# Monochromatic subgraphs in iterated triangulations

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#### Abstract

For integers  $n \ge 0$ , an iterated triangulation  $\operatorname{Tr}(n)$  is defined recursively as follows:  $\operatorname{Tr}(0)$  is the plane triangulation on three vertices and, for  $n \ge 1$ ,  $\operatorname{Tr}(n)$  is the plane triangulation obtained from the plane triangulation  $\operatorname{Tr}(n-1)$  by, for each inner face F of  $\operatorname{Tr}(n-1)$ , adding inside F a new vertex and three edges joining this new vertex to the three vertices incident with F.

In this paper, we show that there exists a 2-edge-coloring of  $\operatorname{Tr}(n)$  such that  $\operatorname{Tr}(n)$  contains no monochromatic copy of the cycle  $C_k$  for any  $k \ge 5$ . As a consequence, the answer to one of two questions asked by Axenovich et al. is negative. We also determine the radius 2 graphs H for which there exists n such that every 2-edge-coloring of  $\operatorname{Tr}(n)$  contains a monochromatic copy of H, extending a result of Axenovich et al. for radius 2 trees.

Mathematics Subject Classifications: 05C55, 05C10, 05D10

# 1 Introduction

For graphs G and H, we write  $G \to H$  if, for any 2-edge-coloring of G, there is a monochromatic copy of H. Otherwise, we write  $G \not\to H$ . We say that H is planar unavoidable if there exists a planar graph G such that  $G \to H$ . Otherwise, we say H is planar avoidable. This notion is introduced and studied in [4].

Deciding if  $G \not\rightarrow H$  is clearly equivalent to asking whether a graph G admits a decomposition (i.e., an edge-decomposition) such that none of the two graphs in the decomposition contains the given graph H. The well-known Four Color Theorem [2,3] (also see [10])

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implies that every planar graph admits a decomposition into two bipartite graphs; so planar unavoidable graphs must be bipartite. A result of Gonçalves [5] says that every planar graph admits a decomposition into two outer planar graphs; so planar unavoidable graphs must be also outer planar. There are a number of interesting results about decomposing planar graphs, see [1, 6-9].

For any positive integer n, let  $P_n$  denote the path on n vertices,  $K_n$  denote the complete graph on n vertices, and  $K_{n,m}$  denote the complete bipartite graph with two parts of sizes n and m. For integer  $n \ge 3$ , we use  $C_n$  to denote the cycle on n vertices. It is shown in [4] that  $P_n$ ,  $C_4$ , and all trees with radius at most 2 are planar unavoidable. This is done by analyzing several sequences of graphs.

In this paper, we investigate one such sequence – the iterated triangulations, which is of particular interest as suggested in [4]. Let  $n \ge 0$  be an integer. An *iterated triangulation*  $\operatorname{Tr}(n)$  is a plane graph defined as follows:  $\operatorname{Tr}(0) \cong K_3$  is the plane triangulation with exactly two 3-faces. For each  $i \ge 0$ , let  $\operatorname{Tr}(i+1)$  be obtained from the plane triangulation  $\operatorname{Tr}(i)$  by adding a new vertex in each of the inner faces of  $\operatorname{Tr}(i)$  and connecting this vertex with edges to the three vertices in the boundary of their respective face. The authors of [4] asked whether for any planar unavoidable graph H there is an integer n such that  $\operatorname{Tr}(n) \to H$ . They also asked whether there exists an integer  $k \ge 3$  such that the even cycle  $C_{2k}$  is planar-unavoidable.

Our first result indicates that a positive answer to one of the above questions implies a negative answer to the other. Let  $H^+$  be the graph with vertex set  $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set  $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_5, v_2v_6\}$ .

**Theorem 1.1.** For all positive integers n,  $\operatorname{Tr}(n) \not\rightarrow C_k$  for  $k \ge 5$ ,  $\operatorname{Tr}(n) \not\rightarrow H^+$ , and  $\operatorname{Tr}(n) \not\rightarrow K_{2,3}$ 

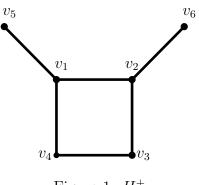


Figure 1:  $H^+$ 

As another direct consequence, we see that if B is a bipartite graph and  $\operatorname{Tr}(n) \to B$ for some n, then every block of B must be a  $C_4$  or  $K_2$ . This can be used to characterize all radius 2 graphs B for which there exists n such that  $\operatorname{Tr}(n) \to B$ , generalizing a result in [4] for radius 2 trees. Before we state this characterization, it is worth mentioning that the authors in [4] show that there is a planar avoidable tree of radius 3. We need some additional notation. A flower  $F_k$  is a collection of k copies of  $C_4$ s sharing a common vertex, which is called the *center*. A k-ary tree of radius 2 means a rooted tree such that every non-leaf vertex has degree k and every leaf has depth 2. A *jellyfish*  $J_k$  is obtained from  $F_k$  and a k-ary tree of radius 2 by identifying the center of  $F_k$  with the root of the k-ary tree. A *bistar*  $B_k$  is obtained from one  $C_4$  and two disjoint  $K_{1,k}$ s by identifying the roots of the  $K_{1,k}$ s with two non-adjacent vertices of  $C_4$ , respectively.

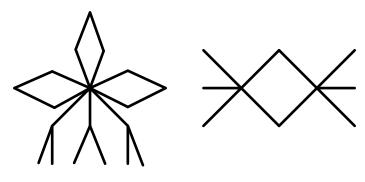


Figure 2:  $J_3$  and  $B_3$ 

**Theorem 1.2.** Let L be a graph with radius 2. Then there exists n such that  $Tr(n) \rightarrow L$  if, and only if, L is a subgraph of a jellyfish or bistar.

We organize this paper as follows. In Section 2, we prove  $\operatorname{Tr}(n) \not\to C_k$  for  $k \ge 5$ and  $\operatorname{Tr}(n) \not\to H^+$  by finding a special edge-coloring scheme for  $\operatorname{Tr}(n)$ . In Section 3, we complete the proof of Theorem 1.1 by using another edge-coloring scheme on  $\operatorname{Tr}(n)$ . From Theorem 1.1, we can derive the following: if L has radius 2 and  $\operatorname{Tr}(n) \to L$  for some n, then L is a subgraph of a jellyfish or bistar. Hence to prove Theorem 1.2, it suffices to show that for any  $k \ge 1$  there exists some n such that  $\operatorname{Tr}(n) \to J_k$  and  $\operatorname{Tr}(n) \to B_k$ . We prove the former statement in Section 4 and the latter one in Section 5 by showing that we can choose n to be linear in k.

# $2 \quad H^+ \text{ and } C_k \text{ for } k \geqslant 5$

In this section, we prove Theorem 1.1 for  $H^+$  and  $C_k$ , with  $k \ge 5$ . First, we describe the 2-edge-coloring of  $\operatorname{Tr}(n)$  that we will use. Let  $\sigma : E(\operatorname{Tr}(n)) \to \{0, 1\}$  be defined inductively for all  $n \ge 1$  as follows:

- (i) Fix an arbitrary triangle T bounding an inner face of Tr(1), and let  $\sigma(e) = 0$  if  $e \in E(T)$  and  $\sigma(e) = 1$  if  $e \in E(\text{Tr}(1)) \setminus E(T)$ .
- (ii) Suppose for some  $1 \leq i < n$ , we have defined  $\sigma(e)$  for all  $e \in E(\operatorname{Tr}(i))$ . We extend  $\sigma$  to  $E(\operatorname{Tr}(i+1))$  as following. Let  $x \in V(\operatorname{Tr}(i)) \setminus V(\operatorname{Tr}(i-1))$  be arbitrary, let  $v_0v_1v_2v_0$  denote the triangle bounding the inner face of  $\operatorname{Tr}(i-1)$  containing x, and fix a labeling so that  $\sigma(xv_1) = \sigma(xv_2)$ .

