Zeta functions with respect to general coined quantum walk of periodic graphs

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Abstract

We define a zeta function of a graph by using the time evolution matrix of a general coined quantum walk on it, and give a determinant expression for the zeta function of a finite graph. Furthermore, we present a determinant expression for the zeta function of an (inifinite) periodic graph.

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1 Introduction

Starting from p-adic Selberg zeta functions, Ihara [12] introduced the Ihara zeta functions of graphs. Ihara [12] showed that the reciprocal of the Ihara zeta function of a regular graph is an explicit polynomial. Serre [17] pointed out that the Ihara zeta function is the zeta function of a regular graph. A zeta function of a regular graph G associated to a unitary representation of the fundamental group of G was developed by Sunada

[18, 19]. Hashimoto [10] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized Ihara's result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial.

The Ihara zeta function of a finite graph was extended to an infinite graph in [1, 3, 6, 7, 8, 9], and its determinant expressions were presented. Bass [1] defined the zeta function for a pair of a tree X and a countable group Γ which acts discretely on X with quotient being a graph of finite groups. Clair and Mokhtari-Sharghi [3] extended Ihara zeta functions to infinite graphs on which a group Γ acts isomorphically and with finite quotient. In [6], Grigorchuk and Żuk defined zeta functions of infinite discrete groups, and of some class of infinite periodic graphs. Guido, Isola and Lapidus [7] defined the Ihara zeta function of a periodic simple graph. Furthermore, Guido, Isola and Lapidus [8] presented a determinant expression for the Ihara zeta function of a periodic graph.

The time evolution matrix of a discrete-time quantum walk in a graph is closely related to the Ihara zeta function of a graph. A discrete-time quantum walk is a quantum analog of the classical random walk on a graph whose state vector is governed by a matrix called the time evolution matrix. Ren et al. [16] gave a relationship between the discrete-time quantum walk and the Ihara zeta function of a graph. Konno and Sato [13] obtained a formula of the characteristic polynomial of the Grover matrix by using the determinant expression for the second weighted zeta function of a graph.

In this paper, we define a zeta function of a periodic graph by using the time evolution matrix of a general coined quantum walk on it, and present its determinant expression. The proof is an analogue of Bass' method [1].

In Section 2, we state a review for the Ihara zeta function of a finite graph and infinite graphs, i.e., a periodic simple graph, a periodic graph. In Section 3, we state about the Grover walk on a graph as a discrete-time quantum walk on a graph. In Section 4, we define a zeta function of a finite graph G by using the time evolution matrix of a general coined quantum walk on G, and present its determinant expression. Furthermore, we give an explicit formula for the characteristic polynomial of the time evolution matrix of a general coined quantum walk on G, and so present its spectrum. In Section 5, we state the definition of a periodic graph. In Section 6, we review a determinant for bounded operators acting on an infinite dimensional Hilbert space and belonging to a von Neumann algebra with a finite trace. In Section 7, we present a determinant expression for the above zeta function of a periodic graph.

2 The Ihara zeta function of a graph

All graphs in this paper are assumed to be simple. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, and let $R(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$ be the set of oriented edges (or arcs) $(u, v), (v, u)$ directed oppositely for each edge uv of G. For $e = (u, v) \in R(G)$, $u = o(e)$ and $v = t(e)$ are called the *origin* and the *terminal* of e, respectively. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path P of length n* in G is a sequence $P = (e_1, \cdots, e_n)$ of n arcs such that $e_i \in R(G)$, $t(e_i) = o(e_{i+1}) (1 \leqslant i \leqslant n-1)$. If $e_i = (v_{i-1}, v_i),\ 1 \leqslant i \leqslant n$, then we also denote P by

$(v_0, v_1, \cdots, v_n)$. Set $\mid P \mid= n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$-*path*. A (v, w)-path is called a v-*closed path* if $v = w$. The *inverse* of a closed path $C = (e_1, \cdots, e_n)$ is the closed path $C^{-1} = (e_n^{-1}, \cdots, e_1^{-1})$.

We say that a path $P = (e_1, \cdots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some $i(1 \leqslant i \leqslant n-1)$. A path without backtracking is called *proper*. Let B^r be the closed path obtained by going r times around a closed path B. Such a closed path is called a *multiple* of B. Multiples of a closed path without backtracking may have a backtracking. Such a closed path is said to have a *tail*. If its length is n, then the closed path can be written as

$$(e_1, \cdots, e_k, f_1, f_2, \cdots, f_{n-2k}, e_k^{-1}, \cdots, e_1^{-1}),$$

where $(f_1, f_2, \cdots, f_{n-2k})$ is a closed path. A closed path is called *reduced* if C has no backtracking nor tail. Furthermore, a closed path C is *primitive* if it is not a multiple of a strictly shorter closed path.

We introduce an equivalence relation between closed paths. Two closed paths $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if there exists an integer k such that $f_j = e_{j+k}$ for all j, where the subscripts are read modulo n. The inverse of C is not equivalent to C if $\mid C \mid\geqslant 3$. Let $[C]$ be the equivalence class which contains a closed path C. Also, $[C]$ is called a *cycle*.

Let $\mathcal{P}$ be the set of primitive, reduced cycles of G. Also, primitive, reduced cycles are called *prime cycles*. Note that each equivalence class of primitive, reduced closed paths of a graph G passing through a vertex v of G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at v.

The *Ihara zeta function* of a graph G is a function of a complex variable u with $|u|$ sufficiently small, defined by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]\in\mathcal{P}} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all prime cycles of G.

