# Monochromatic subgraphs in iterated triangulations 

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#### Abstract

For integers $n \geqslant 0$, an iterated triangulation $\operatorname{Tr}(n)$ is defined recursively as follows: $\operatorname{Tr}(0)$ is the plane triangulation on three vertices and, for $n \geqslant 1, \operatorname{Tr}(n)$ is the plane triangulation obtained from the plane triangulation $\operatorname{Tr}(n-1)$ by, for each inner face $F$ of $\operatorname{Tr}(n-1)$, adding inside $F$ a new vertex and three edges joining this new vertex to the three vertices incident with $F$.

In this paper, we show that there exists a 2 -edge-coloring of $\operatorname{Tr}(n)$ such that $\operatorname{Tr}(n)$ contains no monochromatic copy of the cycle $C_{k}$ for any $k \geqslant 5$. As a consequence, the answer to one of two questions asked by Axenovich et al. is negative. We also determine the radius 2 graphs $H$ for which there exists $n$ such that every 2-edge-coloring of $\operatorname{Tr}(n)$ contains a monochromatic copy of $H$, extending a result of Axenovich et al. for radius 2 trees.


Mathematics Subject Classifications: 05C55, 05C10, 05D10

## 1 Introduction

For graphs $G$ and $H$, we write $G \rightarrow H$ if, for any 2-edge-coloring of $G$, there is a monochromatic copy of $H$. Otherwise, we write $G \nrightarrow H$. We say that $H$ is planar unavoidable if there exists a planar graph $G$ such that $G \rightarrow H$. Otherwise, we say $H$ is planar avoidable. This notion is introduced and studied in [4].

Deciding if $G \nrightarrow H$ is clearly equivalent to asking whether a graph $G$ admits a decomposition (i.e., an edge-decomposition) such that none of the two graphs in the decomposition contains the given graph $H$. The well-known Four Color Theorem [2,3] (also see [10])

[^0]implies that every planar graph admits a decomposition into two bipartite graphs; so planar unavoidable graphs must be bipartite. A result of Gonçalves [5] says that every planar graph admits a decomposition into two outer planar graphs; so planar unavoidable graphs must be also outer planar. There are a number of interesting results about decomposing planar graphs, see [1,6-9].

For any positive integer $n$, let $P_{n}$ denote the path on $n$ vertices, $K_{n}$ denote the complete graph on $n$ vertices, and $K_{n, m}$ denote the complete bipartite graph with two parts of sizes $n$ and $m$. For integer $n \geqslant 3$, we use $C_{n}$ to denote the cycle on $n$ vertices. It is shown in [4] that $P_{n}, C_{4}$, and all trees with radius at most 2 are planar unavoidable. This is done by analyzing several sequences of graphs.

In this paper, we investigate one such sequence - the iterated triangulations, which is of particular interest as suggested in [4]. Let $n \geqslant 0$ be an integer. An iterated triangulation $\operatorname{Tr}(n)$ is a plane graph defined as follows: $\operatorname{Tr}(0) \cong K_{3}$ is the plane triangulation with exactly two 3 -faces. For each $i \geqslant 0$, let $\operatorname{Tr}(i+1)$ be obtained from the plane triangulation $\operatorname{Tr}(i)$ by adding a new vertex in each of the inner faces of $\operatorname{Tr}(i)$ and connecting this vertex with edges to the three vertices in the boundary of their respective face. The authors of [4] asked whether for any planar unavoidable graph $H$ there is an integer $n$ such that $\operatorname{Tr}(n) \rightarrow H$. They also asked whether there exists an integer $k \geqslant 3$ such that the even cycle $C_{2 k}$ is planar-unavoidable.

Our first result indicates that a positive answer to one of the above questions implies a negative answer to the other. Let $H^{+}$be the graph with vertex set $\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ and edge set $\left\{v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{4}, v_{4} v_{1}, v_{1} v_{5}, v_{2} v_{6}\right\}$.

Theorem 1.1. For all positive integers $n, \operatorname{Tr}(n) \nrightarrow C_{k}$ for $k \geqslant 5, \operatorname{Tr}(n) \nrightarrow H^{+}$, and $\operatorname{Tr}(n) \nrightarrow K_{2,3}$


Figure 1: $H^{+}$
As another direct consequence, we see that if $B$ is a bipartite graph and $\operatorname{Tr}(n) \rightarrow B$ for some $n$, then every block of $B$ must be a $C_{4}$ or $K_{2}$. This can be used to characterize all radius 2 graphs $B$ for which there exists $n$ such that $\operatorname{Tr}(n) \rightarrow B$, generalizing a result in [4] for radius 2 trees. Before we state this characterization, it is worth mentioning that the authors in [4] show that there is a planar avoidable tree of radius 3. We need some additional notation. A flower $F_{k}$ is a collection of $k$ copies of $C_{4}$ s sharing a common
vertex, which is called the center. A $k$-ary tree of radius 2 means a rooted tree such that every non-leaf vertex has degree $k$ and every leaf has depth 2 . A jellyfish $J_{k}$ is obtained from $F_{k}$ and a $k$-ary tree of radius 2 by identifying the center of $F_{k}$ with the root of the $k$-ary tree. A bistar $B_{k}$ is obtained from one $C_{4}$ and two disjoint $K_{1, k} \mathrm{~s}$ by identifying the roots of the $K_{1, k} \mathrm{~S}$ with two non-adjacent vertices of $C_{4}$, respectively.



Figure 2: $J_{3}$ and $B_{3}$

Theorem 1.2. Let $L$ be a graph with radius 2. Then there exists $n$ such that $\operatorname{Tr}(n) \rightarrow L$ if, and only if, $L$ is a subgraph of a jellyfish or bistar.