(iii) Let  $x_j \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$  be such that  $x_j$  is inside the face of Tr(i) bounded by the triangle  $xv_jv_{j+1}x$ , where j = 0, 1, 2 and the subscripts are taken modulo 3. Define  $\sigma(xv_0) = \sigma(x_0v_0) = \sigma(x_2v_0) = \sigma(x_jx)$  for all j = 0, 1, 2, and  $\sigma(xv_1) = \sigma(x_0v_1) = \sigma(x_1v_1) = \sigma(x_1v_2) = \sigma(x_2v_2)$ .

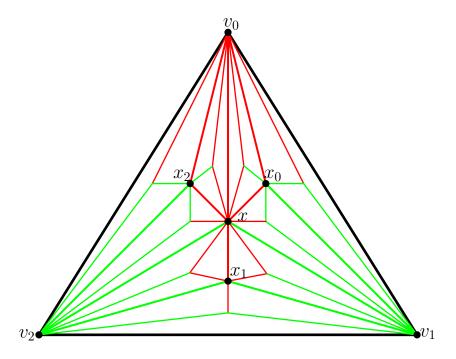


Figure 3: 2 edge-coloring scheme

We now proceed by a sequence of claims to show that  $\sigma$  has no monochromatic  $C_k$  for  $k \ge 5$  nor monochromatic  $H^+$ , thereby proving  $\operatorname{Tr}(n) \not\rightarrow C_k$  for  $k \ge 5$  and  $\operatorname{Tr}(n) \not\rightarrow H^+$ . The first claim is immediate from (iii) so we omit its proof.

Claim 1. For  $1 \leq i \leq n$  and  $x \in V(\operatorname{Tr}(i)) \setminus V(\operatorname{Tr}(i-1)), |\{\sigma(xv) : v \in V(\operatorname{Tr}(i-1))\}| = 2.$ 

**Claim 2.** Let  $v_0v_1v_2v_0$  be a triangle bounding an inner face of  $\operatorname{Tr}(i)$ , where  $0 \leq i < n$ , let  $v \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$  with v inside  $v_0v_1v_2v_0$ . Then, for any  $v_0w \in E(\operatorname{Tr}(n))$  with w inside  $v_0v_1v_2v_0$ ,  $\sigma(v_0w) = \sigma(v_0v)$ .

*Proof.* Let  $v_0 w \in E(\operatorname{Tr}(n))$  with w inside  $v_0 v_1 v_2 v_0$ . Then there exists  $k \ge 0$  with  $i+k+1 \le n$ , such that  $w \in V(\operatorname{Tr}(i+k+1)) \setminus V(\operatorname{Tr}(i+k))$ . We prove Claim 2 by applying induction on k. The basis case is trivial because k = 0 implies w = v.

So assume  $k \ge 1$ . Let  $v_0v_3v_4v_0$  be the triangle bounding an inner face of  $\operatorname{Tr}(i+k-1)$  with w inside  $v_0v_3v_4v_0$ , and let  $v_5 \in V(\operatorname{Tr}(i+k)) \setminus V(\operatorname{Tr}(i+k-1))$  that is inside  $v_0v_3v_4v_0$ . By symmetry, assume w is inside  $v_0v_5v_4v_0$ . By induction hypothesis,  $\sigma(v_0v_5) = \sigma(v_0v)$ .

Suppose  $\sigma(v_4v_5) = \sigma(v_0v_5)$ . Hence by (ii) and (iii),  $\sigma(v_0w) = \sigma(wv_4) = \sigma(v_0v_5)$ . Thus  $\sigma(v_0w) = \sigma(v_0v)$ . Now assume  $\sigma(v_4v_5) \neq \sigma(v_0v_5)$ . Then  $\sigma(v_3v_5) = \sigma(v_0v_5)$  or  $\sigma(v_3v_5) = \sigma(v_4v_5)$ . It follows from (iii) that  $\sigma(v_0w) = \sigma(v_0v_5)$ . Hence,  $\sigma(v_0w) = \sigma(v_0v)$ . **Claim 3.** Let  $v_0v_1v_2v_0$  be a triangle bounding an inner face of  $\operatorname{Tr}(i)$  with  $0 \leq i \leq n-2$ , and let  $v \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$  such that v is inside  $v_0v_1v_2v_0$  and  $\sigma(vv_0) \neq \sigma(vv_1) = \sigma(vv_2)$ . Then for any  $vw \in E(\operatorname{Tr}(n))$  with w inside  $v_0v_1v_2v_0$ ,  $\sigma(vw) = \sigma(vv_0)$ .

*Proof.* To prove Claim 3, let  $\{w_0, w_1, w_2\} \subseteq V(\operatorname{Tr}(i+2)) \setminus V(\operatorname{Tr}(i+1))$  such that  $w_j$  is inside  $vv_jv_{j+1}v$  for j = 0, 1, 2, with subscripts modulo 3. By (ii) and (iii),  $\sigma(vw_0) = \sigma(vw_2) = \sigma(vw_1) = \sigma(vv_0)$ . By Claim 2, there exists some  $j \in \{0, 1, 2\}$  with  $\sigma(vw) = \sigma(vw_j)$ . Hence,  $\sigma(vw) = \sigma(vv_0)$ .

**Claim 4.** Let  $v_0v_1v_2v_0$  be a triangle bounding an inner face of  $\operatorname{Tr}(i)$ , where  $0 \leq i \leq n-2$ , and let  $v \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$  such that v is inside  $v_0v_1v_2v_0$  and  $\sigma(vv_0) \in \{\sigma(vv_1), \sigma(vv_2)\}$ . Then for any  $w \in (N(v) \cap N(v_0)) \setminus \{v_1, v_2\}, \sigma(wv_0) \neq \sigma(wv)$ .

*Proof.* To prove Claim 4, we may assume by symmetry and Claim 1 that  $\sigma(vv_2) \neq \sigma(vv_0) = \sigma(vv_1)$ . Then  $\sigma(wv_0) = \sigma(vv_0)$  by Claim 2, and  $\sigma(wv) = \sigma(vv_2)$  by Claim 3. Hence,  $\sigma(wv_0) \neq \sigma(wv)$ .

**Claim 5.** Suppose upv is a monochromatic path of length two in Tr(n) with  $uv \in E(\text{Tr}(i+1))$  and  $p \in V(\text{Tr}(n)) \setminus V(\text{Tr}(i+1))$ . Then any monochromatic path in Tr(n) between u and v and of the color  $\sigma(up)$  has length at most two.

