The Maximum Number of Cliques in Hypergraphs without Large Matchings

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Abstract

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $\mathcal{F}^{(r)}_{n,k,a}$ be an r-uniform hypergraph on the vertex set $[n]$ with edge set consisting of all the r-element subsets of $[n]$ that contains at least a vertices in $[ak + a - 1]$. For $n \geqslant 2rk$, Frankl proved that $\mathcal{F}^{(r)}_{n,k,1}$ maximizes the number of edges in r-uniform hypergraphs on n vertices with the matching number at most k. Huang, Loh and Sudakov considered a multicolored version of the Erdős matching conjecture, and provided a sufficient condition on the number of edges for a multicolored hypergraph to contain a rainbow matching of size k. In this paper, we show that $\mathcal{F}^{(r)}_{n,k,a}$ maximizes the number of s-cliques in r-uniform hypergraphs on n vertices with the matching number at most k for sufficiently large n, where $a = \lfloor \frac{s-r}{k} \rfloor + 1$. We also obtain a condition on the number of s-clques for a multicolored r-uniform hypergraph to contain a rainbow matching of size k, which reduces to the condition of Huang, Loh and Sudakov when $s = r$.

Mathematics Subject Classifications: 05C15, 05C65, 05C69

1 Introduction

An *r-graph* (or an *r-uniform hypergraph*) is a pair $\mathcal{H} = (V, E)$, where $V = V(\mathcal{H})$ is a finite set of vertices, and $E = E(\mathcal{H}) \subset \binom{V}{r}$ is a family of r-element subsets of V. We often identify $E(\mathcal{H})$ with $\mathcal{H}$. For any $S \subset V(\mathcal{H})$, let $\mathcal{H}[S]$ be the subhypergraph of $\mathcal{H}$ induced by S and let $\mathcal{H} - S$ denote the subhypergraph of $\mathcal{H}$ induced by $V(\mathcal{H}) \setminus S$. For any $S \subset V(\mathcal{H})$ with $|S| < r$, let

$$N_{\mathcal{H}}(S) = \left\{ T \in \binom{V(\mathcal{H})}{r - |S|} : S \cup T \in \mathcal{H} \right\}$$

and $\deg_{\mathcal{H}}(S) = |N_{\mathcal{H}}(S)|$. We call the elements in $N_{\mathcal{H}}(S)$ the neighbors of S in $\mathcal{H}$ and call $\deg_{\mathcal{H}}(S)$ the degree of S in $\mathcal{H}$. For $S = \{v\}$, we often use $H - v$, $N_{\mathcal{H}}(v)$ and $\deg_{\mathcal{H}}(v)$ instead of $\mathcal{H} - \{v\}$, $N_{\mathcal{H}}(\{v\})$ and $\deg_{\mathcal{H}}(\{v\})$, respectively. For any $s \geqslant r$, an s-clique of $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$ on s vertices in which every subset of r vertices is an edge of $\mathcal{H}$. Let $\mathcal{K}_s^r(\mathcal{H})$ denote the family of all the s-cliques of $\mathcal{H}$ and let $K_s^r(\mathcal{H})$ be the cardinality of $\mathcal{K}_s^r(\mathcal{H})$. For any $u \in V(\mathcal{H})$, we use $K_s^r(u, \mathcal{H})$ to denote the number of s-cliques in $\mathcal{H}$ containing u. A *matching* in $\mathcal{H}$ is a collection of pairwise disjoint edges of $\mathcal{H}$. The matching number of $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the size of a maximum matching in $\mathcal{H}$.

Definition 1. Let n, k, r, a be positive integers with $n \geqslant r \geqslant a$. Define

$$\mathcal{F}_{n,k,a}^{(r)} = \left\{ F \in \binom{[n]}{r} : |F \cap [ak + a - 1]| \geqslant a \right\}.$$

Clearly, we have $\nu(\mathcal{F}_{n,k,a}^{(r)}) \leqslant k$. Otherwise, we may assume that $\{E_1, E_2, \ldots, E_{k+1}\}$ is a matching of size $k + 1$ in $\mathcal{F}_{n,k,a}^{(r)}$, then we have

$$|[ak + a - 1]| \geqslant \sum_{i=1}^{k+1} |[ak + a - 1] \cap E_i| \geqslant (k + 1)a,$$

a contradiction.

In 1965, Erdős [3] proposed the following conjecture.

Conjecture 2 (The Erdős matching conjecture [3])**.** Let $\mathcal{H}$ be an r-graph on n vertices with $\nu(\mathcal{H}) \leqslant k$. Then

$$|\mathcal{H}| \leqslant \max \left\{ |\mathcal{F}_{n,k,1}^{(r)}|, |\mathcal{F}_{n,k,r}^{(r)}| \right\}.$$

In 2013, Frankl proved that Conjecture 2 holds for $n \geqslant (2k + 1)r - k$.

Theorem 3 (Frankl [6])**.** *Let $\mathcal{H}$ be an r-graph on n vertices with $\nu(\mathcal{H}) \leqslant k$. If $n \geqslant (2k + 1)r - k$, then $|\mathcal{H}| \leqslant |\mathcal{F}_{n,k,1}^{(r)}|$.*

For recent results on Conjecture 2, we refer the reader to [6, 7, 8]. For ordinary graphs, Alon and Shikhelman [1] introduced a generalization of the usual Turán problem, which is often called the generalized Turán problem. Given two graphs T and H, the *generalized Turán number*, denoted by $ex(n, T, H)$, is defined to be the maximum number of copies of T in an H-free graph on n vertices. The first result in this direction is due to Zykov [19] and independently to Erdős [2], who determined $ex(n, K_s, K_t)$. The second author [18] determined $ex(n, K_s, M_{k+1})$, where M_{k+1} is a matching of size $k + 1$. Recently, the study of the generalized Turán problem has received much attention, see [1, 10, 11, 12, 13, 15, 16, 17].

Motivated by the Erdős matching conjecture and the generalized Turán problem, we determine the maximum number of s-cliques in an r-graph on n vertices with matching number at most k.

Theorem 4. *Let n, k, r, s be integers and $\mathcal{H}$ be an r-graph on n vertices with $\nu(\mathcal{H}) \leqslant k$.*

(I) If $r \leqslant s \leqslant k+r-1$ and $n \geqslant 4(er)^{s-r+2}k$, then $K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,k,1}^{(r)})$;

(II) If $k+r \leqslant s \leqslant (r-1)(k+1)$ and $n \geqslant 4r^2k(er/(a-1))^{s-r+a}$, then $K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,k,a}^{(r)})$, where $a = \lfloor \frac{s-r}{k} \rfloor + 1$;

(III) If $(r-1)k+r \leqslant s \leqslant rk+r-1$ and $n \geqslant rk+r-1$, then $K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,k,r}^{(r)})$. Moreover, if $K_s^r(\mathcal{H}) < K_s^r(\mathcal{F}_{n,k,r}^{(r)})$, then $K_s^r(\mathcal{H}) \leqslant \binom{rk+r-1}{s} - \binom{rk-1}{s-r}$.

Based on the construction $\mathcal{F}_{n,k,a}^{(r)}$, one sees that the upper bounds in Theorem 4 are tight. Let k, r, s be integers with $r \leqslant s \leqslant (r-1)(k+1)$, $a = \lfloor \frac{s-r}{k} \rfloor + 1$ and

$$n^*(k,r,s) = \left(\frac{r}{a}\right)^{\frac{s-r+a}{r-a}} \left(\frac{rk+r-1-s}{s}\right).$$

If $n \leqslant n^*(k,r,s)$, we have

$$\begin{aligned}
K_s^r(\mathcal{F}_{n,k,a}^{(r)}) &\leqslant \binom{ak+a-1}{s-r+a}\binom{n-s+r-a}{r-a} \\
&\leqslant \left(\frac{a}{r}\right)^{s-r+a} \binom{rk+r-1}{s-r+a} \frac{n^{r-a}}{(r-a)!} \\
&< \binom{rk+r-1}{s-r+a} \left(\frac{rk+r-s-1}{s}\right)^{r-a} \\
&\leqslant \binom{rk+r-1}{s} = K_s^r(\mathcal{F}_{n,k,r}^{(r)}).
\end{aligned}$$

Note that $\mathcal{F}_{n,k,r}^{(r)}$ is an r-graph on n vertices with matching number at most k. Since $K_s^r(\mathcal{F}_{n,k,r}^{(r)}) > K_s^r(\mathcal{F}_{n,k,a}^{(r)})$ for $n \leqslant n^*(k,r,s)$, (I) and (II) in Theorem 4 hold if and only if $n \geqslant n_0(k,r,s)$ for some integer $n_0(k,r,s) > n^*(k,r,s)$.

Huang, Loh and Sudakov [14] considered a multi-colored generalization of the Erdős matching conjecture and they proved the following theorem.

Theorem 5 (Huang, Loh and Sudakov [14])**.** *Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be r-graphs on the vertex set $[n]$, where $k \leqslant \frac{n}{3r^2}$, and for any i, $|\mathcal{F}_i| > |\mathcal{F}_{n,k-1,1}^{(r)}|$. Then there exist pairwise disjoint edges $F_1 \in \mathcal{F}_1, \dots, F_k \in \mathcal{F}_k$.*

In this paper, we generalize their result by loosing the conditions on $\mathcal{F}_i$.