We organize this paper as follows. In Section 2, we prove $\operatorname{Tr}(n) \nrightarrow C_{k}$ for $k \geqslant 5$ and $\operatorname{Tr}(n) \nrightarrow H^{+}$by finding a special edge-coloring scheme for $\operatorname{Tr}(n)$. In Section 3, we complete the proof of Theorem 1.1 by using another edge-coloring scheme on $\operatorname{Tr}(n)$. From Theorem 1.1, we can derive the following: if $L$ has radius 2 and $\operatorname{Tr}(n) \rightarrow L$ for some $n$, then $L$ is a subgraph of a jellyfish or bistar. Hence to prove Theorem 1.2, it suffices to show that for any $k \geqslant 1$ there exists some $n$ such that $\operatorname{Tr}(n) \rightarrow J_{k}$ and $\operatorname{Tr}(n) \rightarrow B_{k}$. We prove the former statement in Section 4 and the latter one in Section 5 by showing that we can choose $n$ to be linear in $k$.

## $2 \quad \boldsymbol{H}^{+}$and $C_{k}$ for $k \geqslant 5$

In this section, we prove Theorem 1.1 for $H^{+}$and $C_{k}$, with $k \geqslant 5$. First, we describe the 2-edge-coloring of $\operatorname{Tr}(n)$ that we will use. Let $\sigma: E(\operatorname{Tr}(n)) \rightarrow\{0,1\}$ be defined inductively for all $n \geqslant 1$ as follows:
(i) Fix an arbitrary triangle $T$ bounding an inner face of $\operatorname{Tr}(1)$, and let $\sigma(e)=0$ if $e \in E(T)$ and $\sigma(e)=1$ if $e \in E(\operatorname{Tr}(1)) \backslash E(T)$.
(ii) Suppose for some $1 \leqslant i<n$, we have defined $\sigma(e)$ for all $e \in E(\operatorname{Tr}(i))$. We extend $\sigma$ to $E(\operatorname{Tr}(i+1))$ as following. Let $x \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1))$ be arbitrary, let $v_{0} v_{1} v_{2} v_{0}$ denote the triangle bounding the inner face of $\operatorname{Tr}(i-1)$ containing $x$, and fix a labeling so that $\sigma\left(x v_{1}\right)=\sigma\left(x v_{2}\right)$.
(iii) Let $x_{j} \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ be such that $x_{j}$ is inside the face of $\operatorname{Tr}(i)$ bounded by the triangle $x v_{j} v_{j+1} x$, where $j=0,1,2$ and the subscripts are taken modulo 3. Define $\sigma\left(x v_{0}\right)=\sigma\left(x_{0} v_{0}\right)=\sigma\left(x_{2} v_{0}\right)=\sigma\left(x_{j} x\right)$ for all $j=0,1,2$, and $\sigma\left(x v_{1}\right)=$ $\sigma\left(x_{0} v_{1}\right)=\sigma\left(x_{1} v_{1}\right)=\sigma\left(x_{1} v_{2}\right)=\sigma\left(x_{2} v_{2}\right)$.


Figure 3: 2 edge-coloring scheme
We now proceed by a sequence of claims to show that $\sigma$ has no monochromatic $C_{k}$ for $k \geqslant 5$ nor monochromatic $H^{+}$, thereby proving $\operatorname{Tr}(n) \nrightarrow C_{k}$ for $k \geqslant 5$ and $\operatorname{Tr}(n) \nrightarrow H^{+}$. The first claim is immediate from (iii) so we omit its proof.

Claim 1. For $1 \leqslant i \leqslant n$ and $x \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1)),|\{\sigma(x v): v \in V(\operatorname{Tr}(i-1))\}|=2$.
Claim 2. Let $v_{0} v_{1} v_{2} v_{0}$ be a triangle bounding an inner face of $\operatorname{Tr}(i)$, where $0 \leqslant i<n$, let $v \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ with $v$ inside $v_{0} v_{1} v_{2} v_{0}$. Then, for any $v_{0} w \in E(\operatorname{Tr}(n))$ with $w$ inside $v_{0} v_{1} v_{2} v_{0}, \sigma\left(v_{0} w\right)=\sigma\left(v_{0} v\right)$.

Proof. Let $v_{0} w \in E(\operatorname{Tr}(n))$ with $w$ inside $v_{0} v_{1} v_{2} v_{0}$. Then there exists $k \geqslant 0$ with $i+k+1 \leqslant$ $n$, such that $w \in V(\operatorname{Tr}(i+k+1)) \backslash V(\operatorname{Tr}(i+k))$. We prove Claim 2 by applying induction on $k$. The basis case is trivial because $k=0$ implies $w=v$.

So assume $k \geqslant 1$. Let $v_{0} v_{3} v_{4} v_{0}$ be the triangle bounding an inner face of $\operatorname{Tr}(i+k-1)$ with $w$ inside $v_{0} v_{3} v_{4} v_{0}$, and let $v_{5} \in V(\operatorname{Tr}(i+k)) \backslash V(\operatorname{Tr}(i+k-1))$ that is inside $v_{0} v_{3} v_{4} v_{0}$. By symmetry, assume $w$ is inside $v_{0} v_{5} v_{4} v_{0}$. By induction hypothesis, $\sigma\left(v_{0} v_{5}\right)=\sigma\left(v_{0} v\right)$.