Let G be a connected graph with n vertices $v_1, \cdots, v_n$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The *degree* of a vertex v_i of G is defined by $\deg v_i = \deg_G v_i =\mid \{v_j \mid v_iv_j \in E(G)\} \mid$. If $\deg_G v = k$(constant) for each $v \in V(G)$, then G is called k-*regular*.

Theorem 1 (Bass)**.** *Let G be a connected graph. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G, u)^{-1} = (1 - u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})),$$

where r is the Betti number of G, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ and $d_{ij} = 0, i \neq j, (V(G) = \{v_1, \cdots, v_n\})$.

Let $G = (V(G), E(G))$ be a countable simple graph, and let Γ be a countable discrete subgroup of automorphisms of G, which acts freely on G, and with finite quotient G/Γ.

The graph G is called a *periodic graph*. Then the Ihara zeta function of a periodic simple graph is defined as follows:

$$\mathbf{Z}_{G,\Gamma}(u) = \prod_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} (1 - u^{|C|})^{-1/|\Gamma_{[C]}|},$$

where $\Gamma_{[C]}$ is the stabilizer of $[C]$ in Γ, and $[C]_\Gamma$ runs over all Γ-equivalence classes of prime cycles in G.

Guido, Isola and Lapidus [7] presented a determinant expression for the Ihara zeta function of a periodic simple graph.

Theorem 2 (Guido, Isola and Lapidus)**.** *For a periodic simple graph G,*

$$\mathbf{Z}_{G,\Gamma}(u) = (1 - u^2)^{-(m-n)} \det{}_\Gamma(\mathbf{I} - u\mathbf{A}(G) + (\mathbf{D} - \mathbf{I})u^2)^{-1},$$

where $\det{}_\Gamma$ *is a determinant for bounded operators belonging to a von Neumann algebra with a finite trace.*

Guido, Isola and Lapidus [8] presented a determinant expression for the Ihara zeta function of a periodic graph G and a countable discrete subgroup Γ of aoutomorphisms of G which acts discretely without inversions, and with bounded covolume.

Theorem 3 (Guido, Isola and Lapidus)**.** *For a periodic graph G,*

$$\mathbf{Z}_{G,\Gamma}(u)^{-1} = (1 - u^2)^{\chi^{(2)}(G)} \det{}_\Gamma(\Delta(u)),$$

where $\chi^{(2)}(G)$ is the L^2-Euler characteristic of (G, Γ) (see [2]), and $\Delta(u) = \mathbf{I} - u\mathbf{A} + u^2(\mathbf{D} - \mathbf{I})$.

3 The Grover walk on a graph

Let G be a connected graph with n vertices and m edges, $V(G) = \{v_1, \ldots, v_n\}$ and $R(G) = \{e_1, \ldots, e_m, e_1^{-1}, \ldots, e_m^{-1}\}$. Set $d_j = d_{v_j} = \deg v_j$ for $i = 1, \ldots, n$. The *Grover matrix* $\mathbf{U} = \mathbf{U}(G) = (U_{ef})_{e,f \in R(G)}$ of G is defined by

$$U_{ef} = \begin{cases} 2/d_{t(f)} (= 2/d_{o(e)}) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/d_{t(f)} - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The discrete-time quantum walk with the matrix $\mathbf{U}$ as a time evolution matrix is called the *Grover walk* on G.

Let G be a connected graph with n vertices and m edges. Then the $n \times n$ matrix $\mathbf{T}(G) = (T_{uv})_{u,v \in V(G)}$ is given as follows:

$$T_{uv} = \begin{cases} 1/(\deg{}_G u) & \text{if } (u, v) \in R(G), \\ 0 & \text{otherwise.} \end{cases}$$

Note that the matrix $\mathbf{T}(G)$ is the transition matrix of the simple random walk on G.

Theorem 4 (Konno and Sato)**.** *Let G be a connected graph with n vertices $v_1, \ldots, v_n$ and m edges. Then the characteristic polynomial for the Grover matrix* $\mathbf{U}$ *of G is given by*

$$\begin{aligned} \det(\lambda \mathbf{I}_{2m} - \mathbf{U}) &= (\lambda^2 - 1)^{m-n} \det((\lambda^2 + 1)\mathbf{I}_n - 2\lambda \mathbf{T}(G)) \\ &= \frac{(\lambda^2-1)^{m-n} \det((\lambda^2+1)\mathbf{D} - 2\lambda \mathbf{A}(G))}{d_{v_1} \cdots d_{v_n}}. \end{aligned}$$

From this Theorem, the spectra of the Grover matrix on a graph is obtained by means of those of $\mathbf{T}(G)$ (see [4]). Let $Spec(\mathbf{F})$ be the spectra of a square matrix $\mathbf{F}$.

Corollary 5 (Emms, Hancock, Severini and Wilson)**.** *Let G be a connected graph with n vertices and m edges. The Grover matrix* $\mathbf{U}$ *has $2n$ eigenvalues of the form*

$$\lambda = \lambda_T \pm i\sqrt{1 - \lambda_T^2},$$

where λ_T is an eigenvalue of the matrix $\mathbf{T}(G)$. The remaining $2(m-n)$ eigenvalues of $\mathbf{U}$ *are ± 1 with equal multiplicities.*

4 Spectra for the time evolution matrix of a general coined quantum walk on a graph

We consider a generalization of a coined quantum walk on a graph. We replace the coin operator $\mathbf{C}$ of a coined quantum walk with unitary matrix with two spectra which are distinct from ± 1.