Theorem 6. *Let n, k, r, t be integers such that $r \leqslant t \leqslant k+r-2$ and $n \geqslant 4k(t-r+2)(er)^{t-r+2}$. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ be r-graphs on the vertex set V of size n. If for any $i \in \{1, 2, \dots, k\}$, there exists some $s \in \{r, r+1, \dots, t\}$ such that $K_s^r(\mathcal{F}_i) > K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$. Then there exist pairwise disjoint edges $F_1 \in \mathcal{F}_1, \dots, F_k \in \mathcal{F}_k$.*

To prove the above theorem, we need some estimates on the binomial coefficients, which are listed below. Let a, b and c be integers satisfying $a \geqslant b \geqslant c \geqslant 0$. Then the following inequalities hold:

$$\binom{a}{b} \leqslant \left(\frac{ea}{b}\right)^b, \tag{1}$$

$$\binom{b}{c} \leqslant \left(\frac{b}{a}\right)^c \binom{a}{c}, \tag{2}$$

$$\binom{a}{c} \leqslant \left(\frac{a-c}{b-c}\right)^c \binom{b}{c}, \tag{3}$$

$$\binom{a}{c} \leqslant \left(\frac{ea}{b}\right)^c \binom{b}{c}. \tag{4}$$

When b is close to c, the inequality (4) gives a better upper bound on $\binom{a}{c}$ than the inequality (3). Let p be a positive integer and $x \in (0, \frac{1}{p}]$. Then we have

$$(1+x)^p \leqslant 1 + p^2 x. \tag{5}$$

By the definition of $\mathcal{F}^{(r)}_{n,k,1}$ we have

$$K^r_s(\mathcal{F}^{(r)}_{n,k,1}) = \sum_{j=s-r+1}^{s} \binom{k}{j}\binom{n-k}{s-j}.$$

It is easy to check that

$$K^r_s(\mathcal{F}^{(r)}_{n-1,k-1,1}) + K^r_{s-1}(\mathcal{F}^{(r)}_{n-1,k-1,1}) = K^r_s(\mathcal{F}^{(r)}_{n,k,1}). \tag{6}$$

In Section 2, we prove (I) of Theorem 4. The proofs of (II) and (III) of Theorem 4 will be given in Section 3. Theorem 6 will be proved in Section 4.

2 The maximum number of s-cliques with $s \leqslant k + r - 1$

In this section, we determine the maximum number of s-cliques in an r-graph $\mathcal{H}$ with $\nu(\mathcal{H}) \leqslant k$ when $s \leqslant k + r - 1$. We need the following result due to Huang, Loh and Sudakov [14].

Lemma 7 (Huang, Loh and Sudakov [14]). *Let n, k, r be integers such that $rk \leqslant n$ and $\mathcal{H}$ be an r-graph on n vertices. If $\mathcal{H}$ has k distinct vertices $v_1, v_2, \ldots, v_k$ with* $\deg_{\mathcal{H}}(v_i) > 2(k-1)\binom{n-2}{r-2}$, *then $\mathcal{H}$ contains a matching of size k.*

Theorem 4 (I) will be proved by induction. The following lemma is the basis of the induction.

Lemma 8. *Let r, s be positive integers such that $r \leqslant s$ and $\mathcal{H}$ be an r-graph on n vertices with $\nu(\mathcal{H}) \leqslant s-r+1$. For $n \geqslant 4(s-r+1)(er)^{s-r+2}$, we have $K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,s-r+1,1}^{(r)})$.*

Proof. Let $\mathcal{M} = \{E_1, E_2, \dots, E_p\}$ be a maximum matching in $\mathcal{H}$ and S be the set of vertices that are covered by $\mathcal{M}$. Since $\nu(\mathcal{H}) \leqslant s-r+1$, we have $p \leqslant s-r+1$. Let

$$X = \left\{ x \in S \colon \deg_{\mathcal{H}}(x) > 2(s-r+1)\binom{n-2}{r-2} \right\},$$

$$Y = \left\{ x \in S \colon \deg_{\mathcal{H}}(x) > r(s-r+1)\binom{n-2}{r-2} \right\}.$$

Clearly, $Y \subset X$. By Lemma 7, we have $|X| \leqslant s-r+1$. Thus, $|Y| \leqslant s-r+1$. Now the proof splits into two cases depending on the size of Y.

Case 1. $|Y| = s-r+1$. We claim that every edge of $\mathcal{H}$ contains at least one vertex in Y. Otherwise, assume that E is an edge of $\mathcal{H}$ that is disjoint from Y. Let $Y = \{x_1, x_2, \dots, x_{s-r+1}\}$. For each $i = 1, 2, \dots, s-r+1$, since there are at most $r\binom{n-2}{r-2}$ sets in $N_{\mathcal{H}}(x_i)$ that intersects E, it follows that

$$\begin{aligned} \deg_{\mathcal{H}-E}(x_i) &> r(s-r+1)\binom{n-2}{r-2} - r\binom{n-2}{r-2} \\ &\geqslant 2(s-r)\binom{n-r-2}{r-2}. \end{aligned}$$

By Lemma 7, $\mathcal{H} - E$ contains a matching of size $s-r+1$. Then, $\mathcal{H}$ contains a matching of size $s-r+2$, which contradicts the fact that $\nu(\mathcal{H}) \leqslant s-r+1$. Thus, the claim holds. Therefore, $\mathcal{H}$ is isomorphic to a subhypergraph of $\mathcal{F}_{n,s-r+1,1}^{(r)}$, and we have $K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,s-r+1,1}^{(r)})$.

Case 2. $|Y| \leqslant s-r$. Clearly, each s-clique in $\mathcal{H}$ contains at least $s-r+1$ vertices in S. Otherwise, we obtain a matching of size $p+1$ in $\mathcal{H}$, which contradicts the fact that $\mathcal{M}$ is a maximum matching in $\mathcal{H}$. Now we count the number of s-cliques in $\mathcal{H}$ as follows. First, we choose a set A of $(s-r+1)$ vertices in S and there are at most $\binom{|S|}{s-r+1}$ choices for A. Then choose an $(r-1)$-element subset B of $V(\mathcal{H})$, which may form an s-clique in $\mathcal{H}$ together with A. Consequently, B has to be a common neighbor of all the vertices in A. Since $|A| > |Y|$, there exists some $x \in A$ that falls in $S \setminus Y$. If $A \subset X$, the number of choices for B is at most $(s-r+1)r\binom{n-2}{r-2}$. If A is not contained in X, the number of choices for B is at most $2(s-r+1)\binom{n-2}{r-2}$. Thus we have

$$K_s^r(\mathcal{H}) \leqslant (s-r+1)r\binom{n-2}{r-2}\binom{|X|}{s-r+1} + 2(s-r+1)\binom{n-2}{r-2}\binom{|S|}{s-r+1}.$$

Since $|X| \leqslant s-r+1$ and $|S| = pr \leqslant r(s-r+1)$, we find that

$$K_s^r \leqslant (s-r+1)r\binom{n-2}{r-2} + 2(s-r+1)\binom{n-2}{r-2}\binom{(s-r+1)r}{s-r+1}. \tag{7}$$

Using the inequality (1), we get

$$\binom{(s-r+1)r}{s-r+1} \leqslant (er)^{s-r+1}. \tag{8}$$

Combining (7) and (8), we obtain that

$$K_s^r(\mathcal{H}) \leqslant 3(s-r+1)(er)^{s-r+1}\binom{n-2}{r-2}. \tag{9}$$

By the inequality (3), we have

$$\binom{n-2}{r-2} \leqslant \left(\frac{n-r}{n-s}\right)^{r-2}\binom{n-s+r-2}{r-2}.$$

Therefore, (9) implies that

$$K_s^r(\mathcal{H}) \leqslant 3(s-r+1)(er)^{s-r+1}\left(\frac{n-r}{n-s}\right)^{r-2}\binom{n-s+r-2}{r-2}. \tag{10}$$

Applying (5) gives

$$\left(\frac{n-r}{n-s}\right)^{r-2} = \left(1+\frac{s-r}{n-s}\right)^{r-2} \leqslant 1+\frac{(r-2)^2(s-r)}{n-s}. \tag{11}$$

It follows from (10) and (11) that

$$K_s^r(\mathcal{H}) \leqslant 3(s-r+1)(er)^{s-r+1}\left(1+\frac{(r-2)^2(s-r)}{n-s}\right)\binom{n-s+r-2}{r-2}. \tag{12}$$

Since $n \geqslant 4(s-r+1)(er)^{s-r+2}$, it is easily checked that

$$\frac{(r-2)^2(s-r)}{n-s} \leqslant \frac{1}{3} \tag{13}$$

and

$$4(s-r+1)(er)^{s-r+1} \leqslant \frac{n-s+r-1}{r-1}. \tag{14}$$

Combining (12), (13) and (14), we deduce that

$$K_s^r(\mathcal{H}) \leqslant \frac{n-s+r-1}{r-1}\binom{n-s+r-2}{r-2} = \binom{n-s+r-1}{r-1} = K_s^r(\mathcal{F}^{(r)}_{n,s-r+1,1}).$$

This completes the proof. □

Now we are ready to prove Theorem 4 (I).