Suppose $\sigma\left(v_{4} v_{5}\right)=\sigma\left(v_{0} v_{5}\right)$. Hence by (ii) and (iii), $\sigma\left(v_{0} w\right)=\sigma\left(w v_{4}\right)=\sigma\left(v_{0} v_{5}\right)$. Thus $\sigma\left(v_{0} w\right)=\sigma\left(v_{0} v\right)$. Now assume $\sigma\left(v_{4} v_{5}\right) \neq \sigma\left(v_{0} v_{5}\right)$. Then $\sigma\left(v_{3} v_{5}\right)=\sigma\left(v_{0} v_{5}\right)$ or $\sigma\left(v_{3} v_{5}\right)=\sigma\left(v_{4} v_{5}\right)$. It follows from (iii) that $\sigma\left(v_{0} w\right)=\sigma\left(v_{0} v_{5}\right)$. Hence, $\sigma\left(v_{0} w\right)=\sigma\left(v_{0} v\right)$.

Claim 3. Let $v_{0} v_{1} v_{2} v_{0}$ be a triangle bounding an inner face of $\operatorname{Tr}(i)$ with $0 \leqslant i \leqslant n-2$, and let $v \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ such that $v$ is inside $v_{0} v_{1} v_{2} v_{0}$ and $\sigma\left(v v_{0}\right) \neq \sigma\left(v v_{1}\right)=\sigma\left(v v_{2}\right)$. Then for any $v w \in E(\operatorname{Tr}(n))$ with $w$ inside $v_{0} v_{1} v_{2} v_{0}, \sigma(v w)=\sigma\left(v v_{0}\right)$.

Proof. To prove Claim 3, let $\left\{w_{0}, w_{1}, w_{2}\right\} \subseteq V(\operatorname{Tr}(i+2)) \backslash V(\operatorname{Tr}(i+1))$ such that $w_{j}$ is inside $v v_{j} v_{j+1} v$ for $j=0,1,2$, with subscripts modulo 3. By (ii) and (iii), $\sigma\left(v w_{0}\right)=\sigma\left(v w_{2}\right)=$ $\sigma\left(v w_{1}\right)=\sigma\left(v v_{0}\right)$. By Claim 2, there exists some $j \in\{0,1,2\}$ with $\sigma(v w)=\sigma\left(v w_{j}\right)$. Hence, $\sigma(v w)=\sigma\left(v v_{0}\right)$.

Claim 4. Let $v_{0} v_{1} v_{2} v_{0}$ be a triangle bounding an inner face of $\operatorname{Tr}(i)$, where $0 \leqslant i \leqslant$ $n-2$, and let $v \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ such that $v$ is inside $v_{0} v_{1} v_{2} v_{0}$ and $\sigma\left(v v_{0}\right) \in$ $\left\{\sigma\left(v v_{1}\right), \sigma\left(v v_{2}\right)\right\}$. Then for any $w \in\left(N(v) \cap N\left(v_{0}\right)\right) \backslash\left\{v_{1}, v_{2}\right\}, \sigma\left(w v_{0}\right) \neq \sigma(w v)$.

Proof. To prove Claim 4, we may assume by symmetry and Claim 1 that $\sigma\left(v v_{2}\right) \neq$ $\sigma\left(v v_{0}\right)=\sigma\left(v v_{1}\right)$. Then $\sigma\left(w v_{0}\right)=\sigma\left(v v_{0}\right)$ by Claim 2, and $\sigma(w v)=\sigma\left(v v_{2}\right)$ by Claim 3. Hence, $\sigma\left(w v_{0}\right) \neq \sigma(w v)$.

Claim 5. Suppose $u p v$ is a monochromatic path of length two in $\operatorname{Tr}(n)$ with $u v \in$ $E(\operatorname{Tr}(i+1))$ and $p \in V(\operatorname{Tr}(n)) \backslash V(\operatorname{Tr}(i+1))$. Then any monochromatic path in $\operatorname{Tr}(n)$ between $u$ and $v$ and of the color $\sigma(u p)$ has length at most two.

Proof. Consider any monochromatic path $P=a_{0} a_{1} \ldots a_{r}$ of the color $\sigma(u p)$ with $a_{0}=v$ and $a_{r}=u$. First, suppose $u v \in E(\operatorname{Tr}(0))$. Let $\operatorname{Tr}(0)=u v w u$ and $x \in V(\operatorname{Tr}(1)) \backslash V(\operatorname{Tr}(0))$. By Claim 2, $\sigma(u x)=\sigma(u p)$ and $\sigma(v x)=\sigma(v p)$; so $\sigma(x u)=\sigma(x v)$. Thus, by (i), $\sigma(w x)=\sigma(w u)=\sigma(w v) \neq \sigma(x u)$. Let $v_{0} v_{1} \ldots v_{n}$ be a path in $\operatorname{Tr}(n)$ with $v_{0}=w$, $v_{1}=x$ and for $1 \leqslant i \leqslant n, v_{i} \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1))$ is inside $v_{i-1} u v v_{i-1}$. By (ii) and (iii), $\sigma\left(v_{i} u\right)=\sigma\left(v_{i} v\right)=\sigma(v x)$ for $1 \leqslant i \leqslant n$, and $\sigma\left(v_{i} v_{i+1}\right)=\sigma(x w)$ for $0 \leqslant i \leqslant n-1$. By planarity, $P$ is contained in the closed region bounded by uvwu. So either $P=u v$ or there exists some $1 \leqslant k \leqslant r-1$ such that $a_{k} \in\left\{v_{0}, \ldots, v_{n}\right\}$. We may assume the latter case occurs. If $\left\{a_{k-1}, a_{k+1}\right\}=\{u, v\}$, then $r=2$. Hence without loss of generality, let $a_{k-1} \notin\{u, v\}$. Then by Claim 2 and Claim 3, $\sigma\left(a_{k-1} a_{k}\right)=\sigma\left(v_{i} v_{i+1}\right) \neq \sigma(p u)$ for $i \in\{0,1, \ldots, n-1\}$, a contradiction. Hence $r \leqslant 2$. We remark that this paragraph also shows that such $u v$ in $E(\operatorname{Tr}(0))$ cannot be in a monochromatic $C_{4}$.