For a given connected graph G with n vertices and m edges, let $\mathbf{d} : \ell^2(V(G)) \longrightarrow \ell^2(R(G))$ such that

$$\mathbf{d}\mathbf{d}^* = \mathbf{I}_q,$$

and let $\mathbf{S} = (S_{ef})_{e,f \in R(G)}$ be the $2m \times 2m$ matrix defined by

$$S_{ef} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let

$$\mathbf{C} = a\mathbf{d}^*\mathbf{d} + b(\mathbf{I}_{2m} - \mathbf{d}^*\mathbf{d})$$

and $\mathbf{U} = \mathbf{SC}$(see [11]). Note that $q = \dim\ ker(a - \mathbf{C})$. A discrete-time quantum walk on G with $\mathbf{U}$ as a time evolution matrix is called a *general coined quantum walk* on G. Then we define a zeta function of G by using $\mathbf{U}$ as follows:

$$\zeta(G, u) = \det(\mathbf{I}_{2m} - u\mathbf{U})^{-1} = \det(\mathbf{I}_{2m} - u\mathbf{S}(a\mathbf{d}^*\mathbf{d} + b(\mathbf{I}_{2m} - \mathbf{d}^*\mathbf{d})))^{-1}.$$

Now, we have the following result.

Theorem 6. *Let G be a connected graph n vertices and m edges, $\mathbf{U} = \mathbf{SC}$ the time evolution matrix of a general coined quantum walk on G. Suppose that $\sigma(\mathbf{C}) = \{a, b\}$. Set $q = \dim\ ker(a - \mathbf{C})$. Then, for the unitary matrix $\mathbf{U} = \mathbf{SC}$, we have*

$$\zeta(G,u) = (1 - b^2u^2)^{m-q} \det((1 - abu^2)\mathbf{I}_n - cu\mathbf{d}\mathbf{S}\mathbf{d}^*), c = a - b.$$

Proof. At first, we have

$$\begin{aligned}\zeta(G,u) &= \det(\mathbf{I}_{2m} - u\mathbf{U}) = \det(\mathbf{I}_{2m} - u\mathbf{SC}) \\ &= \det(\mathbf{I}_{2m} - u\mathbf{S}(a\mathbf{d}^*\mathbf{d} + b(\mathbf{I}_{2m} - \mathbf{d}^*\mathbf{d}))) \\ &= \det(\mathbf{I}_{2m} - u\mathbf{S}((a-b)\mathbf{d}^*\mathbf{d} + b\mathbf{I}_{2m}))) \\ &= \det(\mathbf{I}_{2m} - bu\mathbf{S} - cu\mathbf{S}\mathbf{d}^*\mathbf{d}) \\ &= \det(\mathbf{I}_{2m} - cu\mathbf{S}\mathbf{d}^*\mathbf{d}(\mathbf{I}_{2m} - bu\mathbf{S})^{-1}) \det(\mathbf{I}_{2m} - bu\mathbf{S}).\end{aligned}$$

But, if $\mathbf{A}$ and $\mathbf{B}$ are an $m \times n$ matrix and an $n \times m$ matrix, respectively, then we have

$$\det(\mathbf{I}_m - \mathbf{AB}) = \det(\mathbf{I}_n - \mathbf{BA}).$$

Thus, we have

$$\det(\mathbf{I}_{2m} - u\mathbf{U}) = \det(\mathbf{I}_{2m} - u\mathbf{SC}) = \det(\mathbf{I}_n - cu\mathbf{d}(\mathbf{I}_{2m} - bu\mathbf{S})^{-1}\mathbf{S}\mathbf{d}^*) \det(\mathbf{I}_{2m} - bu\mathbf{S}).$$

But, we have

$$\det(\mathbf{I}_{2m} - bu\mathbf{S}) = (1 - b^2u^2)^m.$$

Furthermore, we have

$$(\mathbf{I}_{2m} - bu\mathbf{S})^{-1} = \frac{1}{1 - b^2u^2}(\mathbf{I}_{2m} + u\mathbf{S}).$$

Therefore, it follows that

$$\begin{aligned}&\det(\mathbf{I}_{2m} - u\mathbf{U}) \\ &= (1 - b^2u^2)^m \det(\mathbf{I}_{2m} - \frac{cu}{1 - b^2u^2}\mathbf{S}\mathbf{d}^*\mathbf{d}(\mathbf{I}_{2m} + bu\mathbf{S})) \\ &= (1 - b^2u^2)^m \det(\mathbf{I}_q - \frac{cu}{1 - b^2u^2}\mathbf{d}(\mathbf{I}_{2m} + bu\mathbf{S})\mathbf{S}\mathbf{d}^*) \\ &= (1 - b^2u^2)^{m-n} \det((1 - b^2u^2)\mathbf{I}_q - cu\mathbf{d}\mathbf{S}\mathbf{d}^* - bcu^2\mathbf{d}\mathbf{S}^2\mathbf{d}^*) \\ &= (1 - b^2u^2)^{m-n} \det((1 - b^2u^2)\mathbf{I}_q - cu\mathbf{d}\mathbf{S}\mathbf{d}^* - bcu^2\mathbf{I}_n) \\ &= (1 - b^2u^2)^{m-n} \det((1 - abu^2)\mathbf{I}_q - cu\mathbf{d}\mathbf{S}\mathbf{d}^*).\end{aligned}$$

□

Corollary 7. *Let G be a connected with n vertices and m edges. Then, for the unitary matrix $\mathbf{U} = \mathbf{SC}$, we have*

$$\det(\lambda \mathbf{I}_{2m} - u\mathbf{U}) = (\lambda^2 - b^2)^{m-q} \det((\lambda^2 - ab)\mathbf{I}_q - c\lambda \mathbf{dSd}^*),$$

where $q = \dim \ ker(1 - \mathbf{C})$.