Proof. Consider any monochromatic path  $P = a_0 a_1 \dots a_r$  of the color  $\sigma(up)$  with  $a_0 = v$ and  $a_r = u$ . First, suppose  $uv \in E(\text{Tr}(0))$ . Let Tr(0) = uvwu and  $x \in V(\text{Tr}(1)) \setminus V(\text{Tr}(0))$ . By Claim 2,  $\sigma(ux) = \sigma(up)$  and  $\sigma(vx) = \sigma(vp)$ ; so  $\sigma(xu) = \sigma(xv)$ . Thus, by (i),  $\sigma(wx) = \sigma(wu) = \sigma(wv) \neq \sigma(xu)$ . Let  $v_0v_1 \dots v_n$  be a path in Tr(n) with  $v_0 = w$ ,  $v_1 = x$  and for  $1 \leq i \leq n$ ,  $v_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$  is inside  $v_{i-1}uvv_{i-1}$ . By (ii) and (iii),  $\sigma(v_iu) = \sigma(v_iv) = \sigma(vx)$  for  $1 \leq i \leq n$ , and  $\sigma(v_iv_{i+1}) = \sigma(xw)$  for  $0 \leq i \leq n-1$ . By planarity, P is contained in the closed region bounded by uvwu. So either P = uvor there exists some  $1 \leq k \leq r-1$  such that  $a_k \in \{v_0, \dots, v_n\}$ . We may assume the latter case occurs. If  $\{a_{k-1}, a_{k+1}\} = \{u, v\}$ , then r = 2. Hence without loss of generality, let  $a_{k-1} \notin \{u, v\}$ . Then by Claim 2 and Claim 3,  $\sigma(a_{k-1}a_k) = \sigma(v_iv_{i+1}) \neq \sigma(pu)$  for  $i \in \{0, 1, \dots, n-1\}$ , a contradiction. Hence  $r \leq 2$ . We remark that this paragraph also shows that such uv in E(Tr(0)) cannot be in a monochromatic  $C_4$ .

Thus, we may assume  $uv \notin E(\text{Tr}(0))$ . By symmetry, we may assume that  $v \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$  for some  $0 \leq i < n$  and v is inside the triangle  $u_1 u_2 u_3 u_1$  bounding an inner face of Tr(i) and  $u_1 = u$ . By Claim 4,  $\sigma(u_1 v) \neq \sigma(u_2 v) = \sigma(u_3 v)$ .

If  $a_1$  is inside  $vu_2u_3v$  then there exists  $1 \leq k < r$  such that  $a_k$  is inside  $vu_3u_2v$  and  $a_{k+1} \in \{u_2, u_3\}$ ; so by Claim 2,  $\sigma(a_ka_{k+1}) = \sigma(vu_2) = \sigma(vu_3) \neq \sigma(u_1v) = \sigma(pu)$ , a contradiction.

Therefore, suppose that  $P \neq uv$ , by symmetry, we may assume that  $a_1$  is inside  $u_1vu_2u_1$ . Let  $v_0 = u_2$  and let  $v_1v_2\ldots v_{n-i-1}$  be the path in  $\operatorname{Tr}(n)$  such that, for  $1 \leq \ell \leq n-i-1$ ,  $v_\ell \in V(\operatorname{Tr}(i+\ell+1)) \setminus V(\operatorname{Tr}(i+\ell))$  is inside  $u_1v_{\ell-1}vu_1$ .

By (ii) and (iii),  $\sigma(v_{\ell}u_1) = \sigma(v_{\ell}v) = \sigma(u_1v)$  for  $1 \leq \ell \leq n-i-1$ , and  $\sigma(v_{\ell}v_{\ell+1}) = \sigma(vu_2) \neq \sigma(vu_1)$  for  $0 \leq \ell \leq n-i-2$ . If  $a_1$  is inside  $v_{\ell}v_{\ell+1}vv_{\ell}$  for some  $\ell$  with  $0 \leq \ell \leq n-i-2$ , then exists  $1 \leq k \leq r$  such that  $a_k$  is inside  $v_{\ell}v_{\ell+1}vv_{\ell}$  and  $a_{k+1} \in \{v_{\ell}, v_{\ell+1}\}$ ;

so by Claim 3  $\sigma(a_k a_{k+1}) = \sigma(v_\ell v_{\ell+1})$ , a contradiction. So  $a_1 = v_\ell$  for some  $\ell$  with  $1 \leq \ell \leq n-i-1$ . Then as  $\sigma(a_1 a_2) = \sigma(u_1 v)$  and by Claim 3, we have  $a_2 = u_1$ . Therefore, r = 2, proving Claim 5.

**Claim 6.** If  $C_k$  is monochromatic in Tr(n) then  $k \leq 4$ .

Proof. Let  $C_k = a_1 a_2 \dots a_k a_1$  be a monochromatic cycle in  $\operatorname{Tr}(n)$ . By (i),  $E(C_k) \not\subseteq E(\operatorname{Tr}(0))$ . So we may assume that there exists some  $1 \leq i \leq k$  such that  $a_{i+1} \in V(\operatorname{Tr}(\ell + 1)) \setminus V(\operatorname{Tr}(\ell))$  is inside the triangle  $a_i uva_i$  which bounds an inner face of some  $\operatorname{Tr}(\ell)$ . We may further assume that  $\ell \leq n-2$ , as otherwise, we could consider  $\operatorname{Tr}(n+1)$  instead of  $\operatorname{Tr}(n)$ .<sup>1</sup>

Suppose  $\sigma(a_i a_{i+1}) \in {\sigma(a_{i+1}u), \sigma(a_{i+1}v)}$ . By symmetry, we may assume  $\sigma(a_i a_{i+1}) = \sigma(a_{i+1}u)$ . Then  $a_{i+2} = u$  by Claim 3. Hence, by Claim 5, any monochromatic path in  $C_k$  between  $a_i$  and  $a_{i+2} = u$  has length at most 2. So  $k \leq 4$ .

Thus, we may assume  $\sigma(a_i a_{i+1}) \notin \{\sigma(a_{i+1}u), \sigma(a_{i+1}v)\}$ ; hence,  $\sigma(a_{i+1}u) = \sigma(a_{i+1}v)$ . Let  $w \in V(\operatorname{Tr}(\ell+2)) \setminus V(\operatorname{Tr}(\ell+1))$  be inside the triangle  $a_i u a_{i+1} a_i$ . By (ii) and (iii),  $\sigma(wa_i) = \sigma(wa_{i+1}) = \sigma(a_i a_{i+1})$ . Hence, by Claim 5, the monochromatic path  $C_k - a_i a_{i+1}$ in  $\operatorname{Tr}(n)$  of the color  $\sigma(a_i a_{i+1}) = \sigma(wa_i)$  has length at most 2; so k = 3.

Claim 7. There is no monochromatic  $H^+$  in Tr(n).

Proof. Suppose that there is a monochromatic copy of  $H^+$  on  $\{v_i : 1 \leq i \leq 6\}$  in which  $v_1v_2v_3v_4v_1$  is a 4-cycle and  $v_1v_5, v_2v_6$  are edges. If  $v_1v_2 \in E(\text{Tr}(0))$ , then  $v_1v_2$  satisfies the conditions of Claim 5 and by the footnote from the proof of Claim 5, there is no monochromatic  $C_4$  containing  $v_1v_2$ , a contradiction. So  $v_1v_2 \notin E(\text{Tr}(0))$ . By symmetry, we may assume that  $v_2 \in V(\text{Tr}(i+1)) \setminus V(\text{Tr}(i))$  for some i and that  $v_1uwv_1$  is the triangle bounding the inner face of Tr(i) containing  $v_2$ . Again as before we may assume that  $0 \leq i \leq n-2$ .

If  $\sigma(v_2u) = \sigma(v_2w)$ , then there exists some  $p \in V(\operatorname{Tr}(n)) \setminus V(\operatorname{Tr}(i+1))$  such that  $v_1pv_2$ has the same color as  $\sigma(v_1v_2)$ . But  $v_1v_4v_3v_2$  is a monochromatic path of length 3 in  $\operatorname{Tr}(n)$ between  $v_1$  and  $v_2$  and of the color  $\sigma(v_1v_2)$ , a contradiction to Claim 5.

