(\ell-1)}\Gamma_{\mathcal{F}'}. \qquad (19)
\end{aligned}$$

From Equations (17) to (19) and the induction assumption, we have

$$\begin{aligned}\Gamma_{\mathcal{F}}&\leqslant \Gamma_{\mathcal{F}'}\cdot n^{\ell-1}p^{\ell}+\Gamma_{\mathcal{A}'}\cdot\frac{Cn^{\ell-1}p^{\ell}}{np}\left(1+\frac{1.01C}{np}\right)+C^{R(\ell-1)}\Gamma_{\mathcal{F}'}\\&\leqslant O\left(\frac{n^{R(\ell-1)}p^{R\ell}}{np}\right).\end{aligned}$$

This completes the proof of Lemma 17 and thus Lemma 15 (Here, we have used the assumption that $(np)^\ell = \Omega(n)$). $\square$

$\square$

Proof of Lemma 14. Let $d = (1 + o(1))np = (\beta + o(1))n^\alpha$. From Lemma 8, we have $\mathbf{Pr}[H \subseteq G(n,p)] = (1+o(1))\,\mathbf{Pr}[H \subseteq G_{n,d}]$ for any fixed graph H. Let $R = r_1 + \cdots + r_k$ and $\mathcal{A} = (\mathcal{P}^{(1)})_{r_1} \times \cdots \times (\mathcal{P}^{(k)})_{r_k}$. We write each element $A \in \mathcal{A}$ as a tuple $A = (P_1, \dots, P_R)$ of R paths. Then, from Corollary 16, we have

$$\begin{aligned}
\mathop{\mathbf{E}}_{G_{n,d}}\left[\prod_{i=1}^k \left(X^{(i)}\right)_{r_i}\right] &= \sum_{(P_1,\dots,P_R)\in\mathcal{A}} \mathbf{Pr}[E(P_1 \cup \cdots P_R) \subseteq G_{n,d}] \\
&= (1+o(1)) \sum_{(P_1,\dots,P_R)} \mathbf{Pr}[E(P_1 \cup \cdots P_R) \subseteq G(n,p)] \\
&= (1+o(1)) \mathop{\mathbf{E}}_{G(n,p)}\left[\prod_{i=1}^k \left(X^{(i)}\right)_{r_i}\right] \\
&= (\beta + o(1))^{1/\alpha}.
\end{aligned}$$

$\square$

4 Concentration of $\mathrm{AD}(G(n,p))$

We prove Theorem 3. We use $\mathrm{AD} = \mathrm{AD}(G(n,p))$ and $\mathrm{diam} = \mathrm{diam}(G(n,p))$ as random variables. Let $D = \lceil \mu \rceil = \lfloor \alpha^{-1} \rfloor + 1$. From Corollary 5, we have

$$\begin{aligned}
\mathbf{Pr}\left[|\mathrm{AD} - \mu| > \epsilon\right] &\leqslant \mathbf{Pr}\left[|\mathrm{AD} - \mu| > \epsilon \,|\, \mathrm{diam} = D\right] \mathbf{Pr}[\mathrm{diam} = D] + \mathbf{Pr}[\mathrm{diam} \neq D] \\
&\leqslant \mathbf{Pr}\left[|\mathrm{AD} - \mu| > \epsilon \,|\, \mathrm{diam} = D\right] + o(1)
\end{aligned}$$

for any $\epsilon = \epsilon(n) > 0$. Therefore, we may put the condition that $\mathrm{diam} = D$.

For $i = 1, \dots, D$, let

$$N_i = \left|\left\{\{s,t\} \in \binom{V}{2} : \mathrm{dist}(s,t) = i\right\}\right|.$$

We will prove the following result in Section 4.1:

Lemma 18. *Let $C > 0$ be a sufficiently large constant and $\epsilon = \epsilon(n) := \sqrt{\frac{\log n}{np}}$. Then, $|N_i - M_i| \leqslant C\epsilon M_i$ holds w.h.p. for all $i = 1, \dots, D-1$, where*

$$M_i = \begin{cases} \frac{(np)^i}{n}\binom{n}{2} & \text{if } i < \alpha^{-1}, \\ (1 - \exp(-\beta^{1/\alpha}))\binom{n}{2} & \text{if } i = \alpha^{-1} \in \mathbb{N}. \end{cases}$$

An upper bound of AD. Conditioned on $\mathrm{diam} = D$, it immediately holds that $\mathrm{AD} \leqslant \mathrm{diam} \leqslant D$. Thus, if $\alpha^{-1} \notin \mathbb{N}$, we have

$$\mathrm{AD} \leqslant D = \mu$$

with probability $1 - \exp(-n^{\Omega(n)})$.