Proof. Let $u = 1/\lambda$. Then, by Theorem 6, we have

$$\det(\mathbf{I}_{2m} - 1/\lambda \mathbf{U}) = (1 - b^2/\lambda^2)^{m-q} \det((1 - ab/\lambda^2)\mathbf{I}_q - c/\lambda \mathbf{dSd}^*),$$

and so,

$$\det(\lambda \mathbf{I}_{2m} - \mathbf{U}) = (\lambda^2 - b^2)^{m-q} \det((\lambda^2 - ab)\mathbf{I}_q - c\lambda \mathbf{dSd}^*).$$

□

By Corollary 7, the following result holds.

Corollary 8. *Let G be a connected with n vertices and m edges. Then, the spectra of the unitary matrix $\mathbf{U} = \mathbf{SC}$ are given as follows:*

1. *$2q$ eigenvalues:*
$$\lambda = \frac{c\mu \pm \sqrt{c^2\mu^2 + 4ab}}{2}, \ \mu \in Spec(\mathbf{dSd}^*);$$
2. *The rest eigenvalues are $\pm b$ with the same multiplicity $m - q$.*

Proof. By Corollary 7, we have

$$\begin{aligned} & \det(\lambda \mathbf{I}_{2m} - \mathbf{U}) \\ = \ & (\lambda^2 - b^2)^{m-q} \prod_{\mu \in Spec(\mathbf{dSd}^*)} (\lambda^2 - c\mu\lambda - ab). \end{aligned}$$

Solving $\lambda^2 - 2\mu\lambda + 1 = 0$, we obtain

$$\lambda = \frac{c\mu \pm \sqrt{c^2\mu^2 + 4ab}}{2}.$$

The result follows. □

5 Periodic graphs

Let $G = (V(G), E(G))$ be a simple graph. Assume that G is countable ($V(G)$ and $E(G)$ are countable), and with bounded degree, i.e., $d = \sup_{v \in V(G)} \deg v < \infty$. Let Γ be a countable discrete subgroup of automorphisms of G, which acts

1. without inversions: $\gamma(e) \neq e^{-1}$ for any $\gamma \in \Gamma, e \in R(G)$,
2. discretely: $\Gamma_v = \{\gamma \in \Gamma \mid \gamma v = v\}$ is finite for any $v \in V(G)$,

3. with bounded covolume: $\mathrm{vol}(G/\Gamma) := \sum_{v\in\mathcal{F}_0} \frac{1}{|\Gamma_v|} < \infty$, where $\mathcal{F}_0 \subset V(G)$ contains exactly one representative for each equivalence class in $V(G/\Gamma)$.

Then G is called a *periodic graph* with a countable discrete subgroup Γ of $Aut\ G$. Note that the third condition is equivalent to the following condition:

$$\mathrm{vol}(R(G)/\Gamma) := \sum_{e\in\mathcal{F}_1} \frac{1}{\mid \Gamma_e \mid} < \infty,$$

where a subset $\mathcal{F}_1$ of $R(G)$ contains exactly one representative for each equivalence class in $R(G/\Gamma)$.

Let $\ell^2(V(G))$ be the Hilbert space of functions $f : V(G) \longrightarrow \mathbb{C}$ such that $\| f \| := \sum_{v\in V(G)} \mid f(v) \mid^2 < \infty$. We define the left regular representation λ_0 of Γ on $\ell^2(V(G))$ as follows:

$$(\lambda_0(\gamma)f)(x) = f(\gamma^{-1}x),\ \gamma \in \Gamma,\ f \in \ell^2(V(G)),\ x \in V(G).$$

We state the definition of a von Neumann algebra. Let H be a separable complex Hilbert space, and let $\mathcal{B}(H)$ denote the $\mathbf{C}^*$-algebra of bounded linear operators on H. For a subset $M \subset \mathcal{B}(H)$, the *commutant* of M is $M' = \{T \in \mathcal{B}(H) \mid ST = TS, \forall S \in M\}$. Then a *von Neumann algebra* is a subalgebra $\mathcal{A} \leqslant \mathcal{B}(H)$ such that $\mathcal{A}'' = \mathcal{A}$. It is known that a determinant is defined for a suitable class of operators in a von Neumann algebra with a finite trace (see [5, 7]).