Thus, we may assume $u v \notin E(\operatorname{Tr}(0))$. By symmetry, we may assume that $v \in V(\operatorname{Tr}(i+$ 1)) $\backslash V(\operatorname{Tr}(i))$ for some $0 \leqslant i<n$ and $v$ is inside the triangle $u_{1} u_{2} u_{3} u_{1}$ bounding an inner face of $\operatorname{Tr}(i)$ and $u_{1}=u$. By Claim 4, $\sigma\left(u_{1} v\right) \neq \sigma\left(u_{2} v\right)=\sigma\left(u_{3} v\right)$.

If $a_{1}$ is inside $v u_{2} u_{3} v$ then there exists $1 \leqslant k<r$ such that $a_{k}$ is inside $v u_{3} u_{2} v$ and $a_{k+1} \in\left\{u_{2}, u_{3}\right\}$; so by Claim 2, $\sigma\left(a_{k} a_{k+1}\right)=\sigma\left(v u_{2}\right)=\sigma\left(v u_{3}\right) \neq \sigma\left(u_{1} v\right)=\sigma(p u)$, a contradiction.

Therefore, suppose that $P \neq u v$, by symmetry, we may assume that $a_{1}$ is inside $u_{1} v u_{2} u_{1}$. Let $v_{0}=u_{2}$ and let $v_{1} v_{2} \ldots v_{n-i-1}$ be the path in $\operatorname{Tr}(n)$ such that, for $1 \leqslant \ell \leqslant$ $n-i-1, v_{\ell} \in V(\operatorname{Tr}(i+\ell+1)) \backslash V(\operatorname{Tr}(i+\ell))$ is inside $u_{1} v_{\ell-1} v u_{1}$.

By (ii) and (iii), $\sigma\left(v_{\ell} u_{1}\right)=\sigma\left(v_{\ell} v\right)=\sigma\left(u_{1} v\right)$ for $1 \leqslant \ell \leqslant n-i-1$, and $\sigma\left(v_{\ell} v_{\ell+1}\right)=$ $\sigma\left(v u_{2}\right) \neq \sigma\left(v u_{1}\right)$ for $0 \leqslant \ell \leqslant n-i-2$. If $a_{1}$ is inside $v_{\ell} v_{\ell+1} v v_{\ell}$ for some $\ell$ with $0 \leqslant \ell \leqslant$ $n-i-2$, then exists $1 \leqslant k \leqslant r$ such that $a_{k}$ is inside $v_{\ell} v_{\ell+1} v v_{\ell}$ and $a_{k+1} \in\left\{v_{\ell}, v_{\ell+1}\right\}$;
so by Claim $3 \sigma\left(a_{k} a_{k+1}\right)=\sigma\left(v_{\ell} v_{\ell+1}\right)$, a contradiction. So $a_{1}=v_{\ell}$ for some $\ell$ with $1 \leqslant \ell \leqslant n-i-1$. Then as $\sigma\left(a_{1} a_{2}\right)=\sigma\left(u_{1} v\right)$ and by Claim 3, we have $a_{2}=u_{1}$. Therefore, $r=2$, proving Claim 5 .

Claim 6. If $C_{k}$ is monochromatic in $\operatorname{Tr}(n)$ then $k \leqslant 4$.
Proof. Let $C_{k}=a_{1} a_{2} \ldots a_{k} a_{1}$ be a monochromatic cycle in $\operatorname{Tr}(n)$. By (i), $E\left(C_{k}\right) \nsubseteq$ $E(\operatorname{Tr}(0))$. So we may assume that there exists some $1 \leqslant i \leqslant k$ such that $a_{i+1} \in V(\operatorname{Tr}(\ell+$ 1)) $\backslash V(\operatorname{Tr}(\ell))$ is inside the triangle $a_{i} u v a_{i}$ which bounds an inner face of some $\operatorname{Tr}(\ell)$. We may further assume that $\ell \leqslant n-2$, as otherwise, we could consider $\operatorname{Tr}(n+1)$ instead of $\operatorname{Tr}(n) .{ }^{1}$

Suppose $\sigma\left(a_{i} a_{i+1}\right) \in\left\{\sigma\left(a_{i+1} u\right), \sigma\left(a_{i+1} v\right)\right\}$. By symmetry, we may assume $\sigma\left(a_{i} a_{i+1}\right)=$ $\sigma\left(a_{i+1} u\right)$. Then $a_{i+2}=u$ by Claim 3. Hence, by Claim 5, any monochromatic path in $C_{k}$ between $a_{i}$ and $a_{i+2}=u$ has length at most 2 . So $k \leqslant 4$.

Thus, we may assume $\sigma\left(a_{i} a_{i+1}\right) \notin\left\{\sigma\left(a_{i+1} u\right), \sigma\left(a_{i+1} v\right)\right\}$; hence, $\sigma\left(a_{i+1} u\right)=\sigma\left(a_{i+1} v\right)$. Let $w \in V(\operatorname{Tr}(\ell+2)) \backslash V(\operatorname{Tr}(\ell+1))$ be inside the triangle $a_{i} u a_{i+1} a_{i}$. By (ii) and (iii), $\sigma\left(w a_{i}\right)=\sigma\left(w a_{i+1}\right)=\sigma\left(a_{i} a_{i+1}\right)$. Hence, by Claim 5, the monochromatic path $C_{k}-a_{i} a_{i+1}$ in $\operatorname{Tr}(n)$ of the color $\sigma\left(a_{i} a_{i+1}\right)=\sigma\left(w a_{i}\right)$ has length at most 2 ; so $k=3$.