Proof of Theorem 4 (I). Let n, r be positive integers. We shall proceed by double induction on s and k. Recall the condition $r \leqslant s \leqslant k+r-1$. For $s=r$ and all $k \geqslant 1$, the result follows from Theorem 3. For all $s \geqslant r$ and $k = s-r+1$, the result follows from Lemma 8. Now we assume that the assertion holds for $(s-1, k-1)$ and $(s, k-1)$. Let $\mathcal{H}$ be an r-graph on n vertices with $n \geqslant 4k(er)^{s-r+2}$. Without loss of generality, we assume that $\nu(\mathcal{H}) = k$. Let $\mathcal{M} = \{E_1, E_2, \ldots, E_k\}$ be a maximum matching in $\mathcal{H}$ and S be the set of vertices covered by $\mathcal{M}$.

If there exists a vertex $u \in V(\mathcal{H})$ such that $\nu(\mathcal{H} - u) = k-1$, then by the induction hypothesis we have $K_s^r(\mathcal{H} - u) \leqslant K_s^r(\mathcal{F}^{(r)}_{n-1,k-1,1})$. Again, by the induction hypothesis, we have

$$K_s^r(u, \mathcal{H}) \leqslant K_{s-1}^r(\mathcal{H} - u) \leqslant K_{s-1}^r(\mathcal{F}^{(r)}_{n-1,k-1,1}).$$

From the equality (6), it follows that

$$\begin{aligned} K_s^r(\mathcal{H}) &= K_s^r(\mathcal{H} - u) + K_s^r(u, \mathcal{H}) \\ &\leqslant K_s^r(\mathcal{F}^{(r)}_{n-1,k-1,1}) + K_{s-1}^r(\mathcal{F}^{(r)}_{n-1,k-1,1}) \\ &= K_s^r(\mathcal{F}^{(r)}_{n,k,1}). \end{aligned}$$

Thus, we have shown that Theorem 4 (I) holds if there exists a vertex $u \in V(\mathcal{H})$ such that $\nu(\mathcal{H} - u) = k-1$.

Now we consider the case that $\nu(\mathcal{H} - u) = k$ for any $u \in V(\mathcal{H})$. We claim that the maximum degree in $\mathcal{H}$ is at most $rk\binom{n-2}{r-2}$. Let $u \in V(\mathcal{H})$ and $\mathcal{M}'$ be a matching of size k in $\mathcal{H} - u$. For each edge F in $\mathcal{H}$ with $u \in F$, it is easy to see that $|F \cap (\cup_{E \in \mathcal{M}'} E)| \geqslant 1$. It follows that the maximum degree of $\mathcal{H}$ is at most $rk\binom{n-2}{r-2}$.

Let Y be the set of vertices in S with degree greater than $2k\binom{n-2}{r-2}$. If $|Y| \geqslant k+1$, by Lemma 7 we obtain a matching of size $k+1$ in $\mathcal{H}$, contradicting the assumption that k is the size of a maximum matching. Thus we may assume that $|Y| \leqslant k$. Note that every s-clique in $\mathcal{H}$ contains at least $s-r+1$ vertices in S. We proceed to derive an upper bound on the number of s-cliques in $\mathcal{H}$. First, we choose a set A of $(s-r+1)$ vertices in S. There are at most $\binom{|S|}{s-r+1}$ choices for A. Then choose an $(r-1)$-element subset B of $V(\mathcal{H})$ such that $\mathcal{H}[B \cup A]$ is an s-clique of $\mathcal{H}$. It can be seen that B is a common neighbor of the vertices in A, that is, $B \in N_{\mathcal{H}}(v)$ for any $v \in A$. If $A \subset Y$, then the number of choices for B is at most $rk\binom{n-2}{r-2}$. If there exists a vertex $x \in A \setminus Y$, then the number of choices for B is at most $2k\binom{n-2}{r-2}$. Hence

$$K_s^r(\mathcal{H}) \leqslant kr\binom{n-2}{r-2}\binom{|Y|}{s-r+1} + 2k\binom{n-2}{r-2}\binom{|S|}{s-r+1}.$$

Since $|Y| \leqslant k$ and $|S| = kr$, we find that

$$K_s^r(\mathcal{H}) \leqslant kr\binom{n-2}{r-2}\binom{k}{s-r+1} + 2k\binom{n-2}{r-2}\binom{rk}{s-r+1}. \tag{15}$$

Using the inequality (4), we see that

$$\binom{rk}{s-r+1} \leqslant (er)^{s-r+1}\binom{k}{s-r+1}. \tag{16}$$

Combining (15) and (16), we obtain that

$$\begin{aligned} K_s^r(\mathcal{H}) &\leqslant kr\binom{k}{s-r+1}\binom{n-2}{r-2} + 2k(er)^{s-r+1}\binom{k}{s-r+1}\binom{n-2}{r-2} \\ &\leqslant \left(2k\,(er)^{s-r+1} + rk\right)\binom{k}{s-r+1}\binom{n-2}{r-2} \\ &\leqslant 3k\,(er)^{s-r+1}\binom{k}{s-r+1}\binom{n-2}{r-2}. \end{aligned} \tag{17}$$

Employing (3) and (5), we find that

$$\begin{aligned} \binom{n-2}{r-2} &\leqslant \left(\frac{n-2-(r-2)}{(n-k-1)-(r-2)}\right)^{r-2}\binom{n-k-1}{r-2} \\ &= \left(1+\frac{k-1}{n-k-r+1}\right)^{r-2}\binom{n-k-1}{r-2} \\ &\leqslant \left(1+\frac{(r-2)^2(k-1)}{n-k-r+1}\right)\binom{n-k-1}{r-2}. \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$K_s^r(\mathcal{H}) \leqslant 3(er)^{s-r+1}k\binom{k}{s-r+1}\left(1+\frac{(r-2)^2(k-1)}{n-k-r+1}\right)\binom{n-k-1}{r-2}. \tag{19}$$

Since $n \geqslant 4(er)^{s-r+2}k$, we see that

$$\frac{(r-2)^2(k-1)}{n-k-r+1} \leqslant \frac{1}{3} \tag{20}$$

and

$$4(er)^{s-r+1}k \cdot \frac{r-1}{n-k} \leqslant 1. \tag{21}$$

In view of (19), (20) and (21), we arrive at

$$K_s^r(\mathcal{H}) \leqslant \binom{k}{s-r+1}\binom{n-k}{r-1} \leqslant K_s^r(\mathcal{F}_{n,k,1}^{(r)}).$$

This completes the proof. □

3 The maximum number of s-cliques with $s \geqslant k + r$

In this section, we prove parts (II) and (III) of Theorem 4 by utilizing the shifting method originally due to Erdős-Ko-Rado [4] and further developed by Frankl [5].

Let $\mathcal{H}$ be an r-graph on the vertex set $[n]$. For integers i, j with $1 \leqslant i < j \leqslant n$ and any $E \in \mathcal{H}$, the shifting operator S_{ij} is defined by

$$S_{ij}(E) = \begin{cases} (E \setminus \{j\}) \cup \{i\}, & \text{if } j \in E, i \notin E \text{ and } (E \setminus \{j\}) \cup \{i\} \notin \mathcal{H}; \\ E, & \text{otherwise.} \end{cases}$$

Set

$$S_{ij}(\mathcal{H}) = \{S_{ij}(E) : E \in H\}.$$

An r-graph $\mathcal{H}$ is called a stable r-graph if $S_{ij}(\mathcal{H}) = \mathcal{H}$ holds for all $1 \leqslant i < j \leqslant n$, see Frankl [5]. He showed that any r-graph $\mathcal{H}$ can be shifted to a stable r-graph by applying the shifting operator iteratively.

We aim to determine the maximum number of s-cliques in an r-graph $\mathcal{H}$ with matching number at most k. Frankl [5] proved that $\nu(S_{ij}(\mathcal{H})) \leqslant \nu(\mathcal{H})$. Thus, the shifting operator preserves the property that the matching number is at most k. We shall show that the shifting operator does not decrease the number of s-cliques in an r-graph.

Lemma 9. *Let $\mathcal{H}$ be an r-graph on the vertex set $[n]$. For any $i, j \in [n]$ with $i < j$ and $s \geqslant r$, we have $K_s^r(S_{ij}(\mathcal{H})) \geqslant K_s^r(\mathcal{H})$. Moreover, if each edge of $\mathcal{H}$ is contained in an s-clique of $\mathcal{H}$, then each edge of $S_{ij}(\mathcal{H})$ is contained in an s-clique of $S_{ij}(\mathcal{H})$.*

Proof. Let $K \subset [n]$ with $|K| = s$. If $\mathcal{H}[K]$ is an s-clique but $S_{ij}(\mathcal{H})[K]$ is not an s-clique, then we have $j \in K$ and $i \notin K$ and there is an edge in $\mathcal{H}[K]$ that is shifted by S_{ij}. By the definition of the shifting operator, it follows that $\mathcal{H}[(K - \{j\}) \cup \{i\}]$ is not an s-clique but $S_{ij}(\mathcal{H})[(K \setminus \{j\}) \cup \{i\}]$ is an s-clique. Now, we define a map σ from $\mathcal{K}_s^r(\mathcal{H})$ to $\mathcal{K}_s^r(S_{ij}(\mathcal{H}))$ as follows. If $\mathcal{H}[K] \in \mathcal{K}_s^r(\mathcal{H})$ and $S_{ij}(\mathcal{H})[K] \in \mathcal{K}_s^r(S_{ij}(\mathcal{H}))$, let $\sigma(\mathcal{H}[K]) = S_{ij}(\mathcal{H})[K]$; If $\mathcal{H}[K] \in \mathcal{K}_s^r(\mathcal{H})$ but $S_{ij}(\mathcal{H})[K] \notin \mathcal{K}_s^r(S_{ij}(\mathcal{H}))$, let $\sigma(\mathcal{H}[K]) = S_{ij}(\mathcal{H})[(K - \{j\}) \cup \{i\}]$. It is easy to verify that σ is an injection, and so $K_s^r(S_{ij}(\mathcal{H})) \geqslant K_s^r(\mathcal{H})$.