Now we focus on the case where $\alpha^{-1} \in \mathbb{N}$. Let $\epsilon = C\sqrt{\frac{\log n}{np}}$ for sufficiently large constant $C > 0$. Conditioned on $\mathrm{diam} = D$, Lemma 18 implies

$$\begin{aligned} N_D &= \binom{n}{2} - N_1 - \ldots - N_{D-1} \\ &\leqslant (1 + O(\epsilon)) \exp(-\beta^{1/\alpha}) \binom{n}{2}. \end{aligned}$$

Therefore, conditioned on $\mathrm{diam} = D$, we have

$$\begin{aligned} \binom{n}{2} \cdot \mathrm{AD} &= \sum_{i=1}^{D} i N_i \\ &\leqslant D N_D + (D-1)\left(\binom{n}{2} - N_D\right) \\ &= N_D + (D-1)\binom{n}{2} \\ &\leqslant (1 + O(\epsilon))\mu\binom{n}{2}. \end{aligned}$$

In other words, $\mathrm{AD} \leqslant \mu + O(\epsilon)$ holds w.h.p.

A lower bound of AD. Conditioned on $\mathrm{diam} = D$, we have $N_1 + \cdots + N_D = \binom{n}{2}$ and thus

$$\begin{aligned} \binom{n}{2} \cdot \mathrm{AD} &= \sum_{i=1}^{D} i N_i \\ &= N_1 + 2N_2 + \cdots + (D-1)N_{D-1} + D\left(\binom{n}{2} - N_1 - \cdots - N_{D-1}\right) \\ &= D\binom{n}{2} - (D-1)N_1 - (D-2)N_2 - \cdots - N_{D-1} \\ &\geqslant (1 - O(\epsilon))\mu\binom{n}{2}. \end{aligned}$$

In the last inequality, we used Lemma 18. This completes the proof of Theorem 3.

4.1 Proof of Lemma 18

The proof of Lemma 18 is a slight modification of the proof of Theorem 7.1 of [14].

Consider $G(n,p)$ of $p=(\beta+o(1))n^{-1+\alpha}$. Let $D=\lfloor\alpha^{-1}\rfloor+1$. We consider the breadth first search process on $G(n,p)$ from a fixed vertex. Fix a vertex v. For $k\geqslant 0$, let

$$N_k(v)=\{w\in V:\mathrm{dist}(v,w)=k\}.$$

Note that $N_0(v)=\{v\}$. For sufficiently large constant $C>0$ and $\epsilon:=\sqrt{\frac{\log n}{np}}$, let $\mathcal{F}_k$ be the event of $G(n,p)$ that

$$\left||N_i(v)|-\frac{2M_i}{n}\right|\leqslant\frac{C\epsilon M_i}{n}\ \text{ for all } i=1,\dots,k,$$

where M_i is given in Lemma 18. Note that $\mathcal{F}_0$ must hold. The degree of v is denoted by $\deg(v)$. We denote by $\mathrm{Bin}(m,q)$ the binomial distributed random variable with m trials and success probability q. Note that, if we are given $N_0(v),\dots,N_{k-1}(v)$, the random variable $|N_k(v)|$ is distributed as a binomial random variable, that is,

$$|N_k(v)|\sim\mathrm{Bin}\left(n-\sum_{i=0}^{k-1}|N_i(v)|,\,1-(1-p)^{|N_{k-1}(v)|}\right).$$

Consider $\mathbf{E}[|N_k(v)|\,|\,\mathcal{F}_{k-1}]$. For every $k=1,\dots,D-1$, conditioned on $\mathcal{F}_{k-1}$, we have

$$n\geqslant n-\sum_{i=0}^{k-1}|N_i(v)|\geqslant(1-O(\epsilon))n$$

Here, recall that $(np)^{D-1}=O(n)$. Using the inequality $\mathrm{e}^{-\frac{x}{1-x}}\leqslant 1-x\leqslant\mathrm{e}^{-x}$ for every $x\in[0,1)$ (c.f., Lemma 21.1 of [14]), we obtain

$$1-(1-p)^{|N_{k-1}(v)|}=\begin{cases}(1\pm O(\epsilon))p(np)^{k-1} & \text{if } k=1,\dots,D-2,\\ (1\pm O(\epsilon))\exp(-\beta^{1/\alpha}) & \text{if } k=D-1.\end{cases}$$

Therefore, we have

$$\begin{aligned}\mathbf{E}\left[|N_k(v)|\,|\,\mathcal{F}_{k-1}\right]&=\begin{cases}(1\pm O(\epsilon))(np)^k & \text{if } k=1,\dots,D-2,\\ (1\pm O(\epsilon))\exp(-\beta^{1/\alpha})n & \text{if } k=D-1\end{cases}\\&=(1\pm O(\epsilon))\frac{2M_k}{n}.\end{aligned}$$

From the Chernoff bound (Lemma 6), we have

$$\begin{aligned}\mathbf{Pr}[\mathcal{F}_k\,|\,\mathcal{F}_{k-1}]&\geqslant 1-\exp\left(-\Theta\left(\epsilon^2(np)^k\right)\right)\\&\geqslant 1-O(n^{-2})\end{aligned}$$

if the constant C is sufficiently large (recall that C is the constant in the definition of $\mathcal{F}_k$). Therefore, $\mathcal{F}_{D-1}$ holds with probability $1-O(n^{-2})$ for sufficiently large C. Taking the union bound, it holds w.h.p. that $|N_i(v)|=(1\pm O(\epsilon))\frac{2M_i}{n}$ for all v. Consequently, we have $N_i=\frac{1}{2}\sum_{v\in V}|N_i(v)|=(1\pm O(\epsilon))M_i$, which completes the proof of Lemma 18.

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