Claim 7. There is no monochromatic $H^{+}$in $\operatorname{Tr}(n)$.
Proof. Suppose that there is a monochromatic copy of $H^{+}$on $\left\{v_{i}: 1 \leqslant i \leqslant 6\right\}$ in which $v_{1} v_{2} v_{3} v_{4} v_{1}$ is a 4 -cycle and $v_{1} v_{5}, v_{2} v_{6}$ are edges. If $v_{1} v_{2} \in E(\operatorname{Tr}(0))$, then $v_{1} v_{2}$ satisfies the conditions of Claim 5 and by the footnote from the proof of Claim 5, there is no monochromatic $C_{4}$ containing $v_{1} v_{2}$, a contradiction. So $v_{1} v_{2} \notin E(\operatorname{Tr}(0))$. By symmetry, we may assume that $v_{2} \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ for some $i$ and that $v_{1} u w v_{1}$ is the triangle bounding the inner face of $\operatorname{Tr}(i)$ containing $v_{2}$. Again as before we may assume that $0 \leqslant i \leqslant n-2$.

If $\sigma\left(v_{2} u\right)=\sigma\left(v_{2} w\right)$, then there exists some $p \in V(\operatorname{Tr}(n)) \backslash V(\operatorname{Tr}(i+1))$ such that $v_{1} p v_{2}$ has the same color as $\sigma\left(v_{1} v_{2}\right)$. But $v_{1} v_{4} v_{3} v_{2}$ is a monochromatic path of length 3 in $\operatorname{Tr}(n)$ between $v_{1}$ and $v_{2}$ and of the color $\sigma\left(v_{1} v_{2}\right)$, a contradiction to Claim 5 .

Hence, $\sigma\left(v_{1} v_{2}\right) \in\left\{\sigma\left(v_{2} u\right), \sigma\left(v_{2} w\right)\right\}$ and by symmetry, we may assume $\sigma\left(v_{1} v_{2}\right)=$ $\sigma\left(v_{2} u\right)$. Then by Claim $1, \sigma\left(v_{1} v_{2}\right) \neq \sigma\left(v_{2} w\right)$ and thus $\sigma\left(v_{2} v_{3}\right)=\sigma\left(v_{2} v_{6}\right) \neq \sigma\left(v_{2} w\right)$. This shows $w \notin\left\{v_{3}, v_{6}\right\}$. So there exists $y \in\left\{v_{3}, v_{6}\right\} \backslash\{u, w\}$. By Claim 3, $\sigma\left(v_{2} y\right)=\sigma\left(v_{2} w\right)$, a contradiction.

This completes the proof of Theorem 1.1 for $H^{+}$and $C_{k}$, with $k \geqslant 5$.

## 3 Monochromatic $\boldsymbol{K}_{2,3}$

In this section, we prove Theorem 1.1 for $K_{2,3}$ using a different coloring scheme on $\operatorname{Tr}(n)$ described below. Let $\sigma: E(\operatorname{Tr}(n)) \rightarrow\{0,1\}$ be defined inductively as follows:

[^1](i) Fix a triangle $T$ bounding an inner face of $\operatorname{Tr}(1)$, and let $\sigma(e)=0$ if $e \in E(T)$ and $\sigma(e)=1$ if $e \in E(\operatorname{Tr}(1)) \backslash E(T)$.
(ii) Suppose for some $1 \leqslant i<n$, we have defined $\sigma(e)$ for all $e \in E(\operatorname{Tr}(i))$. We now extend $\sigma$ to $E(\operatorname{Tr}(i+1))$. Let $x \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1))$ be arbitrary, let $v_{0} v_{1} v_{2} v_{0}$ denote the triangle bounding the inner face of $\operatorname{Tr}(i-1)$ containing $x$, with $v_{0}, v_{1}, v_{2}$ on the triangle in clockwise order, and let $\sigma\left(x v_{1}\right)=\sigma\left(x v_{2}\right)$.
(iii) Let $x_{j} \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ such that $x_{j}$ is inside the face of $\operatorname{Tr}(i)$ bounded by the triangle $x v_{j} v_{j+1} x$, where $j=0,1,2$ and the subscripts are taken modulo 3 . Define $\sigma\left(v_{0} x\right)=\sigma\left(v_{0} x_{0}\right)=\sigma\left(v_{0} x_{2}\right)=\sigma\left(x x_{2}\right)=\sigma\left(x_{1} v_{1}\right)$, and $\sigma\left(v_{2} x\right)=\sigma\left(v_{2} x_{1}\right)=$ $\sigma\left(v_{2} x_{2}\right)=\sigma\left(x x_{1}\right)=\sigma\left(x x_{0}\right)=\sigma\left(x_{0} v_{1}\right)$.


Figure 4: 2 edge-coloring scheme
Note that in (ii) we have $\left|\left\{\sigma\left(x v_{j}\right): j=0,1,2\right\}\right|=2$ and that in (iii) we have $\sigma\left(x_{j} v_{j}\right) \neq$ $\sigma\left(x_{j} v_{j+1}\right)$ for $j=0,1,2$. Hence, inductively, we have
(1) For $1 \leqslant i \leqslant n$ and $x \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1)),|\{\sigma(x v): v \in V(\operatorname{Tr}(i-1))\}|=2$.
(2) If $x_{1} x_{2} x_{3} x_{1}$ is a triangle which bounds an inner face of $\operatorname{Tr}(i)$ for some $1 \leqslant i \leqslant n-2$, and if $x \in V(\operatorname{Tr}(n)) \backslash V(\operatorname{Tr}(i+1))$ is inside $x_{1} x_{2} x_{3} x_{1}$ with $x x_{1}, x x_{2} \in E(\operatorname{Tr}(n))$, then $\sigma\left(x x_{1}\right) \neq \sigma\left(x x_{2}\right)$.