Suppose that each edge of $\mathcal{H}$ is contained in an s-clique of $\mathcal{H}$ and there exists an edge $E \in S_{ij}(\mathcal{H})$ that is not contained in any s-clique of $S_{ij}(\mathcal{H})$. If $E \in \mathcal{H}$, let $\mathcal{H}[K]$ be an s-clique of $\mathcal{H}$ containing E, where K is a subset of $[n]$ with $|K| = s$. Since $E \in S_{ij}(\mathcal{H})$, we have $S_{ij}(\mathcal{H})[(K \setminus \{j\}) \cup \{i\}]$ is an s-clique containing E, a contradiction. If $E \notin \mathcal{H}$, then $E' = (E \setminus \{i\}) \cup \{j\}$ is an edge of $\mathcal{H}$. Let K be a subset of $[n]$ such that $\mathcal{H}[K]$ is an s-clique in $\mathcal{H}$ containing E'. Clearly, we have $j \in E'$ and $i \notin K$. Then $S_{ij}(\mathcal{H})[(K \setminus \{j\}) \cup \{i\}]$ is an s-clique in $S_{ij}(\mathcal{H})$ containing E, a contradiction. Thus, if each edge of $\mathcal{H}$ is contained in an s-clique of $\mathcal{H}$, then each edge of $S_{ij}(\mathcal{H})$ is contained in an s-clique of $S_{ij}(\mathcal{H})$. This completes the proof. $\square$

Let us recall a basic property of the shifting operator, which will be used later. Let $E_1 = \{a_1, a_2, \ldots, a_r\}$ and $E_2 = \{b_1, b_2, \ldots, b_r\}$ be two different r-element subsets of $[n]$. As used in [9], we write $E_1 \prec E_2$ if there exists a permutation $\sigma_1 \sigma_2 \cdots \sigma_r$ of $[r]$ such that

$a_j \leqslant b_{\sigma_j}$ for all $j = 1, \ldots, r$. Frankl [5] showed that if $\mathcal{H}$ is a stable r-graph on $[n]$, $E \in \mathcal{H}$ and S is an r-element subset of $[n]$ with $S \prec E$, then $S \in \mathcal{H}$.

The following proposition gives a characterization of stable r-graphs with matching number at most k.

Proposition 10. *Let n, k, r, s be positive integers with $k + r \leqslant s \leqslant rk + r - 1$ and $n \geqslant rk + r - 1$. Let $\mathcal{H}$ be a stable r-graph on the vertex set $[n]$ with $\nu(\mathcal{H}) \leqslant k$. If every edge of $\mathcal{H}$ is contained in at least one s-clique in $\mathcal{H}$, then for every edge $E \in \mathcal{H}$, we have $|E \cap [rk + a - 1]| \geqslant a$, where $a = \lfloor \frac{s-r}{k} \rfloor + 1$.*

Proof. It is easily checked that $2 \leqslant a \leqslant r$ and $(a-1)k + r \leqslant s \leqslant ak + r - 1$. Suppose that there is an edge $E = \{x_1, x_2, \ldots, x_r\} \in \mathcal{H}$ with $x_1 < x_2 < \cdots < x_r$ such that $|E \cap [rk + a - 1]| < a$. Then $|E \cap [rk + a - 1]| < a$ implies that $x_a \geqslant rk + a$. Let K be an s-clique in $\mathcal{H}$ containing E and $X = \{x_a, x_{a+1}, \ldots, x_r\}$. Since

$$|V(K) \setminus X| = s - (r - a + 1) \geqslant (a-1)k + r - (r - a + 1) = (a-1)(k+1),$$

there exist $k+1$ disjoint $(a-1)$-element sets $S_1, S_2, \ldots, S_{k+1}$ in $V(K) \setminus X$. Moreover, $S_i \cup X$ is an edge of $\mathcal{H}$ for each $i = 1, 2, \ldots, k+1$. For any

$$T \subset [rk + a - 1] \setminus (\cup_{i=1}^{k+1} S_i)$$

with $|T| = r - a + 1$, $S_i \cup T$ forms an edge of $\mathcal{H}$ for each $i = 1, 2, \ldots, k+1$, since $\mathcal{H}$ is stable and $S_i \cup T \prec S_i \cup X$. Noting that

$$|[rk + a - 1] \setminus (\cup_{i=1}^{k+1} S_i)| \geqslant rk + a - 1 - (a-1)(k+1) = (r - a + 1)k,$$

there are k disjoint $(r-a+1)$-element sets $T_1, T_2, \ldots, T_k$ in $[rk + a - 1] \setminus (\cup_{i=1}^{k+1} S_i)$. Thus, $S_1 \cup T_1$, $S_2 \cup T_2$, $\ldots$, $S_k \cup T_k$, $S_{k+1} \cup X$ constitute $k+1$ disjoint edges in $\mathcal{H}$, which contradicts the fact that $\nu(\mathcal{H}) \leqslant k$. This completes the proof. $\square$

Moreover, we need a result similar to Lemma 7, which can be proved by the greedy algorithm.

Lemma 11. *Let n, k, r, a be integers such that $rk \leqslant n$, $a < r$ and $\mathcal{H}$ be an r-graph on n vertices. If $V(\mathcal{H})$ has k disjoint a-element subset $A_1, A_2, \ldots, A_k$ with $\deg(A_i) > r(k-1)\binom{n-a-2}{r-a-1}$, then $\mathcal{H}$ contains a matching of size k.*

Proof. Using the greedy algorithm, we can find a matching of size k in $\mathcal{H}$. Since

$$|N_{\mathcal{H}}(A_1)| > r(k-1)\binom{n-a-1}{r-a-1} > |\cup_{j=2}^{k} A_j| \binom{n-a-1}{r-a-1},$$

we can choose B_1 from $N_{\mathcal{H}}(A_1)$ such that B_1 is disjoint from $\cup_{j=2}^{k} A_j$. For $i \in \{2, \ldots, k\}$, suppose that $B_1, B_2, \ldots, B_{i-1}$ have been chosen such that $A_1 \cup B_1$, $A_2 \cup B_2$, $\ldots$, $A_{i-1} \cup B_{i-1}$, A_i, A_{i+1}, $\ldots$, A_k are pairwise disjoint. Since

$$|N_{\mathcal{H}}(A_i)| > r(k-1)\binom{n-a-1}{r-a-1} > \left(\sum_{j=1}^{i-1} |A_j \cup B_j| + \sum_{j=i+1}^{k} |A_j|\right)\binom{n-a-1}{r-a-1},$$

we can choose B_i from $N_{\mathcal{H}}(A_i)$ such that B_i is disjoint from $(\cup_{j=1}^{k} A_j) \cup (\cup_{j=1}^{i-1} B_j)$. Finally, we end up with a matching of size k in $\mathcal{H}$. □

We first prove Theorem 4 (III), because it will be needed in the proof of Theorem 4 (II).

Proof of Theorem 4 (III). Let $\mathcal{H}^*$ be the subhypergraph obtained from $\mathcal{H}$ by deleting all the edges in $\mathcal{H}$ that are not contained in any s-clique in $\mathcal{H}$. Frankl [5] proved that any r-graph can be shifted to a stable r-graph by applying the shifting operator iteratively. By Lemma 9, we may assume that $\mathcal{H}^*$ is stable. Since $\lfloor \frac{s-r}{k} \rfloor + 1 = r$, by Proposition 10 we obtain that $|E \cap [rk + r - 1]| \geqslant r$ for every $E \in \mathcal{H}^*$. So $\mathcal{H}^*$ is a subhypergraph of $\mathcal{F}_{n,k,r}^{(r)}$, and thus

$$K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{H}^*) \leqslant K_s^r(\mathcal{F}_{n,k,r}^{(r)}).$$

If $K_s^r(\mathcal{H}) < K_s^r(\mathcal{F}_{n,k,r}^{(r)})$, then $\mathcal{H}^*$ has to be a proper subhypergraph of $\mathcal{F}_{n,k,r}^{(r)}$. It follows that there is an r-element subset T of $[rk + r - 1]$ such that $T \notin \mathcal{H}^*$. Then none of the s-element subsets of $[rk + r - 1]$ containing T can be an s-clique of $\mathcal{H}^*$. Note that there are exactly $\binom{rk-1}{s-r}$ s-element subsets of $[rk + r - 1]$ containing T. Thus,

$$K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{H}^*) \leqslant \binom{rk + r - 1}{s} - \binom{rk - 1}{s - r},$$

as claimed. □

We are ready to prove Theorem 4 (II).