Hence,  $\sigma(v_1v_2) \in \{\sigma(v_2u), \sigma(v_2w)\}$  and by symmetry, we may assume  $\sigma(v_1v_2) = \sigma(v_2u)$ . Then by Claim 1,  $\sigma(v_1v_2) \neq \sigma(v_2w)$  and thus  $\sigma(v_2v_3) = \sigma(v_2v_6) \neq \sigma(v_2w)$ . This shows  $w \notin \{v_3, v_6\}$ . So there exists  $y \in \{v_3, v_6\} \setminus \{u, w\}$ . By Claim 3,  $\sigma(v_2y) = \sigma(v_2w)$ , a contradiction.

This completes the proof of Theorem 1.1 for  $H^+$  and  $C_k$ , with  $k \ge 5$ .

## 3 Monochromatic $K_{2,3}$

In this section, we prove Theorem 1.1 for  $K_{2,3}$  using a different coloring scheme on Tr(n) described below. Let  $\sigma : E(\text{Tr}(n)) \to \{0,1\}$  be defined inductively as follows:

<sup>&</sup>lt;sup>1</sup>This is fair because  $\operatorname{Tr}(n+1) \not\rightarrow C_k$  implies  $\operatorname{Tr}(n) \not\rightarrow C_k$ .

- (i) Fix a triangle T bounding an inner face of Tr(1), and let  $\sigma(e) = 0$  if  $e \in E(T)$  and  $\sigma(e) = 1$  if  $e \in E(\text{Tr}(1)) \setminus E(T)$ .
- (ii) Suppose for some  $1 \leq i < n$ , we have defined  $\sigma(e)$  for all  $e \in E(\text{Tr}(i))$ . We now extend  $\sigma$  to E(Tr(i+1)). Let  $x \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$  be arbitrary, let  $v_0v_1v_2v_0$  denote the triangle bounding the inner face of Tr(i-1) containing x, with  $v_0, v_1, v_2$  on the triangle in clockwise order, and let  $\sigma(xv_1) = \sigma(xv_2)$ .
- (iii) Let  $x_j \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$  such that  $x_j$  is inside the face of  $\operatorname{Tr}(i)$  bounded by the triangle  $xv_jv_{j+1}x$ , where j = 0, 1, 2 and the subscripts are taken modulo 3. Define  $\sigma(v_0x) = \sigma(v_0x_0) = \sigma(v_0x_2) = \sigma(xx_2) = \sigma(x_1v_1)$ , and  $\sigma(v_2x) = \sigma(v_2x_1) = \sigma(v_2x_2) = \sigma(xx_1) = \sigma(xx_0) = \sigma(x_0v_1)$ .

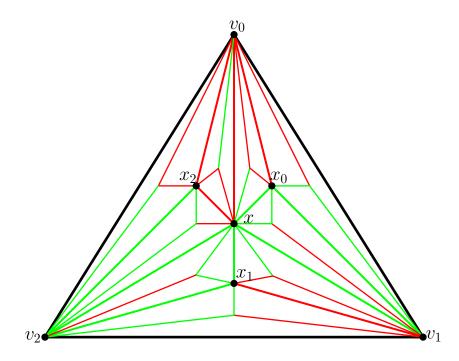


Figure 4: 2 edge-coloring scheme

Note that in (ii) we have  $|\{\sigma(xv_j) : j = 0, 1, 2\}| = 2$  and that in (iii) we have  $\sigma(x_jv_j) \neq \sigma(x_jv_{j+1})$  for j = 0, 1, 2. Hence, inductively, we have

- (1) For  $1 \leq i \leq n$  and  $x \in V(\operatorname{Tr}(i)) \setminus V(\operatorname{Tr}(i-1)), |\{\sigma(xv) : v \in V(\operatorname{Tr}(i-1))\}| = 2.$
- (2) If  $x_1x_2x_3x_1$  is a triangle which bounds an inner face of  $\operatorname{Tr}(i)$  for some  $1 \leq i \leq n-2$ , and if  $x \in V(\operatorname{Tr}(n)) \setminus V(\operatorname{Tr}(i+1))$  is inside  $x_1x_2x_3x_1$  with  $xx_1, xx_2 \in E(\operatorname{Tr}(n))$ , then  $\sigma(xx_1) \neq \sigma(xx_2)$ .

These two claims are straightforward so we omit their proofs.

(3) For any  $x_1x_2 \in E(\operatorname{Tr}(n)), |\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 0\}| \leq 2$  and  $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 1\}| \leq 2.$ 

First, suppose  $x_1x_2 \in E(\text{Tr}(0))$ . Then by (i) and (2),  $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 0\}| \leq 1$  and  $|\{x \in N(x_1) \cap N(x_2) : \sigma(xx_1) = \sigma(xx_2) = 1\}| \leq 1$ .

So we may assume that  $x_1vwx_1$  bounds an inner face of  $\operatorname{Tr}(i)$  and  $x_2 \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$  inside  $x_1vwx_1$ . Let  $v_1 \in \operatorname{Tr}(i+2)$  be inside  $x_1vx_2x_1$  and  $w_1 \in \operatorname{Tr}(i+2)$  be inside  $x_1wx_2x_1$ . By (iii),  $\sigma(w_1x_1) \neq \sigma(w_1x_2)$  or  $\sigma(v_1x_1) \neq \sigma(v_1x_2)$ . By (2), for any  $x \in V(\operatorname{Tr}(n)) \setminus V(\operatorname{Tr}(i+2))$  inside  $x_1vwx_1$  with  $xx_1, xx_2 \in E(\operatorname{Tr}(n))$ , we have  $\sigma(xx_1) \neq \sigma(xx_2)$ . Hence, if (3) fails, then we may assume by symmetry between  $w_1$  and  $v_1$  that  $\sigma(vx_1) = \sigma(vx_2) = \sigma(wx_1) = \sigma(wx_2) = \sigma(v_1x_1) = \sigma(v_1x_2)$ , and  $\sigma(w_1x_1) \neq \sigma(w_1x_2)$ . Then, by (1),  $\sigma(x_1x_2) \neq \sigma(x_2v) = \sigma(x_2w)$ . Now by (iii), at least one of the two edges  $v_1x_1$  and  $v_1x_2$  has the same color as  $x_1x_2$ , a contradiction. This proves (3).

(4) If  $x_1x_2x_3x_4x_1$  is a 4-cycle in  $\operatorname{Tr}(n)$ , then  $x_1x_3 \in E(\operatorname{Tr}(n))$  or  $x_2x_4 \in E(\operatorname{Tr}(n))$ .

We may assume that  $\{x_1, x_2, x_3, x_4\} \subseteq V(\operatorname{Tr}(i+1))$  and  $x_j \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$  for some  $0 \leq i < n$  and  $j \in [4]$ . Let uvwu be the triangle bounding an inner face of  $\operatorname{Tr}(i)$  such that  $x_j$  is inside it. Then  $\{x_{j-1}, x_{j+1}\} \subseteq \{u, v, w\}$ , implying that  $x_{j-1}x_{j+1} \in E(\operatorname{Tr}(n))$ .  $\Box$ 

(5) There is no monochromatic  $K_{2,3}$  in Tr(n).