These two claims are straightforward so we omit their proofs.
(3) For any $x_{1} x_{2} \in E(\operatorname{Tr}(n)),\left|\left\{x \in N\left(x_{1}\right) \cap N\left(x_{2}\right): \sigma\left(x x_{1}\right)=\sigma\left(x x_{2}\right)=0\right\}\right| \leqslant 2$ and $\left|\left\{x \in N\left(x_{1}\right) \cap N\left(x_{2}\right): \sigma\left(x x_{1}\right)=\sigma\left(x x_{2}\right)=1\right\}\right| \leqslant 2$.

First, suppose $x_{1} x_{2} \in E(\operatorname{Tr}(0))$. Then by (i) and (2), $\mid\left\{x \in N\left(x_{1}\right) \cap N\left(x_{2}\right): \sigma\left(x x_{1}\right)=\right.$ $\left.\sigma\left(x x_{2}\right)=0\right\} \mid \leqslant 1$ and $\left|\left\{x \in N\left(x_{1}\right) \cap N\left(x_{2}\right): \sigma\left(x x_{1}\right)=\sigma\left(x x_{2}\right)=1\right\}\right| \leqslant 1$.

So we may assume that $x_{1} v w x_{1}$ bounds an inner face of $\operatorname{Tr}(i)$ and $x_{2} \in V(\operatorname{Tr}(i+$ 1)) $\backslash V(\operatorname{Tr}(i))$ inside $x_{1} v w x_{1}$. Let $v_{1} \in \operatorname{Tr}(i+2)$ be inside $x_{1} v x_{2} x_{1}$ and $w_{1} \in \operatorname{Tr}(i+2)$ be inside $x_{1} w x_{2} x_{1}$. By (iii), $\sigma\left(w_{1} x_{1}\right) \neq \sigma\left(w_{1} x_{2}\right)$ or $\sigma\left(v_{1} x_{1}\right) \neq \sigma\left(v_{1} x_{2}\right)$. By (2), for any $x \in V(\operatorname{Tr}(n)) \backslash V(\operatorname{Tr}(i+2))$ inside $x_{1} v w x_{1}$ with $x x_{1}, x x_{2} \in E(\operatorname{Tr}(n))$, we have $\sigma\left(x x_{1}\right) \neq$ $\sigma\left(x x_{2}\right)$. Hence, if (3) fails, then we may assume by symmetry between $w_{1}$ and $v_{1}$ that $\sigma\left(v x_{1}\right)=\sigma\left(v x_{2}\right)=\sigma\left(w x_{1}\right)=\sigma\left(w x_{2}\right)=\sigma\left(v_{1} x_{1}\right)=\sigma\left(v_{1} x_{2}\right)$, and $\sigma\left(w_{1} x_{1}\right) \neq \sigma\left(w_{1} x_{2}\right)$. Then, by (1), $\sigma\left(x_{1} x_{2}\right) \neq \sigma\left(x_{2} v\right)=\sigma\left(x_{2} w\right)$. Now by (iii), at least one of the two edges $v_{1} x_{1}$ and $v_{1} x_{2}$ has the same color as $x_{1} x_{2}$, a contradiction. This proves (3).
(4) If $x_{1} x_{2} x_{3} x_{4} x_{1}$ is a 4-cycle in $\operatorname{Tr}(n)$, then $x_{1} x_{3} \in E(\operatorname{Tr}(n))$ or $x_{2} x_{4} \in E(\operatorname{Tr}(n))$.

We may assume that $\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\} \subseteq V(\operatorname{Tr}(i+1))$ and $x_{j} \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$ for some $0 \leqslant i<n$ and $j \in[4]$. Let $u v w u$ be the triangle bounding an inner face of $\operatorname{Tr}(i)$ such that $x_{j}$ is inside it. Then $\left\{x_{j-1}, x_{j+1}\right\} \subseteq\{u, v, w\}$, implying that $x_{j-1} x_{j+1} \in E(\operatorname{Tr}(n))$.
(5) There is no monochromatic $K_{2,3}$ in $\operatorname{Tr}(n)$.

For, suppose $\operatorname{Tr}(n)$ has a monochromatic copy of $K_{2,3}$ on $\left\{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}\right\}$ with $v_{4} v_{i}, v_{5} v_{i} \in$ $E(\operatorname{Tr}(n))$ for all $i=1,2,3$. Then $v_{4} v_{5} \notin E(\operatorname{Tr}(n))$ by (3) and, hence, it follows from (4) that $v_{1} v_{2}, v_{2} v_{3}, v_{3} v_{1} \in E(\operatorname{Tr}(n))$. By planarity, $v_{1} v_{2} v_{3} v_{1}$ bounds an inner face of $\operatorname{Tr}(i)$ for some $i$ with $1 \leqslant i<n$ and, by the symmetry between $v_{4}$ and $v_{5}$, we may assume that $v_{4}$ is inside $v_{1} v_{2} v_{3} v_{1}$. Then $v_{4} \in V(\operatorname{Tr}(i+1)) \backslash V(\operatorname{Tr}(i))$. However, this contradicts (1), as $\sigma\left(v_{4} v_{1}\right)=\sigma\left(v_{4} v_{2}\right)=\sigma\left(v_{4} v_{3}\right)$. We have completed the proof of Theorem 1.1.