For the Hilbert space $\ell^2(V(G))$, we consider a von Neumann algebra. Let $\mathcal{B}(\ell^2(V(G)))$ be the $\mathbb{C}^*$-algebra of bounded linear operators on $\ell^2(V(G))$. A bounded linear operator A of $\mathcal{B}(\ell^2(V(G)))$ acts on $\ell^2(V(G))$ by

$$A(f)(v) = \sum_{w\in V(G)} A(v,w)f(w),\ v \in V(G),\ f \in \ell^2(V(G)).$$

Then the von Neumann algebra $\mathcal{N}_0(G,\Gamma)$ of bounded operators on $\ell^2(V(G))$ commuting with the action of Γ is defined as follows:

$$\mathcal{N}_0(G,\Gamma) = \{\lambda_0(\gamma) \mid \gamma \in \Gamma\}' = \{T \in \mathcal{B}(\ell^2(V(G))) \mid \lambda_0(\gamma)T = T\lambda_0(\gamma), \forall \gamma \in \Gamma\}.$$

The von Neumann algebra $\mathcal{N}_0(G,\Gamma)$ inherits a trace by

$$\mathrm{Tr}_\Gamma(A) = \sum_{x\in\mathcal{F}_0} \frac{1}{\mid \Gamma_x \mid} A(x,x),\ A \in \mathcal{N}_0(G,\Gamma).$$

Let the adjacency matrix $\mathbf{A} = \mathbf{A}(G)$ of G be defined by

$$(\mathbf{A}f)(v) = \sum_{(v,w)\in R(G)} f(w),\ f \in \ell^2(V(G)).$$

By [14, 15], we have

$$\| \mathbf{A} \| \leqslant d = \sup_{v\in V(G)} \deg_G v < \infty,$$

and so $\mathbf{A} \in \mathcal{N}_0(G,\Gamma)$.

Similarly to $\ell^2(V(G))$, we consider the Hilbert space $\ell^2(R(G))$ of functions $f : R(G) \to \mathbb{C}$ such that $\| \omega \| := \sum_{e \in R(G)} \mid \omega(e) \mid {}^2 < \infty$. We define the left regular representation λ_1 of Γ on $\ell^2(R(G))$ as follows:

$$(\lambda_1(\gamma)\omega)(e) = \omega(\gamma^{-1}e),\ \gamma \in \Gamma,\ \omega \in \ell^2(R(G)),\ e \in R(G).$$

Then the von Neumann algebra $\mathcal{N}_1(G,\Gamma) = \{\lambda_1(\gamma) \mid \gamma \in \Gamma\}'$ of bounded operators on $\ell^2(R(G))$ commuting with the action of Γ, inherits a trace by

$$\mathrm{Tr}_\Gamma(A) = \sum_{e \in \mathcal{F}_1} \frac{1}{\mid \Gamma_e \mid} A(e,e),\ A \in \mathcal{N}_1(G,\Gamma).$$

6 An analytic determinant for von Neumann algebras with a finite trace

In an excellent paper [5], Fuglede and Kadison defined a positive-valued determinant for a von Neumann algebra with trivial center and finite trace τ. For an invertible operator A with polar decomposition $A = UH$, the Fuglede-Kadison determinant of A is defined by

$$Det(A) = \exp \circ \tau \circ \log H,$$

where $\log H$ may be defined via functional calculus.

Guido, Isola and Lapidus [7] extended the Fuglede-Kadison determinant to a determinant which is an analytic function. Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace τ. Then, for $A \in \mathcal{A}$, let

$$\det{}_\tau(A) = \exp \circ \tau \circ \log A,$$

where

$$\log(A) := \frac{1}{2\pi i} \int_\Lambda \log \lambda (\lambda - A)^{-1} d\lambda,$$

and Λ is the boundary of a connected, simply connected region Ω containing the spectrum $\sigma(A)$ of A. Then the following lemma holds (see Lemma 5.1 of [7]).

Lemma 9 (Guido, Isola and Lapidus)**.** *Let $\mathcal{A}, \Omega, \Gamma$ be as above, and ϕ, ψ two branches of the logarithm such that both domains contain Ω. Then*

$$\exp \circ \tau \circ \phi(A) = \exp \circ \tau \circ \psi(A).$$

Next, we consider a determinant on some subset of $\mathcal{A}$. Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace, and $\mathcal{A}_0 = \{A \in \mathcal{A} \mid 0 \notin \text{conv } \sigma(A)\}$, where conv $\sigma(A)$ is the convex hull of $\sigma(A)$. For any $A \in \mathcal{A}_0$, we set

$$\det{}_\tau(A) = \exp \circ \tau \circ (\frac{1}{2\pi i} \int_\Lambda \log \lambda (\lambda - A)^{-1} d\lambda),$$

where Λ is the boundary of a connected, simply connected region Ω containing the spectrum conv $\sigma(A)$, and log is a branch of the logarithm whose domain contains Ω. Then the above determinant is well-defined and analytic on $\mathcal{A}_0$ (see Corollary 5.3 of [7]). Furthermore, Guido, Isola and Lapidus of [7, 8] showed that $\det_\tau$ has the following properties.

Proposition 10 (Guido, Isola and Lapidus)**.** *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace, $A \in \mathcal{A}_0$. Then*

1. $\det_\tau(zA) = z^{\tau(I)} \det_\tau(A)$ *for any* $z \in \mathbb{C} \setminus \{0\}$.
2. *If A is normal, and $A = UH$ is its polar decomposition,*
$$\det_\tau(A) = \det_\tau(U) \det_\tau(H).$$
3. *If A is positive, $\det_\tau(A) = Det(A)$, where $Det(A)$ is the Fuglede-Kadison determinant of A.*

Proposition 11 (Guido, Isola and Lapidus)**.** *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra with a finite trace. Then*