Proof of Theorem 4 (II). By Lemma 9, we may assume that $\mathcal{H}$ is a stable r-graph on $[n]$ and each edge of $\mathcal{H}$ is contained in an s-clique. Note that $a = \lfloor \frac{s-r}{k} \rfloor + 1$, so that $(a-1)k + r \leqslant s \leqslant ak + r - 1$. By Proposition 10, we have $|E \cap [rk + a - 1]| \geqslant a$ for every $E \in \mathcal{H}$. Define an a-graph $\mathcal{H}^*$ on $[rk + a - 1]$ as

$$\mathcal{H}^* = \left\{ A \in \binom{[rk + a - 1]}{a} : \deg_{\mathcal{H}}(A) > rk \binom{n - a - 1}{r - a - 1} \right\}.$$

Now we prove the following two claims, leading to a description of $\mathcal{H}^*$.
Claim 1. $\mathcal{H}^*$ is stable.

Suppose to the contrary that $\mathcal{H}^*$ is not stable. Then, there exist i and j such that $1 \leqslant i < j \leqslant n$ and $S_{ij}(\mathcal{H}^*) \neq \mathcal{H}^*$. This ensures the existence of an edge $A \in \mathcal{H}^*$ such that $S_{ij}(A) \neq A$. By the definition of $\mathcal{H}^*$, we have $|N_{\mathcal{H}}(A)| = \deg_{\mathcal{H}}(A) > rk\binom{n-a-1}{r-a-1}$. Let $A' = (A \setminus \{j\}) \cup \{i\}$. Since $S_{ij}(A) \neq A$, we find that $j \in A$, $i \notin A$ and $A' \notin \mathcal{H}^*$. Let $B \in N_{\mathcal{H}}(A)$. If $i \notin B$, since $A \cup B \in \mathcal{H}$ and $\mathcal{H}$ is stable, it follows that $A' \cup B \in \mathcal{H}$. If $i \in B$, since $A \cup B \in \mathcal{H}$, we see that $A' \cup (B \setminus \{i\}) \cup \{j\} = A \cup B \in \mathcal{H}$. Now we define a map τ from $N_{\mathcal{H}}(A)$ to $N_{\mathcal{H}}(A')$. If $i \notin B$, let $\tau(B) = B$; if $i \in B$, let $\tau(B) = (B \setminus \{i\}) \cup \{j\}$. It can be seen that τ is injective and $|N_{\mathcal{H}}(A')| \geqslant |N_{\mathcal{H}}(A)| > rk\binom{n-a-1}{r-a-1}$, which contradicts the fact that $A' \notin \mathcal{H}^*$. This proves the claim.
Claim 2. $\nu(\mathcal{H}^*) \leqslant k$.

Suppose to the contrary that $\nu(\mathcal{H}^*) \geqslant k+1$. Then, there exist $k+1$ disjoint edges $A_1, A_2, \ldots, A_{k+1}$ in $\mathcal{H}^*$. Since $\deg_{\mathcal{H}}(A_i) \geqslant rk\binom{n-a-1}{r-a-1}$ for each $i = 1, 2, \ldots, k+1$, by Lemma 11 there exists a matching of size $k+1$ in $\mathcal{H}$, which contradicts the fact that $\nu(\mathcal{H}) \leqslant k$. Thus the claim holds.

Since $|E \cap [rk+a-1]| \geqslant a$ for every edge $E \in \mathcal{H}$, every s-clique in $\mathcal{H}$ has at least $s-r+a$ vertices in $[rk-a+1]$. Now we consider the maximum number of $(s-r+a)$-cliques in $\mathcal{H}^*$. Since $\mathcal{H}^*$ is an a-graph and $(a-1)k+a \leqslant s-r+a \leqslant ak+a-1$, by Theorem 4 (III) we have

$$K^a_{s-r+a}(\mathcal{H}^*) \leqslant K^a_{s-r+a}(\mathcal{F}^{(a)}_{n,k,a}) = \binom{ak+a-1}{s-r+a}.$$

Moreover, if $K^a_{s-r+a}(\mathcal{H}^*) < K^a_{s-r+a}(\mathcal{F}^{(a)}_{n,k,a})$, we have

$$K^a_{s-r+a}(\mathcal{H}^*) \leqslant \binom{ak+a-1}{s-r+a} - \binom{ak-1}{s-r}.$$

Next we consider two cases depending on the value of $K^a_{s-r+a}(\mathcal{H}^*)$.

Case 1. $K^a_{s-r+a}(\mathcal{H}^*) = K^a_{s-r+a}(\mathcal{F}^{(a)}_{n,k,a})$. If there exists an edge $E \in E(\mathcal{H})$ with $|E \cap [ak+a-1]| \leqslant a-1$, then there are k disjoint edges $A_1, A_2, \ldots, A_k$ in $\mathcal{H}^* - E$. Noting that there are at most $r\binom{n-a-1}{r-a-1}$ sets T in $N_{\mathcal{H}}(A_i)$ such that $|T \cap E| \geqslant 1$ and that $\deg_{\mathcal{H}}(A_i) > rk\binom{n-a-1}{r-a-1}$ for each $i = 1, 2, \ldots, s-r+1$, we have

$$\begin{aligned}\deg_{\mathcal{H}-E}(A_i) &> rk\binom{n-a-1}{r-a-1} - r\binom{n-a-1}{r-a-1}\\ &\geqslant r(k-1)\binom{n-r-a-1}{r-a-1}.\end{aligned}$$

By Lemma 11, there is a matching $\mathcal{M}$ of size k in $\mathcal{H}-E$. Then $\mathcal{M} \cup \{E\}$ forms a matching of size $k+1$ in $\mathcal{H}$, which contradicts the fact that $\nu(\mathcal{H}) \leqslant k$. Thus, $\mathcal{H}$ is a subhypergraph of $\mathcal{F}^{(r)}_{n,k,a}$ and so $K^r_s(\mathcal{H}) \leqslant K^r_s(\mathcal{F}^{(r)}_{n,k,a})$.

Case 2. $K^a_{s-r+a}(\mathcal{H}^*) \leqslant \binom{ak+a-1}{s-r+a} - \binom{ak-1}{s-r}$. By Proposition 10, we have $|E \cap [rk+a-1]| \geqslant a$ for any $E \in \mathcal{H}$. Thus, for each s-clique K in $\mathcal{H}$, $|V(K) \cap [rk+a-1]| \geqslant s-r+a$. We aim to derive an upper bound on the number of s-cliques in $\mathcal{H}$. First, we choose an $(s-r+a)$-element subset S of $[rk+a-1]$. Then choose an $(r-a)$-element subset T such that $\mathcal{H}[S \cup T]$ forms an s-clique of $\mathcal{H}$. It can be seen that T is a common neighbor of the a-element subsets of S, that is, $T \in N_{\mathcal{H}}(R)$ for any $R \in \binom{S}{a}$. If $\mathcal{H}^*[S]$ is not an $(s-r+a)$-clique in $\mathcal{H}^*$, there exists an a-element subset A of S such that $A \notin \mathcal{H}^*$. Hence the number of choices for T is at most $\deg_{\mathcal{H}}(A) \leqslant rk\binom{n-a-1}{r-a-1}$. If $\mathcal{H}^*[S]$ is an $(s-r+a)$-clique in $\mathcal{H}^*$, then the number of choices for T is at most $\binom{n-(s-r+a)}{r-a}$. Therefore,

$$K^r_s(\mathcal{H}) \leqslant K^a_{s-r+a}(\mathcal{H}^*)\binom{n-s+r-a}{r-a} + rk\binom{n-a-1}{r-a-1}\binom{rk+a-1}{s-r+a}$$

$$\leqslant \left(\binom{ak+a-1}{s-r+a} - \binom{ak-1}{s-r}\right)\binom{n-s+r-a}{r-a}$$
$$+ rk\binom{n-a-1}{r-a-1}\binom{rk+a-1}{s-r+a}. \tag{22}$$

If $s = ak + r - 1$, substituting $s = ak + r - 1$ into (22), we obtain that

$$K_s^r(\mathcal{H}) \leqslant rk\binom{n-a-1}{r-a-1}\binom{rk+a-1}{ak+a-1}. \tag{23}$$

It follows from (3) and (5) that

$$\binom{n-a-1}{r-a-1} \leqslant \left(\frac{n-r}{n-ak-r+1}\right)^{r-a-1}\binom{n-ak-a}{r-a-1}$$
$$\leqslant \left(1 + \frac{(ak-1)(r-a-1)^2}{n-ak-r+1}\right)\binom{n-ak-a}{r-a-1}. \tag{24}$$

Since $n \geqslant 4r^2(er/a)^{s-r+a}k$, we see that

$$\frac{(ak-1)(r-a-1)^2}{n-ak-r+1} \leqslant 1. \tag{25}$$

By (24) and (25), we get

$$\binom{n-a-1}{r-a-1} \leqslant 2\binom{n-ak-a}{r-a-1}. \tag{26}$$

Combining (23) and (26), we obtain that

$$K_s^r(\mathcal{H}) \leqslant 2rk\binom{rk+a-1}{ak+a-1}\binom{n-ak-a}{r-a-1}. \tag{27}$$

Applying (1) and (27) gives

$$K_s^r(\mathcal{H}) \leqslant 2rk\left(\frac{e(rk+a-1)}{ak+a-1}\right)^{ak+a-1}\binom{n-ak-a}{r-a-1}$$
$$\leqslant 2kr\left(\frac{er}{a}\right)^{ak+a-1}\frac{r-a}{n-ak-a+1}\binom{n-ak-a+1}{r-a}. \tag{28}$$