For, suppose  $\operatorname{Tr}(n)$  has a monochromatic copy of  $K_{2,3}$  on  $\{v_1, v_2, v_3, v_4, v_5\}$  with  $v_4v_i, v_5v_i \in E(\operatorname{Tr}(n))$  for all i = 1, 2, 3. Then  $v_4v_5 \notin E(\operatorname{Tr}(n))$  by (3) and, hence, it follows from (4) that  $v_1v_2, v_2v_3, v_3v_1 \in E(\operatorname{Tr}(n))$ . By planarity,  $v_1v_2v_3v_1$  bounds an inner face of  $\operatorname{Tr}(i)$  for some i with  $1 \leq i < n$  and, by the symmetry between  $v_4$  and  $v_5$ , we may assume that  $v_4$  is inside  $v_1v_2v_3v_1$ . Then  $v_4 \in V(\operatorname{Tr}(i+1)) \setminus V(\operatorname{Tr}(i))$ . However, this contradicts (1), as  $\sigma(v_4v_1) = \sigma(v_4v_2) = \sigma(v_4v_3)$ . We have completed the proof of Theorem 1.1.

## 4 Monochromatic $J_k$

In this section we prove that  $Tr(100k) \rightarrow J_k$  holds for any positive integer k.

We need the following result, which is Lemma 9 in [4]. The original statement in [4] states  $Tr(16) \rightarrow C_4$ , but the same proof in [4] actually gives the following stronger version.

**Lemma 4.1.** If xyzx bounds the outer face of Tr(16), then any 2-edge-coloring of Tr(16) gives a monochromatic  $C_4$  that intersects  $\{x, y\}$ .

Note that if the triangle xyzx bounds the outer face of Tr(n) and  $v \in V(Tr(1)) \setminus V(Tr(0))$  then the subgraph of Tr(n) contained in the closed disc bounded by vxyv is isomorphic to Tr(n-1). Hence, the following is an easy consequence of Lemma 4.1.

**Corollary 4.2.** If xyzx bounds the outer face of Tr(17) then any 2-edge-coloring of Tr(17) gives a monochromatic  $C_4$  that intersects  $\{x, y\}$  and avoids z.

#### **Lemma 4.3.** For any positive integer k, $Tr(38k) \rightarrow F_k$

Proof. Let  $\sigma : E(\operatorname{Tr}(38k)) \to \{0,1\}$  be an arbitrary 2-edge coloring. Let uvwu be the triangle bounding the outer face of  $\operatorname{Tr}(38k)$ . Let  $x_0 := w$  and, for  $1 \leq \ell \leq 2k$ , let  $x_l \in V(\operatorname{Tr}(\ell)) \setminus V(\operatorname{Tr}(\ell-1))$  such that  $x_\ell$  is inside  $x_{\ell-1}uvx_{\ell-1}$ . Let  $y_{i,0} := x_i$  for  $i \in \{0, 1, \ldots, 2k-1\}$  and, for  $\ell \in \{1, \ldots, 36k\}$ , let  $y_{i,\ell} \in V(\operatorname{Tr}(i+1+\ell)) \setminus V(\operatorname{Tr}(i+\ell))$  such that  $y_{i,\ell}$  is inside  $y_{i,\ell-1}ux_{i+1}y_{i,\ell-1}$ .

Suppose for each  $0 \leq i \leq 2k - 1$  there exists a monochromatic  $C_4$  inside  $x_i u x_{i+1} x_i$  that contains u and avoids  $x_i$ . By pigeonhole principle, at least k of these  $C_4$ s are of the same color, which form a monochromatic  $F_k$  centered at u.

Hence, we may assume that there exists some  $i \in \{0, 1, \ldots, 2k - 1\}$  such that no monochromatic  $C_4$  inside  $x_i u x_{i+1} x_i$  contains u and avoids  $x_i$ . Since  $i \leq 2k - 1$ ,  $x_i u x_{i+1} x_i$  bounds the outer face of a Tr(36k) that is contained in Tr(38k).

Now for each  $h \in \{0, 1, ..., 2k - 1\}$ , we view the region enclosed by  $u, x_{i+1}$  and  $y_{i,18h}$  without the closed region enclosed by  $u, x_{i+1}$  and  $y_{i,18(h+1)}$  as a Tr(17). Note that these copies of Tr(17) share  $u, x_{i+1}$  as the only common vertices. Taking  $y_{i,18h}$  to be the vertex z in Corollary 4.2, we conclude from Corollary 4.2 that there is a monochromatic  $C_4$  in the Tr(17). We denote this  $C_4$  by  $G_h$ . Then  $x_{i+1} \in V(G_h)$  and  $\{u, y_{i,18h}\} \cap V(G_h) = \emptyset$ . By pigeonhole principle, at least k of these  $C_{48}$  are of the same color, which clearly form a monochromatic  $F_k$  centered at  $x_{i+1}$ .

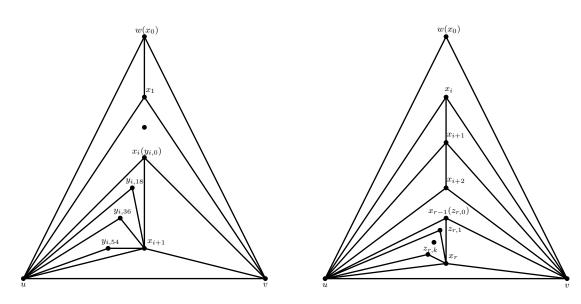


Figure 5: Lemma 4.3

Figure 6: Lemma 4.4

**Lemma 4.4.** Let k be a positive integer and let uvwu bound the outer face of Tr(9k+2). Suppose  $\sigma : E(\text{Tr}(9k+2)) \rightarrow \{0,1\}$  is a 2-edge-coloring such that  $|\{\sigma(ux) : x \in V(\text{Tr}(9k+2))\}| = 1$  and there is no monochromatic  $C_4$  containing u. Then Tr(9k+2) contains monochromatic  $J_k$  centered at v.

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Proof. Without loss of generality, assume  $\sigma(uv) = 0$ . Then  $\sigma(uy) = 0$  for all  $y \in N(u)$ . Let  $x_0 := w$  and, for  $1 \leq i \leq 8k+1$ , let  $x_i \in V(\operatorname{Tr}(i)) \setminus V(\operatorname{Tr}(i-1))$  such that  $x_i$  is inside  $x_{i-1}uvx_{i-1}$ . Since no monochromatic  $C_4$  in  $\operatorname{Tr}(9k+2)$  contains u, there is at most one  $i \in \{0, 1, 2, \ldots, 8k+1\}$  such that  $\sigma(x_iv) = 0$ . Hence, there exists  $i \in \{0, 1, \ldots, 4k+2\}$  such that  $\sigma(vx_i) = 1$  for  $j \in \{i, i+1, \ldots, i+4k-1\}$ . We now make the following claim.

# **Claim.** The subgraph of Tr(9k+2) contained in the closed disc bounded by $vx_i \dots x_{i+3k-1}v$ has a monochromatic $F_k$ of color 1 and centered at v, which we denote by $F_v$ .

To show this, it suffices to show that for each r with  $0 \leq r \leq k-1$ , the subgraph of  $\operatorname{Tr}(9k+2)$  inside  $vx_{i+3r}x_{i+3r+1}x_{i+3r+2}v$  (inclusive) contains a monochromatic  $C_4$  of color 1 and containing v, as the union of such  $C_4$  is an  $F_k$  centered at v. So fix an arbitrary r, with  $0 \leq r \leq k-1$ . Note that  $\sigma(x_{i+3r}x_{i+3r+1}) = 1$  or  $\sigma(x_{i+3r+1}x_{i+3r+2}) = 1$ , for  $0 \leq r \leq k-1$ ; for, otherwise,  $x_{i+3r}x_{i+3r+1}x_{i+3r+2}ux_{i+3r}$  is a monochromatic  $C_4$  of color 0 and containing u, a contradiction. Without loss of generality, assume  $\sigma(x_{i+3r}x_{i+3r+1}) = 1$ .