1. *For $A, B \in \mathcal{A}$ and sufficiently small $u \in \mathbb{C}$,*
$$\det_\tau((I + uA)(I + uB)) = \det_\tau(I + uA) \det_\tau(I + uB).$$
2. *If $A \in \mathcal{A}$ has a bounded inverse, and $T \in \mathcal{A}_0$, then*
$$\det_\tau(ATA^{-1}) = \det_\tau(T).$$
3. *If*
$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{A}),$$
with $T_{ii} \in \mathcal{A}$ such that $\sigma(T_{ii}) \subset B_1(1) := \{z \in \mathbb{C} \mid \mid z - 1 \mid < 1\}$ for $i = 1, 2$, then
$$\det_\tau(T) = \det_\tau(T_{11}) \det_\tau(T_{22}).$$

Corollary 12 (Guido, Isola and Lapidus)**.** *Let Γ be a discrete group, π_1, π_2 unitary representations of Γ, τ_1, τ_2 finite traces on $\pi_1(\Gamma)'$ and $\pi_2(\Gamma)'$, respectively. Let $\pi = \pi_1 \oplus \pi_2$, $\tau = \tau_1 + \tau_2$ and $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \in \pi(\Gamma)'$, with $\sigma(T_{ii}) \subset B_1(1) := \{z \in \mathbf{C} \mid \mid z - 1 \mid < 1\}$ for $i = 1, 2$, then*
$$\det_\tau(T) = \det_{\tau_1}(T_{11}) \det_{\tau_2}(T_{22}).$$

7 A zeta function with respect to a general coined quantum walk of an infinite periodic graph

We define a zeta function with respect to a general coined quantum walk of an infinite periodic graph.

Let G be a periodic graph with a countable discrete subgroup Γ of $Aut\ G$. Moreover, let

$$\mathbf{I}_V = Id_{\ell^2(V(G))}, \mathbf{I}_R = Id_{\ell^2(R(G))}.$$

Then, let $\mathbf{d} : \ell^2(V(G)) \longrightarrow \ell^2(R(G))$ such that

$$\mathbf{d}\mathbf{d}^* = \mathbf{I}_V.$$

Furthermore, let

$$\mathbf{C} = a\mathbf{d}^*\mathbf{d} + b(\mathbf{I}_R - \mathbf{d}^*\mathbf{d})$$

and $\mathbf{U} = \mathbf{S}\mathbf{C}$, where $\mathbf{S}$ is the operator on $\ell^2(R(G))$ such that

$$(\mathbf{S}\omega)(e) = \omega(e^{-1}),\ \omega \in \ell^2(R(G)),\ e \in R(G).$$

Now, we consider the following determinant:

$$\det{}_\Gamma(B) = \exp \circ \mathrm{Tr}_\Gamma \circ \log B$$

for $B \in \mathcal{N}_1(G, \Gamma)_0$. Then a zeta function with respect to a general coined quantum walk of G is defined as follows:

$$\zeta(G, \Gamma, u) = \det{}_\Gamma(\mathbf{I}_R - u\mathbf{U})^{-1} = \det{}_\Gamma(\mathbf{I}_R - u\mathbf{S}(a\mathbf{d}^*\mathbf{d} + b(\mathbf{I}_R - \mathbf{d}^*\mathbf{d})))^{-1},$$

where $u \in \mathbb{C}$ are sufficiently small so that the infinite product converges.

Then we have the following result.

Theorem 13. *Let G be a periodic graph with a countable discrete subgroup Γ of Aut G. Then*

$$\zeta(G, \Gamma, u) = (1 - b^2u^2)^{\mathrm{Tr}_\Gamma(\mathbf{I}_V) - \frac{1}{2}\mathrm{Tr}_\Gamma(\mathbf{I}_R)} \det{}_\Gamma((1 - abu^2)\mathbf{I}_V - cu\mathbf{d}\mathbf{S}\mathbf{d}^*),$$

where $\mathrm{Tr}_\Gamma(\mathbf{I}_R) = \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|}$ *and* $\mathrm{Tr}_\Gamma(\mathbf{I}_V) = \sum_{v \in \mathcal{F}_0} \frac{1}{|\Gamma_v|}$ *(see [2]).*

Proof. The argument is an analogue of the method of Bass [1].

Let G be a periodic graph with a countable discrete subgroup Γ of $Aut\ G$.

Now we consider the direct sum of the unitary representations λ_0 and λ_1: $\lambda(\gamma) := \lambda_0(\gamma) \oplus \lambda_1(\gamma) \in \mathcal{B}(\ell^2(V(G)) \oplus \ell^2(R(G)))$. Then the von Neumann algebra $\lambda(\Gamma)' := \{S \in \mathcal{B}(\ell^2(V(G)) \oplus \ell^2(R(G))) \mid S\lambda(\gamma) = \lambda(\gamma)S, \gamma \in \Gamma\}$ consists of operators

$$S = \begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix},$$

where $S_{ij}\lambda_j(\gamma) = \lambda_i(\gamma)S_{ij}, \gamma \in \Gamma, i, j = 0, 1$, so that $S_{ii} \in \Lambda_i \equiv \mathcal{N}_i(G, \Gamma), i = 0, 1$. Thus, $\lambda(\Gamma)'$ inherits a trace given by

$$\mathrm{Tr}_\Gamma \begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix} := \mathrm{Tr}_\Gamma(S_{00}) + \mathrm{Tr}_\Gamma(S_{11}).$$