Under the condition $s = ak+r-1$, we have $ak+a-1 = s-r+a$. Since $n \geqslant 4r^2(er/a)^{s-r+a}k$, it can be checked that

$$2kr\left(\frac{er}{a}\right)^{ak+a-1}\frac{r-a}{n-ak-a+1} \leqslant 1. \tag{29}$$

Combining (28) and (29), we get

$$K_s^r(\mathcal{H}) \leqslant \binom{n-ak-a+1}{r-a} = K_s^r(\mathcal{F}_{n,k,a}^{(r)}).$$

It remains to consider the case $(a-1)k+r \leqslant s < ak+r-1$. We have

$$\begin{aligned}\binom{ak+a-1}{s-r+a} &= \frac{ak+a-1}{s-r+a} \cdot \frac{ak+a-2}{s-r+a-1} \cdots \frac{ak}{s-r+1} \cdot \binom{ak-1}{s-r} \\ &\leqslant \left(\frac{ak-1}{s-r}\right)^a \binom{ak-1}{s-r} \\ &\leqslant \left(\frac{ak-1}{(a-1)k}\right)^a \binom{ak-1}{s-r} \\ &\leqslant \left(\frac{a}{a-1}\right)^a \binom{ak-1}{s-r}. \end{aligned} \tag{30}$$

Employing (4) and (30), we find that

$$\begin{aligned}\binom{rk+a-1}{s-r+a} &\leqslant \left(\frac{e(rk+a-1)}{ak+a-1}\right)^{s-r+a} \binom{ak+a-1}{s-r+a} \\ &\leqslant \left(\frac{er}{a}\right)^{s-r+a} \left(\frac{a}{a-1}\right)^a \binom{ak-1}{s-r} \\ &\leqslant \left(\frac{er}{a-1}\right)^{s-r+a} \binom{ak-1}{s-r}. \end{aligned} \tag{31}$$

It follows from (3) and (5) that

$$\begin{aligned}\binom{n-s+r-a}{r-a} &\leqslant \left(\frac{n-s+r-a-(r-a)}{n-ak-a+1-(r-a)}\right)^{r-a} \binom{n-ak-a+1}{r-a} \\ &= \left(1+\frac{ak+r-1-s}{n-ak-r+1}\right)^{r-a} \binom{n-ak-a+1}{r-a} \\ &\leqslant \left(1+\frac{(r-a)^2(ak+r-1-s)}{n-ak-r+1}\right) \binom{n-ak-a+1}{r-a}. \end{aligned} \tag{32}$$

The condition $(a-1)k+r \leqslant s < ak+r-1$ implies $ak+r-1-s \leqslant k$. Since $n \geqslant 2(ak+r-1)$, from (32) we see that

$$\begin{aligned}\binom{n-s+r-a}{r-a} &\leqslant \left(1+\frac{(r-a)^2 k}{n-ak-r+1}\right) \binom{n-ak-a+1}{r-a} \\ &\leqslant \left(1+\frac{2r^2 k}{n}\right) \binom{n-ak-a+1}{r-a}. \end{aligned} \tag{33}$$

Substituting (33) into (22), we obtain that

$$K_s^r(\mathcal{H}) \leqslant \left(\binom{ak+a-1}{s-r+a} - \binom{ak-1}{s-r}\right)\left(1+\frac{2r^2k}{n}\right)\binom{n-ak-a+1}{r-a} + rk\binom{n-a-1}{r-a-1}\binom{rk+a-1}{s-r+a}$$

$$\leqslant \left(\binom{ak+a-1}{s-r+a} - \binom{ak-1}{s-r}\right)\binom{n-ak-a+1}{r-a} + \frac{2r^2k}{n}\binom{ak+a-1}{s-r+a} \cdot \binom{n-ak-a+1}{r-a} + rk\binom{n-a-1}{r-a-1}\binom{rk+a-1}{s-r+a}. \tag{34}$$

Since each edge of $\mathcal{F}_{n,k,a}^{(r)}$ has at least a vertices in $[ak+a-1]$, it is easy to see that each s-clique of $\mathcal{F}_{n,k,a}^{(r)}$ has at least $s-r+a$ vertices in $[ak+a-1]$. Thus,

$$K_s^r(\mathcal{F}_{n,k,a}^{(r)}) = \sum_{i=s-r+a}^{s}\binom{ak+a-1}{i}\binom{n-ak-a+1}{s-i} > \binom{ak+a-1}{s-r+a}\binom{n-ak-a+1}{r-a}. \tag{35}$$

Using the inequalities (34) and (35), we find that

$$K_s^r(\mathcal{H}) < K_s^r(\mathcal{F}_{n,k,a}^{(r)}) - \binom{ak-1}{s-r}\binom{n-ak-a+1}{r-a} + \frac{2r^2k}{n}\binom{ak+a-1}{s-r+a}\binom{n-ak-a+1}{r-a} + rk\binom{n-a-1}{r-a-1}\binom{rk+a-1}{s-r+a}. \tag{36}$$

Substituting (30), (31), (26) into (36), we deduce that

$$K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,k,a}^{(r)}) - \binom{ak-1}{s-r}\binom{n-ak-a+1}{r-a} + \frac{2r^2k}{n}\left(\frac{a}{a-1}\right)^a\binom{ak-1}{s-r} \cdot \binom{n-ak-a+1}{r-a} + \left(\frac{er}{a-1}\right)^{s-r+a}\binom{ak-1}{s-r}\cdot rk\cdot 2\binom{n-ak-a}{r-a-1}$$

$$= K_s^r(\mathcal{F}_{n,k,a}^{(r)}) - \binom{ak-1}{s-r}\binom{n-ak-a+1}{r-a} \cdot \left(1-\left(\frac{a}{a-1}\right)^a\frac{2r^2k}{n} - \left(\frac{er}{a-1}\right)^{s-r+a}\frac{2rk(r-a)}{n-ak-a+1}\right) \tag{37}$$

Since $n \geqslant 4r^2k(er/(a-1))^{s-r+a}$, we obtain that

$$1-\left(\frac{a}{a-1}\right)^a\frac{2r^2k}{n} - \left(\frac{er}{a-1}\right)^{s-r+a}\frac{2rk(r-a)}{n-ak-a+1} > 0. \tag{38}$$

Combining (37) and (38), we arrive at $K_s^r(\mathcal{H}) \leqslant K_s^r(\mathcal{F}_{n,k,a}^{(r)})$. This completes the proof. □

4 Proof of Theorem 6

Huang, Loh and Sudakov [14] considered a multicolored generalization of the Erdős matching conjecture and provided a sufficient condition on the number of edges for a multicolored hypergraph to contain a rainbow matching of size k, as stated in Lemma 12 below. Theorem 1.6 can be considered as a generalization of Theorem 1.5. The proof of Theorem 6 also relies on Lemma 12.

Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be r-graphs on $[n]$. We say that $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ contains a rainbow matching if there exist k pairwise disjoint sets $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \ldots, F_k \in \mathcal{F}_k$.

Lemma 12 (Huang, Loh and Sudakov [14])**.** *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be r-graphs on $[n]$ such that $|\mathcal{F}_i| > (k-1)\binom{n-1}{r-1}$, and $n \geqslant rk$. Then $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ contains a rainbow matching.*

Theorem 6 will be proved by induction. The following lemma is the basis of the induction.

Lemma 13. *Let n, k and r be integers such that $n \geqslant 4k^2(er)^k$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be r-graphs on $[n]$. If for all $i \in \{1, 2, \ldots, k\}$, there exists some $s \in \{r, r+1, \ldots, k+r-2\}$ such that $K_s^r(\mathcal{F}_i) > K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$. Then the family $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ contains a rainbow matching.*

Proof. Let $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ be a family of r-graphs that does not contain any rainbow matching. We may further assume that this family attains the maximum value of $\sum_{i=1}^k |\mathcal{F}_i|$. We shall prove the lemma by showing that there exists some i such that

$$K_s^r(\mathcal{F}_i) \leqslant K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$$

for any $s \in \{r, r+1, \ldots, k+r-2\}$.

Let l be the number of r-graphs in the family $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ that are not complete r-graphs. Without loss of generality, we may assume that $\mathcal{F}_1, \ldots, \mathcal{F}_l$ are such non-complete r-graphs and $\mathcal{F}_{l+1}, \ldots, \mathcal{F}_k$ are complete r-graphs. For $l = 1$, if $\mathcal{F}_1$ is not an empty r-graph, by the definition of l, there exist disjoint edges $F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2, \ldots, F_k \in \mathcal{F}_k$, contradicting the assumption that $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ does not contain any rainbow matching. If $\mathcal{F}_1$ is an empty r-graph, we have

$$K_s^r(\mathcal{F}_1) = 0 \leqslant K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$$

for any $s \in \{r, r+1, \ldots, k+r-2\}$. Thus, we may assume that $2 \leqslant l \leqslant k$.

For $i = 1, 2, \ldots, l$, let X_i be the set of vertices $v \in [n]$ such that $\deg_{\mathcal{F}_i}(v) > 2(l-1)\binom{n-2}{r-2}$ and let Y_i be the set of vertices $v \in [n]$ such that $\deg_{\mathcal{F}_i}(v) \geqslant r(k-1)\binom{n-2}{r-2}$. It is clear that $Y_i \subseteq X_i$.