## 4 Monochromatic $\boldsymbol{J}_{k}$

In this section we prove that $\operatorname{Tr}(100 k) \rightarrow J_{k}$ holds for any positive integer $k$.
We need the following result, which is Lemma 9 in [4]. The original statement in [4] states $\operatorname{Tr}(16) \rightarrow C_{4}$, but the same proof in [4] actually gives the following stronger version.

Lemma 4.1. If $x y z x$ bounds the outer face of $\operatorname{Tr}(16)$, then any 2-edge-coloring of $\operatorname{Tr}(16)$ gives a monochromatic $C_{4}$ that intersects $\{x, y\}$.

Note that if the triangle $x y z x$ bounds the outer face of $\operatorname{Tr}(n)$ and $v \in V(\operatorname{Tr}(1)) \backslash$ $V(\operatorname{Tr}(0))$ then the subgraph of $\operatorname{Tr}(n)$ contained in the closed disc bounded by vxyv is isomorphic to $\operatorname{Tr}(n-1)$. Hence, the following is an easy consequence of Lemma 4.1.

Corollary 4.2. If $x y z x$ bounds the outer face of $\operatorname{Tr}(17)$ then any 2-edge-coloring of $\operatorname{Tr}(17)$ gives a monochromatic $C_{4}$ that intersects $\{x, y\}$ and avoids $z$.

Lemma 4.3. For any positive integer $k, \operatorname{Tr}(38 k) \rightarrow F_{k}$
Proof. Let $\sigma: E(\operatorname{Tr}(38 k)) \rightarrow\{0,1\}$ be an arbitrary 2-edge coloring. Let uvwu be the triangle bounding the outer face of $\operatorname{Tr}(38 k)$. Let $x_{0}:=w$ and, for $1 \leqslant \ell \leqslant 2 k$, let $x_{l} \in V(\operatorname{Tr}(\ell)) \backslash V(\operatorname{Tr}(\ell-1))$ such that $x_{\ell}$ is inside $x_{\ell-1} u v x_{\ell-1}$. Let $y_{i, 0}:=x_{i}$ for $i \in$ $\{0,1, \ldots, 2 k-1\}$ and, for $\ell \in\{1, \ldots, 36 k\}$, let $y_{i, \ell} \in V(\operatorname{Tr}(i+1+\ell)) \backslash V(\operatorname{Tr}(i+\ell))$ such that $y_{i, \ell}$ is inside $y_{i, \ell-1} u x_{i+1} y_{i, \ell-1}$.

Suppose for each $0 \leqslant i \leqslant 2 k-1$ there exists a monochromatic $C_{4}$ inside $x_{i} u x_{i+1} x_{i}$ that contains $u$ and avoids $x_{i}$. By pigeonhole principle, at least $k$ of these $C_{4} \mathrm{~s}$ are of the same color, which form a monochromatic $F_{k}$ centered at $u$.

Hence, we may assume that there exists some $i \in\{0,1, \ldots, 2 k-1\}$ such that no monochromatic $C_{4}$ inside $x_{i} u x_{i+1} x_{i}$ contains $u$ and avoids $x_{i}$. Since $i \leqslant 2 k-1, x_{i} u x_{i+1} x_{i}$ bounds the outer face of a $\operatorname{Tr}(36 k)$ that is contained in $\operatorname{Tr}(38 k)$.

Now for each $h \in\{0,1, \ldots, 2 k-1\}$, we view the region enclosed by $u, x_{i+1}$ and $y_{i, 18 h}$ without the closed region enclosed by $u, x_{i+1}$ and $y_{i, 18(h+1)}$ as a $\operatorname{Tr}(17)$. Note that these copies of $\operatorname{Tr}(17)$ share $u, x_{i+1}$ as the only common vertices. Taking $y_{i, 18 h}$ to be the vertex $z$ in Corollary 4.2, we conclude from Corollary 4.2 that there is a monochromatic $C_{4}$ in the $\operatorname{Tr}(17)$. We denote this $C_{4}$ by $G_{h}$. Then $x_{i+1} \in V\left(G_{h}\right)$ and $\left\{u, y_{i, 18 h}\right\} \cap V\left(G_{h}\right)=\emptyset$. By pigeonhole principle, at least $k$ of these $C_{4} \mathrm{~S}$ are of the same color, which clearly form a monochromatic $F_{k}$ centered at $x_{i+1}$.


Figure 5: Lemma 4.3


Figure 6: Lemma 4.4

Lemma 4.4. Let $k$ be a positive integer and let uvwu bound the outer face of $\operatorname{Tr}(9 k+2)$. Suppose $\sigma: E(\operatorname{Tr}(9 k+2)) \rightarrow\{0,1\}$ is a 2-edge-coloring such that $\mid\{\sigma(u x): x \in V(\operatorname{Tr}(9 k+$ $2))\} \mid=1$ and there is no monochromatic $C_{4}$ containing $u$. Then $\operatorname{Tr}(9 k+2)$ contains monochromatic $J_{k}$ centered at $v$.

Proof. Without loss of generality, assume $\sigma(u v)=0$. Then $\sigma(u y)=0$ for all $y \in N(u)$. Let $x_{0}:=w$ and, for $1 \leqslant i \leqslant 8 k+1$, let $x_{i} \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1))$ such that $x_{i}$ is inside $x_{i-1} u v x_{i-1}$. Since no monochromatic $C_{4}$ in $\operatorname{Tr}(9 k+2)$ contains $u$, there is at most one $i \in\{0,1,2, \ldots, 8 k+1\}$ such that $\sigma\left(x_{i} v\right)=0$. Hence, there exists $i \in\{0,1, \ldots, 4 k+2\}$ such that $\sigma\left(v x_{j}\right)=1$ for $j \in\{i, i+1, \ldots, i+4 k-1\}$. We now make the following claim.