Let  $y \in V(\text{Tr}(i+3r+2)) \setminus V(\text{Tr}(i+3r+1))$  such that y is inside  $x_{i+3r}x_{i+3r+1}vx_{i+3r}$ . If there are two edges in  $\{yx_{i+3r}, yx_{i+3r+1}, yv\}$  of color 0, then one can easily find a monochromatic  $C_4$  of color 0 and containing u, a contradiction. Hence, at least two of  $\{\sigma(yx_{i+3r}), \sigma(yx_{i+3r+1}), \sigma(yv)\}$  are 1. So  $\{y, x_{i+3r}, x_{i+3r+1}, v\}$  induces a subgraph which contains a monochromatic  $C_4$  of color 1. This proves the claim.  $\Box$ 

Note that for  $i + 3k \leq r \leq i + 4k - 1$ ,  $ux_rx_{r-1}u$  bounds the outer face of a  $\operatorname{Tr}(k+1)$ . Let  $z_{r,0} := x_{r-1}$  and, for  $r \in \{i + 3k, i + 3k + 1, \ldots, i + 4k - 1\}$  and  $\ell \in \{1, 2, \ldots, k\}$ , let  $z_{r,\ell} \in V(Tr(r+\ell)) \setminus V(Tr(r+\ell-1))$  such that  $z_{r,\ell}$  is inside  $z_{r,\ell-1}x_ruz_{r,\ell-1}$ . Because  $\sigma(uz_{r,j}) = 0$  (by assumption) and  $\operatorname{Tr}(9k+2)$  has no monochromatic  $C_4$  containing u, there is at most one  $y \in \{z_{r,1}, z_{r,2}, \ldots, z_{r,k}\}$  such that  $\sigma(yx_r) = 0$ . So there exists k-1 vertices in  $\{z_{r,1}, \ldots, z_{r,k}\}$  which together with  $x_r v$  form a monochromatic  $K_{1,k}$  of color 1 centered at  $x_r$ , which we denote by  $H_r$ . Now  $H_{i+3k}, H_{i+3k+1}, \ldots, H_{i+4k-1}$  form a monochromatic k-ary radius 2 tree rooted at v of color 1. This radius 2 tree and  $F_v$  form a monochromatic  $J_k$  of color 1, completing the proof of Lemma 4.4.

Now we are ready to prove the main result of this section, that is  $\operatorname{Tr}(100k) \to J_k$ . Let  $\sigma : E(\operatorname{Tr}(100k)) \to \{0, 1\}$  be arbitrary. We show that  $\sigma$  always contains a monochromatic  $J_k$ . By Lemma 4.3,  $\operatorname{Tr}(76k)$  contains monochromatic copy of  $F_{2k}$ , say F, and, without loss of generality, assume it is of color 1. Let the  $C_4$ s in F be  $xa_{i,1}a_{i,2}a_{i,3}x$  for  $i \in [2k]$ . For  $i \in [2k]$ , let  $b_i \in V(\operatorname{Tr}(76k+1)) \setminus V(\operatorname{Tr}(76k))$  such that  $b_i$  is inside  $xa_{i,1}a_{i,2}a_{i,3}x$  and  $a_{i,1}a_{i,2}b_ia_{i,1}$  bounds an inner face of  $\operatorname{Tr}(76k+1)$ . Let  $A_i$  be the family of all vertices  $a \in N(a_{i,1})$  inside  $a_{i,1}a_{i,2}b_ia_{i,1}$  and satisfying  $\sigma(aa_{i,1}) = 1$ .

(1) There exists some  $i \in \{k+1, k+2, \dots, 2k\}$  such that  $|A_i| < k$ .

Otherwise, suppose that  $|A_i| \ge k$  for all  $i \in \{k+1, k+2, \ldots, 2k\}$ . Then let  $Z_i := \{z_{i,1}, z_{i,2}, \ldots, z_{i,k-1}\} \subseteq A_i$ . Now, for each  $i \in \{k+1, \ldots, 2k\}, \{x, a_{i,1}\} \cup Z_i$  induces a graph containing a monochromatic  $K_{1,k}$ . Those  $K_{1,k}$ s form a monochromatic radius-two

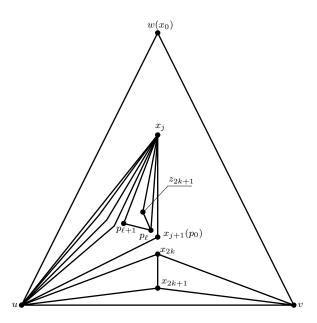


Figure 7

*k*-ary tree of color 1 and rooted at x, which we denote by  $T_x$ . The k four-cycles  $xa_{i,1}a_{i,2}a_{i,3}x$  for  $i \in [k]$  form a monochromatic  $F_k$ . Now  $F_k \cup T_x$  is a monochromatic  $J_k$ .  $\Box$ 

Let  $u := a_{i,1}$ . By (1), there exists an edge  $vw \in \text{Tr}(78k)$  such that uvwu bounds an inner face of Tr(78k) and  $\sigma(uy) = 0$  for any  $y \in N(u)$  in the closed disc bounded by uvwu.

Let G be the subgraph of Tr(78k) contained in the closed disc bounded by uvwu (see Figure 5). Clearly G is isomorphic to a copy of Tr(22k). In the rest of the proof, we should only discuss the graph G and all Tr(i) will be referred to this copy of Tr(22k). Let  $x_0 := w$  and for  $i \in [4k]$ , let  $x_i \in V(\text{Tr}(i)) \setminus V(\text{Tr}(i-1))$  such that  $x_i$  is inside  $ux_{i-1}vu$ .

(2) G contains a monochromatic copy of  $F_k$ , say F', which has color 0 and center u and is disjoint from the union of closed regions bounded by  $ux_ix_{i+1}u$  over all  $0 \le i \le 2k-1$ .

If for each  $i \in \{k, k+1, \ldots, 2k-1\}$  there exists a monochromatic  $C_4$  inside  $ux_{2i}x_{2i+1}u$  and containing u, then these k monochromatic  $C_4$ s of color 0 form a desired monochromatic  $F_k$  centered at u and thus (2) holds. Otherwise, since  $ux_{2i}x_{2i+1}u$  bounds the outer face of a Tr(9k+2), it follows from Lemma 4.4 that there exists a monochromatic  $J_k$  in G.  $\Box$ 

For  $j \in \{0, 1, ..., 2k-1\}$ , let  $B_j$  be the family of all vertices  $x \in N(x_j)$  inside  $ux_jx_{j+1}u$ and satisfying  $\sigma(xx_j) = 0$ .

(3) There exists some  $j \in \{0, 1, \dots, k-1\}$  such that  $|B_j| < k$ .