Claim 3. The family $\{X_1, X_2, \ldots, X_l\}$ does not contain a system of distinct representatives.

Suppose to the contrary that there exists a system of distinct representatives in $\{X_1, X_2, \ldots, X_l\}$. Assume that $x_1 \in X_1, x_2 \in X_2, \ldots, x_l \in X_l$ are l distinct vertices.

Let $X = \{x_1, x_2, \dots, x_l\}$. For $i = 1, 2, \dots, l$, define

$$\mathcal{H}_i = \left\{ T \in \binom{[n] \setminus X}{r-1} : T \cup \{x_i\} \in \mathcal{F}_i \right\}.$$

For any $i, j \in [l]$ with $i \neq j$, there are at most $\binom{n-2}{r-2}$ edges of $\mathcal{F}_i$ containing both x_i and x_j. Thus, for $i = 1, 2, \dots, l$,

$$|\mathcal{H}_i| \geqslant \deg_{\mathcal{F}_i}(v_i) - (l-1)\binom{n-2}{r-2} > (l-1)\binom{n-2}{r-2} \geqslant (l-1)\binom{n-l-1}{r-2}.$$

Since $\mathcal{H}_i$ is an $(r-1)$-graph on $n-l$ vertices, by Lemma 12, there exist l disjoint edges $E_1 \in \mathcal{H}_1, E_2 \in \mathcal{H}_2, \dots, E_l \in \mathcal{H}_l$. It follows that $\{E_1 \cup \{x_1\}, E_2 \cup \{x_2\}, \dots, E_l \cup \{x_l\}\}$ forms a rainbow matching in $\{\mathcal{F}_1, \dots, \mathcal{F}_l\}$. Since $\mathcal{F}_{l+1}, \dots, \mathcal{F}_k$ are all complete r-graphs, there exists a rainbow matching in $\{\mathcal{F}_1, \dots, \mathcal{F}_k\}$, a contradiction. This proves the claim.

The following claim shows that if $|X_i|$ and $|Y_i|$ are both small, then the lemma follows.

Claim 4. If there exists $i \in \{1, 2, \dots, l\}$ such that $|X_i| \leqslant l-1$ and $|Y_i| \leqslant l-2$, then $K_s^r(\mathcal{F}_i) \leqslant K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$ for all $r \leqslant s \leqslant k+r-2$.

Since $\mathcal{F}_i$ is not a complete r-graph, by the maximality of $\sum_{i=1}^{k} |\mathcal{F}_i|$, we see that $\{\mathcal{F}_1, \dots, \mathcal{F}_{i-1}, \mathcal{F}_{i+1}, \dots, \mathcal{F}_k\}$ contains a rainbow matching. Let

$$\mathcal{M} = \{E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_k\}$$

be such a rainbow matching and S be the set of vertices that are covered by $\mathcal{M}$. For each $s \in \{r, r+1, \dots, k+r-2\}$, every s-clique in $\mathcal{F}_i$ has at least $s-r+1$ vertices in S. To derive an upper bound on the number of s-cliques in $\mathcal{F}_i$, we first choose an $(s-r+1)$-element subset A of S. Then choose an $(r-1)$-element subset B of V, such that $\mathcal{F}_i[B \cup A]$ is an s-clique of $\mathcal{F}_i$. It can be seen that B is a common neighbor in $\mathcal{F}_i$ of the vertices in A, that is, $B \in N_{\mathcal{F}_i}(v)$ for any $v \in A$.

If $A \subset Y_i$, the number of choices for B is at most $\binom{n-s+r-1}{r-1}$. If $A \subset X_i$ and $A \nsubseteq Y_i$, the number of choices for B is at most $r(k-1)\binom{n-2}{r-2}$. If $A \nsubseteq X_i$, the number of choices for B is at most $2(l-1)\binom{n-2}{r-2}$. Thus,

$$K_s^r(\mathcal{F}_i) \leqslant \binom{|Y_i|}{s-r+1}\binom{n-(s-r+1)}{r-1} + r(k-1)\binom{n-2}{r-2}\binom{|X_i|}{s-r+1} + 2(l-1)\binom{n-2}{r-2}\binom{|S|}{s-r+1}.$$

Since $|Y_i| \leqslant l-2 \leqslant k-2$, $|X_i| \leqslant l-1 \leqslant k-1$ and $|S| = r(k-1)$, we find that

$$K_s^r(\mathcal{F}_i) \leqslant \binom{k-2}{s-r+1}\binom{n-s+r-1}{r-1} + r(k-1)\binom{n-2}{r-2}\binom{k-1}{s-r+1} + 2(k-1)\binom{n-2}{r-2}\binom{r(k-1)}{s-r+1}. \tag{39}$$

If $s=k+r-2$, by the inequality (39) we have

$$\begin{aligned}K_s^r(\mathcal{F}_i) &\leqslant r(k-1)\binom{n-2}{r-2}+2(k-1)\binom{n-2}{r-2}\binom{r(k-1)}{k-1}\\ &\leqslant 3(k-1)\binom{r(k-1)}{k-1}\binom{n-2}{r-2}. \qquad (40)\end{aligned}$$

Using the inequality (1), we get

$$\binom{r(k-1)}{k-1}\leqslant (er)^{k-1}. \qquad (41)$$

Employing (3) and (5), we see that

$$\begin{aligned}\binom{n-2}{r-2} &\leqslant \left(\frac{n-2-(r-2)}{n-k-(r-2)}\right)^{r-2}\binom{n-k}{r-2}\\ &\leqslant \left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right)\binom{n-k}{r-2}. \qquad (42)\end{aligned}$$

Substituting (41) and (42) into (40), we obtain that

$$K_s^r(\mathcal{F}_i)\leqslant 3k(er)^{k-1}\left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right)\frac{r-1}{n-k+1}\cdot\binom{n-k+1}{r-1}. \qquad (43)$$

Since $n\geqslant 4k^2(er)^k$, we have

$$\frac{(r-2)^2(k-2)}{n-k-r+2}\leqslant 1 \qquad (44)$$

and

$$\frac{3k(er)^k}{n-k+1}\leqslant 1. \qquad (45)$$

In view of (43), (44) and (45), we arrive at

$$K_s^r(\mathcal{F}_i)\leqslant\binom{n-k+1}{r-1}\leqslant K_s^r(\mathcal{F}_{n,k-1,1}^{(r)}).$$

It remains to consider the case $r\leqslant s\leqslant k+r-3$. Applying (3) and (5) gives

$$\begin{aligned}\binom{n-s+r-1}{r-1} &\leqslant \left(1+\frac{(k+r-2-s)}{n-k-r+2}\right)^{r-1}\binom{n-k+1}{r-1}\\ &\leqslant \left(1+\frac{(r-1)^2(k+r-2-s)}{n-k-r+2}\right)\binom{n-k+1}{r-1}\end{aligned}$$

$$\leqslant \left(1+\frac{2r^2k}{n}\right)\binom{n-k+1}{r-1}, \tag{46}$$

and

$$\begin{aligned}\binom{n-2}{r-2} &\leqslant \left(\frac{n-2-(r-2)}{n-k-(r-2)}\right)^{r-2}\binom{n-k}{r-2}\\ &\leqslant \left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right)\binom{n-k}{r-2}. \end{aligned}\tag{47}$$

Combining (39) and (46), we deduce that

$$\begin{aligned}K_s^r(\mathcal{F}_i) \leqslant \left(1+\frac{2r^2k}{n}\right)\binom{k-2}{s-r+1}\binom{n-k+1}{r-1} + r(k-1)\binom{k-1}{s-r+1}\binom{n-2}{r-2}\\ + 2(k-1)\binom{r(k-1)}{s-r+1}\binom{n-2}{r-2}. \end{aligned}\tag{48}$$

Using the inequality (1), we get

$$\binom{r(k-1)}{s-r+1} \leqslant (er)^{s-r+1}\binom{k-1}{s-r+1}. \tag{49}$$

Substituting (47) and (49) into (48), we obtain that

$$\begin{aligned}K_s^r(\mathcal{F}_i) \leqslant \left(1+\frac{2r^2k}{n}\right)\binom{k-2}{s-r+1}\binom{n-k+1}{r-1}\\ + 3(er)^{s-r+1}(k-1)\binom{k-1}{s-r+1}\left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right)\binom{n-k}{r-2}. \end{aligned}\tag{50}$$

Under the condition $n \geqslant 4k^2(er)^k$, we find that

$$(k-1)(r-1)\left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right) \leqslant erk. \tag{51}$$

It follows from (50) and (51) that

$$\begin{aligned}K_s^r(\mathcal{F}_i) &\leqslant \left(\binom{k-1}{s-r+1}-\binom{k-2}{s-r}\right)\left(1+\frac{2r^2k}{n}\right)\binom{n-k+1}{r-1} + \frac{3(er)^{s-r+2}k}{n-k+1}\\ &\quad\cdot\binom{n-k+1}{r-1}\binom{k-1}{s-r+1}\\ &\leqslant \binom{k-1}{s-r+1}\binom{n-k+1}{r-1}\left(1+\frac{3(er)^{s-r+2}k}{n-k+1}+\frac{2r^2k}{n}-\frac{s-r+1}{k-1}\right). \end{aligned}\tag{52}$$

Again, under the condition $n \geqslant 4k^2(er)^k$, we also have

$$\frac{3(er)^{s-r+2}k}{n-k+1} + \frac{2r^2k}{n} - \frac{s-r+1}{k-1} < 0. \tag{53}$$

Combining (52) and (53), we obtain

$$K_s^r(\mathcal{F}_i) \leqslant \binom{k-1}{s-r+1}\binom{n-k+1}{r-1} \leqslant K_t^r(\mathcal{F}_{n,s-r+1,1}^{(r)}). \tag{54}$$

This complete the proof of Claim 4.