We introduce two operators as follows:

$$\mathbf{L} = \begin{bmatrix} (1-b^2u^2)\mathbf{I}_V & -c\mathbf{d} - bcu\mathbf{d}\mathbf{S} \\ 0 & \mathbf{I}_R \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \mathbf{I}_V & c\mathbf{d} + bcu\mathbf{d}\mathbf{S} \\ u\mathbf{S}\mathbf{d}^* & (1-b^2u^2)\mathbf{I}_R \end{bmatrix},$$

where $c = a - b$. Then we have

$$\begin{aligned} \mathbf{LM} &= \begin{bmatrix} (1-b^2u^2)\mathbf{I}_V - cu\mathbf{d}\mathbf{S}d^* - bcu^2\mathbf{d}\mathbf{S}^2\mathbf{d}^* & 0 \\ u\mathbf{S}\mathbf{d}^* & (1-b^2u^2)\mathbf{I}_R \end{bmatrix} \\ &= \begin{bmatrix} (1-abu^2)\mathbf{I}_V - cu\mathbf{d}\mathbf{S}\mathbf{d}^* & 0 \\ u\mathbf{S}\mathbf{d}^* & (1-b^2u^2)\mathbf{I}_R \end{bmatrix}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \mathbf{ML} &= \begin{bmatrix} (1-b^2u^2)\mathbf{I}_V & 0 \\ u(1-b^2u^2)\mathbf{S}\mathbf{d}^* & -cu\mathbf{S}\mathbf{d}^*\mathbf{d} - bcu^2\mathbf{S}\mathbf{d}^*\mathbf{d}\mathbf{S} + (1-b^2u^2)\mathbf{I}_R \end{bmatrix} \\ &= \begin{bmatrix} (1-b^2u^2)\mathbf{I}_V & 0 \\ u(1-b^2u^2)\mathbf{S}\mathbf{d}^* & (\mathbf{I}_R - u(c\mathbf{S}\mathbf{d}^*\mathbf{d} + b\mathbf{S}))(\mathbf{I}_R + ub\mathbf{S}) \end{bmatrix}. \end{aligned}$$

Here, note that $\mathbf{S}^2 = \mathbf{I}_R$.

For $\mid t \mid, \mid u \mid$ sufficiently small, we have

$$\sigma(\Delta(u)), \sigma((1-b^2t^2)\mathbf{I}_V), \sigma((1-b^2t^2)\mathbf{I}_R), \sigma((\mathbf{I}_R - u(c\mathbf{S}\mathbf{d}^*\mathbf{d} + b\mathbf{S}))(\mathbf{I}_R + ub\mathbf{S}))$$

$$\in B_1(1) = \{z \in \mathbf{C} \mid \mid z - 1 \mid < 1\}.$$

Similar to the proof of Proposition 3.8 in [8], $\sigma(\mathbf{LM})$ and $\sigma(\mathbf{ML})$ are contained in $B_1(1)$. Thus, $\mathbf{L}$ and $\mathbf{M}$ are invertible, with bounded inverse, for $\mid t \mid, \mid u \mid$ sufficiently small.

By 1 of Proposition 10, 1 of Proposition 11 and Corollary 12, we have

$$\begin{aligned} \det{}_\Gamma(\mathbf{LM}) &= \det{}_\Gamma((1-b^2u^2)\mathbf{I}_V - cu\mathbf{d}\mathbf{S}\mathbf{d}^* - bcu^2\mathbf{d}\mathbf{S}^2\mathbf{d}^*)\det{}_\Gamma((1-b^2u^2)\mathbf{I}_R) \\ &= (1-b^2u^2)^{\mathrm{Tr}_\Gamma(\mathbf{I}_R)}\det{}_\Gamma((1-abu^2)\mathbf{I}_V - cu\mathbf{d}\mathbf{S}\mathbf{d}^*) \end{aligned}$$

and

$$\begin{aligned} \det{}_\Gamma(\mathbf{ML}) &= \det{}_\Gamma((1-b^2u^2)\mathbf{I}_V)\det{}_\Gamma(\mathbf{I}_R - u(c\mathbf{S}\mathbf{d}^*\mathbf{d} + b\mathbf{S}))\det{}_\Gamma(\mathbf{I}_R + ub\mathbf{S}) \\ &= (1-b^2u^2)^{\mathrm{Tr}_\Gamma(\mathbf{I}_V)}\det{}_\Gamma(\mathbf{I}_R - u(c\mathbf{S}\mathbf{d}^*\mathbf{d} + b\mathbf{S}))\det{}_\Gamma(\mathbf{I}_R + ub\mathbf{S}). \end{aligned}$$

Let an orientation of G be a choice of one oriented edge for each pair of edges in $R(G)$, which is called positively oriented. We denote by E^+G the set of positively oriented edges. Moreover, let $E^-G := \{e^{-1} \mid e \in E^+G\}$. An element of E^-G is called a negatively oriented. Note that $R(G) = E^+G \cup E^-G$.