Suppose to the contrary that there exist subsets  $Z_j \subseteq B_j$  of size k for all  $j \in \{0, 1, \dots, k-1\}$ . Then each  $Z_j \cup \{u, x_j\}$  induces a graph containing a monochromatic  $K_{1,k}$  which is

centered at  $x_j$  and has color 0. These  $K_{1,k}$ s together with F' form a monochromatic  $J_k$  of color 0. This proves (3).  $\Box$ 

Let  $p_0 := x_{j+1}$  and for  $1 \leq \ell \leq 4k$ , let  $p_\ell \in V(\operatorname{Tr}(j+\ell+1)) \setminus V(\operatorname{Tr}(j+\ell))$  such that  $p_\ell$  is inside  $ux_jp_{\ell-1}u$ . By (3), there exists some  $0 \leq \ell \leq 4k-1$  such that  $\sigma(zx_j) = 1$  for any  $z \in N(x_j)$  in the closed disc bounded by  $x_jp_\ell p_{\ell+1}x_j$ .

(4) There is a monochromatic  $F_k$  inside  $x_j p_\ell p_{\ell+1} x_j$ , say F'', with color 1 and center  $x_j$ .

Let  $z_0 := p_{\ell+1}$  and for  $s \in [2k+1]$ , let  $z_s \in V(\operatorname{Tr}(j+\ell+s+2)) \setminus V(\operatorname{Tr}(j+\ell+s+1))$  such that  $z_s$  is inside  $x_j z_{s-1} p_\ell x_j$ . Note that each  $x_j z_{2s} z_{2s+1} x_j$  bounds a  $\operatorname{Tr}(9k+3)$ . If for each  $s \in [k]$  there exists a monochromatic  $C_4$  of color 1 inside  $x_j z_{2s} z_{2s+1} x_j$  and containing  $x_j$ , then these monochromatic copies of  $C_4$  form the desired monochromatic  $F_k$  centered at  $x_j$ . Otherwise, it follows from Lemma 4.4 that there exists a monochromatic  $J_k$ .  $\Box$ 

As  $|B_j| < k$ , there exists a subset  $A \subseteq \{p_1, p_2, \ldots, p_{4k}\}$  of size 2k such that  $\sigma(\alpha x_j) = 1$ for each  $\alpha \in A$  and moreover, there is no neighbors of A belonging to V(F''). Let  $A := \{\alpha_1, \ldots, \alpha_{2k}\}$ . Note that for each  $h \in [2k]$ , we have  $\sigma(\alpha_h u) = 0$  and  $\sigma(\alpha_h x_j) = 1$ .

It is easy to see that there exist pairwise disjoint sets  $N_h \subseteq N(\alpha_h)$  of size 2k for  $h \in [2k]$ . Then there exists  $M_h \subseteq N_h$  such that  $|M_h| = k - 1$  and  $\sigma(x\alpha_h)$  is the same for all  $x \in M_h$ . This gives 2k monochromatic copies of  $K_{1,k-1}$  with centers  $\alpha_h$  for  $h \in [2k]$ . At least k of them (say with centers  $\alpha_h$  for  $h \in [k]$ ) have the same color. If this color is 0, these copies together with  $\{u\alpha_h : h \in [k]\}$  and F' give a monochromatic  $J_k$  with color 0 and center u. Otherwise, this color is 1. Then these copies together with  $\{x_j\alpha_h : h \in [k]\}$  and F'' give a monochromatic  $J_k$  with color 1 and center u. This proves  $\operatorname{Tr}(100k) \to J_k$ .

### 5 Monochromatic bistar

In this section we prove  $Tr(6k + 30) \rightarrow B_k$ . We first establish the following lemma.

**Lemma 5.1.** Let uvwu be the triangle bounding the outer face of Tr(k + 10). Let  $\sigma$ :  $E(\text{Tr}(k + 10)) \rightarrow \{0, 1\}$  such that  $|\{\sigma(ux) : x \in V(\text{Tr}(k + 10))\}| = 1$  and there is no monochromatic  $C_4$  containing u. Then Tr(k + 10) contains a monochromatic  $B_k$ .

*Proof.* Without loss of generality, let  $\sigma(uv) = 0$ . Let  $x_0 := w$  and, for  $i \in [6]$ , let  $x_i \in V(\operatorname{Tr}(i)) \setminus V(\operatorname{Tr}(i-1))$  such that  $x_i$  is inside  $uvx_{i-1}u$ .

Since  $\operatorname{Tr}(k+10)$  has no monochromatic  $C_4$  containing u, we see that  $|\{0 \leq i \leq 6 : \sigma(vx_i) = 0\}| \leq 1$ . So there exists some  $i \in \{0, 1, 2, 3, 4\}$  such that  $\sigma(vx_i) = \sigma(vx_{i+1}) = \sigma(vx_{i+2}) = 1$ . We have either  $\sigma(x_ix_{i+1}) = 1$  or  $\sigma(x_{i+1}x_{i+2}) = 1$ ; as otherwise  $ux_ix_{i+1}x_{i+2}u$  is a monochromatic  $C_4$  of color 0 and containing u, a contradiction. We consider two cases.

Case 1.  $\sigma(x_i x_{i+1}) = \sigma(x_{i+1} x_{i+2}).$ 

In this case, we have  $\sigma(x_i x_{i+1}) = \sigma(x_{i+1} x_{i+2}) = 1$ . So  $x_i x_{i+1} x_{i+2} v x_i$  is a monochromatic  $C_4$  of color 1. Let  $y_0 := x_{i+1}$  and for  $\ell \in [k+1]$ , let  $y_\ell \in V(\operatorname{Tr}(i+1+\ell)) \setminus V(\operatorname{Tr}(i+\ell))$ 

such that  $y_{\ell}$  is inside  $uy_{\ell-1}x_iu$ . Similarly let  $z_0 := x_{i+1}$  and for  $\ell \in [k+1]$ , let  $z_{\ell} \in V(\operatorname{Tr}(i+2+\ell)) \setminus V(\operatorname{Tr}(i+1+\ell))$  such that  $z_{\ell}$  is inside  $uz_{\ell-1}x_{i+2}u$ .

Since  $\operatorname{Tr}(k+10)$  has no monochromatic  $C_4$  containing u, this shows that  $|\{\ell \in [k+1] : \sigma(x_i y_\ell) = 0\}| \leq 1$  and  $|\{\ell \in [k+1] : \sigma(x_{i+2} z_\ell) = 0\}| \leq 1$ . Therefore, there exist  $Y \subseteq \{y_\ell : \ell \in [k+1]\}$  and  $Z \subseteq \{z_\ell : \ell \in [k+1]\}$  such that |Y| = |Z| = k,  $\sigma(yx_i) = 1$  for each  $y \in Y$  and  $\sigma(zx_{i+2}) = 1$  for each  $z \in Z$ . Hence,  $\operatorname{Tr}(k+10)$  has two monochromatic  $K_{1,k}$ s of color 1 with centers  $x_i, x_{i+1}$  and leave sets Y, Z, respectively. These two  $K_{1,k}$ s together with  $vx_ix_{i+1}x_{i+2}v$  form a monochromatic  $B_k$  of color 1.

Case 2.  $\sigma(x_i x_{i+1}) \neq \sigma(x_{i+1} x_{i+2}).$ 

Without loss of generality, let  $\sigma(x_i x_{i+1}) = 0$  and  $\sigma(x_{i+1} x_{i+2}) = 1$ . Let  $y \in V(\operatorname{Tr}(i+2)) \setminus V(\operatorname{Tr}(i+1))$  be inside  $ux_i x_{i+1} u$ . Because  $\sigma(uy) = 0$  and  $\operatorname{Tr}(k+10)$  has no monochromatic  $C_4$  containing  $u, \sigma(yx_i) = \sigma(yx_{i+1}) = 1$ . Therefore,  $yx_{i+1}vx_i y$  is a monochromatic  $C_4$  of color 1. Let  $y_0 := y$  and, for  $\ell \in [k+1]$ , let  $y_\ell \in V(\operatorname{Tr}(i+2+\ell)) \setminus V(\operatorname{Tr}(i+1+\ell))$  such that  $y_\ell$  is inside  $uy_{\ell-1}x_i u$ . Let  $z_0 := y$  and, for  $\ell \in [k+1]$ , let  $z_\ell \in V(\operatorname{Tr}(i+2+\ell)) \setminus V(\operatorname{Tr}(i+1+\ell))$  such that  $z_\ell$  is inside  $uz_{\ell-1}x_{i+1}u$ .