By Claim 3 and Hall's Marriage Theorem, there exists $I \subset [l]$ such that $|\cup_{i\in I} X_i| < |I|$. By Claim 4, we only need to consider the case when $|X_i| \geqslant l$ or $|X_i| \geqslant |Y_i| \geqslant l-1$ for any $i = 1,\dots,l$. Thus, we may assume that $X_1 = X_2 = \cdots = X_l = Y_1 = Y_2 = \cdots = Y_l = \{x_1, x_2, \dots, x_{l-1}\}$.

Claim 5. For $i \in \{1, 2, \dots, l\}$ and $E \in \mathcal{F}_i$, we have $E \cap \{x_1, x_2, \dots, x_{l-1}\} \neq \emptyset$.

We may assume that there exists $E \in \mathcal{F}_l$ such that $E \cap \{x_1, x_2, \dots, x_{l-1}\} = \emptyset$. Since $\deg_{\mathcal{F}_i}(x_i) \geqslant r(k-1)\binom{n-2}{r-2}$ for $i = 1,\dots,l-1$, there exist disjoint edges $E_1 \in \mathcal{F}_1, \dots, E_{l-1} \in \mathcal{F}_{l-1}$ such that $(\bigcup_{i=1}^{l-1} E_i) \cap E = \emptyset$. Now $E_1,\dots,E_{l-1},E$ forms a rainbow matching in $\{\mathcal{F}_1,\dots,\mathcal{F}_{l-1},\mathcal{F}_l\}$. Since $\mathcal{F}_{l+1},\dots,\mathcal{F}_k$ are all complete r-graphs, one can find a rainbow matching in $\{\mathcal{F}_1,\dots,\mathcal{F}_k\}$, a contradiction. This completes the proof of Claim 5.

Claim 5 implies that $\mathcal{F}_i$ is isomorphic to a subhypergraph of $\mathcal{F}_{n,l-1,1}^{(r)}$ for any $i = 1,\dots,l$. Consequently,

$$K_s^r(\mathcal{F}_i) \leqslant K_s^r(\mathcal{F}_{n,l-1,1}^{(r)}) \leqslant K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$$

for $r \leqslant s \leqslant k+r-2$. Therefore, this completes the proof. □

We are now in a position to prove Theorem 6. Notice that Theorem 6 is implied by Lemma 13 for sufficiently large n. That is, when $n \geqslant 4k^2(er)^k$, if there exists $s \in \{r, r+1, \dots, t\}$ such that $K_s^r(\mathcal{F}_i) > K_s^r(\mathcal{F}_{n,k-1,1}^{(r)})$ for $i = 1,\dots,k$, then the family $\{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$ contains a rainbow matching. In the following proof of Theorem 6, the lower bound on n is improved to $n \geqslant 4k(t-r+2)(er)^{t-r+1}$ for $t \leqslant k+r-2$.

Proof of Theorem 6. We proceed by induction on k. By Lemma 13, the theorem holds for $k = t-r+2$ and $n \geqslant 4k(t-r+2)(er)^{t-r+2}$. Now assume that the theorem holds for $k-1$.

Suppose that there exist $v \in V$ and $i \in [k]$ such that $\{\mathcal{F}_1 - v, \dots, \mathcal{F}_{i-1} - v, \mathcal{F}_{i+1} - v, \dots, \mathcal{F}_k - v\}$ does not contain any rainbow matching. By the induction hypothesis, there exists $j \in [k] \setminus \{i\}$ satisfying $K_s^r(\mathcal{F}_j - v) \leqslant K_s^r(\mathcal{F}_{n-1,k-2,1}^{(r)})$ for all $r \leqslant s \leqslant t$. For $s = r$, we have

$$K_r^r(\mathcal{F}_j) \leqslant K_r^r(\mathcal{F}_j - v) + \binom{n-1}{r-1} \leqslant K_r^r(\mathcal{F}_{n,k-1,1}^{(r)}).$$

For $r+1 \leqslant s \leqslant t$, by the equality (6) we find that

$$K_s^r(\mathcal{F}_j) = K_s^r(\mathcal{F}_j - v) + K_s^r(v, \mathcal{F}_j)$$

$$\begin{aligned}
&\leqslant K_s^r(\mathcal{F}_j - v) + K_{s-1}^r(\mathcal{F}_j - v)\\
&\leqslant K_s^r(\mathcal{F}^{(r)}_{n-1,k-2,1}) + K_{s-1}^r(\mathcal{F}^{(r)}_{n-1,k-2,1})\\
&= K_s^r(\mathcal{F}^{(r)}_{n,k-1,1}).
\end{aligned}$$

Suppose that for any $v \in V$ and $i \in [k]$, $\{\mathcal{F}_1 - v, \ldots, \mathcal{F}_{i-1} - v, \mathcal{F}_{i+1} - v, \ldots, \mathcal{F}_k - v\}$ contains a rainbow matching. This assumption implies that the maximum degree of each $\mathcal{F}_i$ is at most $r(k-1)\binom{n-2}{r-2}$, otherwise we may find a rainbow matching using the greedy algorithm. For $i = 1, 2, \ldots, k$, let X_i be the set of vertices $u \in V$ such that $d_{\mathcal{F}_i}(v) > 2(k-1)\binom{n-2}{r-2}$. By the same argument as in Claim 3 of Lemma 13, we deduce that there exists j such that $|X_j| \leqslant k-1$. Let $\mathcal{M} = \{E_1, \ldots, E_{j-1}, E_{j+1}, \ldots, E_k\}$ be a rainbow matching in $\{\mathcal{F}_1, \ldots, \mathcal{F}_{j-1}, \mathcal{F}_{j+1}, \ldots, \mathcal{F}_k\}$ and let S be the set of vertices covered by $\mathcal{M}$. Note that each s-clique in $\mathcal{F}_j$ has at least $s-r+1$ vertices in S.

We wish to derive an upper bound on the number of s-cliques in $\mathcal{F}_j$. First, we choose a set A of $(s-r+1)$ vertices in S. There are at most $\binom{|S|}{s-r+1}$ choices for A. Then choose an $(r-1)$-element subset B of $[n]$ such that $\mathcal{F}_j[B \cup A]$ is an s-clique of $\mathcal{F}_j$. It can be seen that in the hypergraph $\mathcal{F}_j$, B is a common neighbor of the vertices in A, that is, $B \in N_{\mathcal{F}_j}(v)$ for any $v \in A$. If A is a subset of X_j, the number of choices for B is at most $r(k-1)\binom{n-2}{r-2}$. If A is not a subset of X_j, the number of choices for B is at most $2(k-1)\binom{n-2}{r-2}$. Thus,

$$K_s^r(\mathcal{F}_i) \leqslant r(k-1)\binom{n-2}{r-2}\binom{k-1}{s-r+1} + 2(k-1)\binom{n-2}{r-2}\binom{r(k-1)}{s-r+1}. \tag{55}$$

The inequality (4) yields

$$\binom{r(k-1)}{s-r+1} \leqslant (er)^{s-r+1}\binom{k-1}{s-r+1}. \tag{56}$$

It follows from (55) and (56) that

$$\begin{aligned}
K_s^r(\mathcal{F}_i) &\leqslant \left(r(k-1) + 2(er)^{s-r+1}(k-1)\right)\binom{k-1}{s-r+1}\binom{n-2}{r-2}\\
&\leqslant 3(er)^{s-r+1}(k-1)\binom{k-1}{s-r+1}\binom{n-2}{r-2}.
\end{aligned} \tag{57}$$

Using the inequalities (3) and (5), we see that

$$\begin{aligned}
\binom{n-2}{r-2} &\leqslant \left(\frac{n-r}{n-k-r+2}\right)^{r-2}\binom{n-k}{r-2}\\
&= \left(1 + \frac{k-2}{n-k-r+2}\right)^{r-2}\binom{n-k}{r-2}
\end{aligned}$$

$$\leqslant \left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right)\binom{n-k}{r-2}. \tag{58}$$

Combining (57) and (58), we obtain that

$$K_s^r(\mathcal{F}_i) \leqslant 3(er)^{s-r+1}(k-1)\binom{k-1}{s-r+1}\left(1+\frac{(r-2)^2(k-2)}{n-k-r+2}\right)\binom{n-k}{r-2} \tag{59}$$

Since $n \geqslant 4k(t-r+2)(er)^{t-r+2}$, we find that

$$\frac{(r-2)^2(k-2)}{n-k-r+2} \leqslant \frac{1}{3} \tag{60}$$

and

$$4(er)^{s-r+1}(k-1)\cdot\frac{r-1}{n-k+1} \leqslant 1. \tag{61}$$

In view of (59), (60) and (61), we conclude that

$$K_s^r(\mathcal{F}_i) \leqslant \binom{k-1}{s-r+1}\binom{n-k+1}{r-1} \leqslant K_s^r(\mathcal{F}_{n,k-1,1}^{(r)}).$$

This completes the proof. □

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