The operator $\mathbf{S}$ maps $\ell^2(E^+G)$ to $\ell^2(E^-G)$. Then we obtain a representation ρ of $\mathcal{B}(\ell^2(R(G)))$ onto $Mat_2\mathcal{B}(\ell^2(E^+G))$, under

$$\rho(\mathbf{S}) = \begin{bmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix}, \rho(\mathbf{I}_R) = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}.$$

By 1 and 3 of Proposition 11,

$$\begin{aligned} \det{}_\Gamma(\mathbf{I}_R + bu\mathbf{S}) &= \det{}_\Gamma \begin{bmatrix} \mathbf{I} & -bu\mathbf{I} \\ 0 & \mathbf{I} \end{bmatrix} \det{}_\Gamma \begin{bmatrix} \mathbf{I} & bu\mathbf{I} \\ bu\mathbf{I} & \mathbf{I} \end{bmatrix} \\ &= \det{}_\Gamma \begin{bmatrix} (1-b^2u^2)\mathbf{I} & 0 \\ * & \mathbf{I} \end{bmatrix} = (1-b^2u^2)^{\frac{1}{2}\mathrm{Tr}_\Gamma(\mathbf{I}_R)}. \end{aligned}$$

For $\mid t \mid, \mid u \mid$ sufficiently small, we have

$$\mathbf{ML} = \mathbf{MLMM}^{-1},$$

and so, by 2 of Proposition 11,

$$\det{}_\Gamma(\mathbf{LM}) = \det{}_\Gamma(\mathbf{ML}).$$

Therefore, it follows that

$$(1-b^2u^2)^{\mathrm{Tr}_\Gamma(\mathbf{I}_R)} \det{}_\Gamma((1-abu^2)\mathbf{I}_V - cu\mathbf{dSd}^*)$$

$$= (1-b^2u^2)^{\frac{1}{2}\mathrm{Tr}_\Gamma(\mathbf{I}_R)+\mathrm{Tr}_\Gamma(\mathbf{I}_V)} \det{}_\Gamma(\mathbf{I}_R - u\mathbf{S}(c\mathbf{d}^*\mathbf{d} + b\mathbf{I}_R)),$$

and so

$$\det{}_\Gamma(\mathbf{I}_R - u\mathbf{SC}) = \det{}_\Gamma(\mathbf{I}_R - u\mathbf{S}(c\mathbf{d}^*\mathbf{d} + b\mathbf{I}_R))$$

$$= (1-b^2u^2)^{\frac{1}{2}\mathrm{Tr}_\Gamma(\mathbf{I}_R)-\mathrm{Tr}_\Gamma(\mathbf{I}_V)} \det{}_\Gamma((1-abu^2)\mathbf{I}_V - cu\mathbf{dSd}^*).$$

Hence the result follows by the definition of Tr_Γ. □

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References

[1] H. Bass, The Ihara-Selberg zeta function of a tree lattice, *Internat. J. Math.*, 3: 717–797, 1992.

[2] J. Cheeger and M. Gromov, L_2-cohomology and group cohomology, *Topology*, 25: 189–215, 1986.

[3] B. Clair and S. Mokhtari-Sharghi, Zeta functions of discrete groups acting on trees, *J. Algebra*, 237: 591–620, 2001.

[4] D.Emms, E. R. Hancock, S. Severini andR. C. Wilson, A matrix representation of graphs and its spectrum as a graph invariant, *Electron. J. Combin.*, 13: #R34, 2006.

[5] B. Fuglede and R. Kadison, Determinant theory in infinite factors, *Ann. Math.*, 55: 520–530, 1952.

[6] R. I. Grigorchuk and A. Żuk, The Ihara zeta function of infinite graphs, the KNS spectral measure and integrable maps, In *Random Walk and Geometry*, Proc. Workshop (Vienna, 2001), V. A. Kaimanovich et at., eds., pages 141–180. de Gruyter, Berkin, 2004.

[7] D. Guido, T. Isola and M. L. Lapidus, Ihara zeta functions for periodic simple graphs, In *C*-algebras and elliptic theory II*, Edited by D.Burghelea, R.Melrose, A.Mishchenko, E.Troitsky. *Trends in Mathematics*, pages 103–121. Birkhauser Verlag, Basel, 2008.

[8] D. Guido, T. Isola and M. L. Lapidus, Ihara's zeta function for periodic graphs and its approximation in the amenable case, *Journal of Functional Analysis*, 255: 1339–1361, 2008.

[9] D. Guido, T. Isola and M. L. Lapidus, Bartholdi zeta function for periodic simple graphs, In *Analysis on Graphs and its Applications*, *Proc. Sympos. Pure Math.*, 77: 109–122, 2008.

[10] K. Hashimoto, Zeta Functions of Finite Graphs and Representations of p-Adic Groups, *Adv. Stud. Pure Math.* Vol. 15, pages 211–280, Academic Press, New York, 1989.

[11] Yu. Ide, N. Konno, I. Sato, E. Segawa and Y. Shikano, The spectra of the unitary matrix of a general coined quantum walk on a graph, preprint.

[12] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, *J. Math. Soc. Japan*, 18: 219–235, 1966.

[13] N. Konno and I. Sato, On the relation between quantum walks and zeta functions, *Quantum Inf. Process.*, 11: 341–349, 2012.

[14] B. Mohar, The spectrum of an infinite graph, *Linear Algebra Appl.*, 48: 245–256, 1982.

[15] B. Mohar and W. Woess, A survey on spectra of infinite graphs, *Bull. London Math. Soc.*, 21: 209–234, 1989.

[16] P. Ren, T. Aleksic, D. Emms, R. C. Wilson and E. R. Hancock, Quantum walks, Ihara zeta functions and cospectrality in regular graphs, *Quantum Inf. Process.*, 10: 405–417, 2011.

[17] J. -P. Serre, *Trees,* Springer-Verlag, New York, 1980.

[18] T. Sunada, L-Functions in Geometry and Some Applications, In *Lecture Notes in Math.*, Vol. 1201, pages 266–284. Springer-Verlag, New York, 1986.

[19] T. Sunada, Fundamental Groups and Laplacians (in Japanese), Kinokuniya, Tokyo, 1988.