The remaining proof is similar as in Case 1. We observe that  $|\{\ell \in [k+1] : \sigma(x_i y_\ell) = 0\}| \leq 1$  and  $|\{\ell \in [k+1] : \sigma(x_{i+1} z_\ell) = 0\}| \leq 1$ . Therefore, there exist  $Y \subseteq \{y_\ell : \ell \in [k+1]\}$  and  $Z \subseteq \{z_\ell : \ell \in [k+1]\}$  such that |Y| = |Z| = k,  $\sigma(yx_i) = 1$  for  $y \in Y$ , and  $\sigma(zx_{i+1}) = 1$  for  $z \in Z$ . Hence, Tr(k+10) has two monochromatic  $K_{1,k}$ s of color 1 with centers  $x_i, x_{i+1}$  and leave sets Y, Z, respectively. These two  $K_{1,k}$ s together with  $yx_{i+1}vx_iy$  form a monochromatic  $B_k$  of color 1. This proves Lemma 5.1.

We are ready to prove  $\operatorname{Tr}(6k + 30) \to B_k$ . Let  $\sigma : E(\operatorname{Tr}(6k + 30)) \to \{0, 1\}$ . By Lemma 4.1, each copy of  $\operatorname{Tr}(16)$  with the same outer face as of  $\operatorname{Tr}(6k + 30)$  contains a monochromatic  $C_4$ , say  $u_1u_2u_3u_4u_1$  of color 1 (see Figure 6). For each  $i \in \{1, 3\}$ , let  $v_iw_i$ be an edge in  $\operatorname{Tr}(18)$  such that  $u_iv_iw_iu_i$  is a triangle inside  $u_1u_2u_3u_4u_1$ . Note that  $u_iv_iw_iu_i$ bounds the outer face of a  $\operatorname{Tr}(6k + 12)$ . Let  $A_i$  be the family of all vertices  $x \in N(u_i)$ inside  $u_iv_iw_iu_i$  and satisfying  $\sigma(xu_i) = 1$ . If  $|A_1| \ge k$  and  $|A_3| \ge k$ , then together with the monochromatic 4-cycle  $u_1u_2u_3u_4u_1$ , it is easy to form a monochromatic  $B_k$  of color 1.

Hence by symmetry, we may assume that  $|A_1| < k$ . Then there exists an edge vwin  $\operatorname{Tr}(18 + k)$  such that  $u_1vwu_1$  bounds an inner face of  $\operatorname{Tr}(18 + k)$  and  $\sigma(u_1x) = 0$ for all  $x \in N(u_1)$  in the closed disc bounded by  $u_1vwu_1$ . We may assume that the induced subgraph contained in the closed disc bounded by  $u_1vwu_1$  has a monochromatic  $C_4$  say  $u_1xyzu_1$  (as otherwise, it contains a  $B_k$  by Lemma 5.1). Furthermore, we have  $\{x, y, z\} \subseteq V(\operatorname{Tr}(2k + 28)).$ 

Let  $\{p_0, q_0\} \subseteq V(\operatorname{Tr}(2k+29)) \setminus V(\operatorname{Tr}(2k+28))$  such that both  $xyp_0x$  and  $yzq_0y$  bound two inner faces of  $\operatorname{Tr}(2k+29)$ . For  $\ell \in [3k]$ , let  $p_\ell \in V(\operatorname{Tr}(2k+29+\ell)) \setminus V(\operatorname{Tr}(2k+28+\ell))$ such that  $p_\ell$  is inside  $xp_{\ell-1}yx$ . Similarly, for  $\ell \in [3k]$ , let  $q_\ell \in V(\operatorname{Tr}(2k+29+\ell)) \setminus V(\operatorname{Tr}(2k+28+\ell))$  $(2k+\ell)$  such that  $q_\ell$  is inside  $yq_{\ell-1}zy$ . Moreover, let

$$B_1 := \{ p \in N(x) : p \text{ is inside } xp_0yx \text{ and } \sigma(xp) = 0 \},\$$
$$B_2 := \{ q \in N(z) : q \text{ is inside } yq_0zy \text{ and } \sigma(zq) = 0 \}.$$

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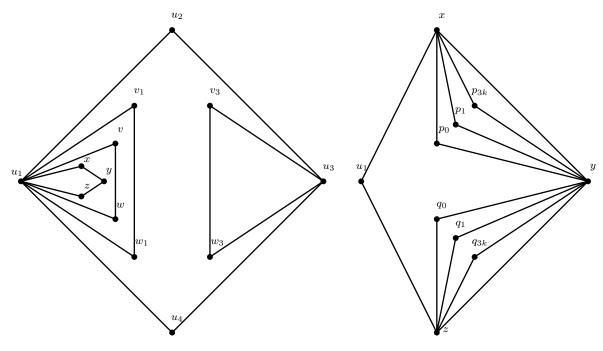


Figure 8

If  $|B_1| \ge k$  and  $|B_2| \ge k$ , we can find two monochromatic  $K_{1,k}$ s of color 0, one inside  $xp_0yx$  rooted at x and one inside  $yq_0zy$  rooted at z; these two  $K_{1,k}$ s and  $u_1xyzu_1$  form a monochromatic  $B_k$  of color 0. So we may assume, without loss of generality, that  $|B_1| < k$ .

Let  $C := \{\ell \in [3k] : \sigma(yp_\ell) = 0\}$ . We claim |C| < k. Suppose to the contrary that  $|C| \ge k$ . Then there is a monochromatic  $K_{1,k}$  with root y and leaves in C of color 0. Since  $\sigma(u_1p) = 0$  for all  $p \in N(u_1)$  inside  $u_1vw$ , there is also a monochromatic  $K_{1,k}$  with root  $u_1$  and leaves inside  $u_1xyzu_1$  of color 0. Now these two  $K_{1,k}$ s and  $u_1xyzu_1$  form a monochromatic  $B_k$  of color 0.

So  $|B_1| < k$  and |C| < k. Then there exist  $p_h, p_s$  with  $h, s \in [3k]$  such that  $\sigma(p_h x) = \sigma(p_h y) = \sigma(p_s x) = \sigma(p_s y) = 1$ . Because  $xp_0p_1x$  bounds an inner face of Tr(2k + 30), it also bounds the outer face of a Tr(4k). As  $|B_1| < k$ , there exists a monochromatic  $K_{1,k}$  of color 1 with the root x and k leaves inside  $xp_0p_1x$ . Similarly, as |C| < k, there exists a monochromatic  $K_{1,k}$  of color 1 with root y and k leavers inside  $yp_0p_1y$ . Now these two  $K_{1,k}s$  and the 4-cycle  $xp_hyp_sx$  form a monochromatic  $B_k$  of color 1. This proves that  $\text{Tr}(6k + 30) \rightarrow B_k$  and thus completes the proof of Theorem 1.2.

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