Claim. The subgraph of $\operatorname{Tr}(9 k+2)$ contained in the closed disc bounded by $v x_{i} \ldots x_{i+3 k-1} v$ has a monochromatic $F_{k}$ of color 1 and centered at $v$, which we denote by $F_{v}$.

To show this, it suffices to show that for each $r$ with $0 \leqslant r \leqslant k-1$, the subgraph of $\operatorname{Tr}(9 k+2)$ inside $v x_{i+3 r} x_{i+3 r+1} x_{i+3 r+2} v$ (inclusive) contains a monochromatic $C_{4}$ of color 1 and containing $v$, as the union of such $C_{4}$ is an $F_{k}$ centered at $v$. So fix an arbitrary $r$, with $0 \leqslant r \leqslant k-1$. Note that $\sigma\left(x_{i+3 r} x_{i+3 r+1}\right)=1$ or $\sigma\left(x_{i+3 r+1} x_{i+3 r+2}\right)=1$, for $0 \leqslant r \leqslant k-1$; for, otherwise, $x_{i+3 r} x_{i+3 r+1} x_{i+3 r+2} u x_{i+3 r}$ is a monochromatic $C_{4}$ of color 0 and containing $u$, a contradiction. Without loss of generality, assume $\sigma\left(x_{i+3 r} x_{i+3 r+1}\right)=1$.

Let $y \in V(\operatorname{Tr}(i+3 r+2)) \backslash V(\operatorname{Tr}(i+3 r+1))$ such that $y$ is inside $x_{i+3 r} x_{i+3 r+1} v x_{i+3 r}$. If there are two edges in $\left\{y x_{i+3 r}, y x_{i+3 r+1}, y v\right\}$ of color 0 , then one can easily find a monochromatic $C_{4}$ of color 0 and containing $u$, a contradiction. Hence, at least two of $\left\{\sigma\left(y x_{i+3 r}\right), \sigma\left(y x_{i+3 r+1}\right), \sigma(y v)\right\}$ are 1. So $\left\{y, x_{i+3 r}, x_{i+3 r+1}, v\right\}$ induces a subgraph which contains a monochromatic $C_{4}$ of color 1 . This proves the claim.

Note that for $i+3 k \leqslant r \leqslant i+4 k-1, u x_{r} x_{r-1} u$ bounds the outer face of a $\operatorname{Tr}(k+1)$. Let $z_{r, 0}:=x_{r-1}$ and, for $r \in\{i+3 k, i+3 k+1, \ldots, i+4 k-1\}$ and $\ell \in\{1,2, \ldots, k\}$, let $z_{r, \ell} \in V(\operatorname{Tr}(r+\ell)) \backslash V(\operatorname{Tr}(r+\ell-1))$ such that $z_{r, \ell}$ is inside $z_{r, \ell-1} x_{r} u z_{r, \ell-1}$. Because $\sigma\left(u z_{r, j}\right)=0$ (by assumption) and $\operatorname{Tr}(9 k+2)$ has no monochromatic $C_{4}$ containing $u$, there is at most one $y \in\left\{z_{r, 1}, z_{r, 2}, \ldots, z_{r, k}\right\}$ such that $\sigma\left(y x_{r}\right)=0$. So there exists $k-1$ vertices in $\left\{z_{r, 1}, \ldots, z_{r, k}\right\}$ which together with $x_{r} v$ form a monochromatic $K_{1, k}$ of color 1 centered at $x_{r}$, which we denote by $H_{r}$. Now $H_{i+3 k}, H_{i+3 k+1}, \ldots, H_{i+4 k-1}$ form a monochromatic $k$-ary radius 2 tree rooted at $v$ of color 1 . This radius 2 tree and $F_{v}$ form a monochromatic $J_{k}$ of color 1, completing the proof of Lemma 4.4.

Now we are ready to prove the main result of this section, that is $\operatorname{Tr}(100 k) \rightarrow J_{k}$. Let $\sigma: E(\operatorname{Tr}(100 k)) \rightarrow\{0,1\}$ be arbitrary. We show that $\sigma$ always contains a monochromatic $J_{k}$. By Lemma 4.3, $\operatorname{Tr}(76 k)$ contains monochromatic copy of $F_{2 k}$, say $F$, and, without loss of generality, assume it is of color 1 . Let the $C_{4} \mathrm{~s}$ in $F$ be $x a_{i, 1} a_{i, 2} a_{i, 3} x$ for $i \in[2 k]$. For $i \in[2 k]$, let $b_{i} \in V(\operatorname{Tr}(76 k+1)) \backslash V(\operatorname{Tr}(76 k))$ such that $b_{i}$ is inside $x a_{i, 1} a_{i, 2} a_{i, 3} x$ and $a_{i, 1} a_{i, 2} b_{i} a_{i, 1}$ bounds an inner face of $\operatorname{Tr}(76 k+1)$. Let $A_{i}$ be the family of all vertices $a \in N\left(a_{i, 1}\right)$ inside $a_{i, 1} a_{i, 2} b_{i} a_{i, 1}$ and satisfying $\sigma\left(a a_{i, 1}\right)=1$.
(1) There exists some $i \in\{k+1, k+2, \ldots, 2 k\}$ such that $\left|A_{i}\right|<k$.

Otherwise, suppose that $\left|A_{i}\right| \geqslant k$ for all $i \in\{k+1, k+2, \ldots, 2 k\}$. Then let $Z_{i}:=$ $\left\{z_{i, 1}, z_{i, 2}, \ldots, z_{i, k-1}\right\} \subseteq A_{i}$. Now, for each $i \in\{k+1, \ldots, 2 k\},\left\{x, a_{i, 1}\right\} \cup Z_{i}$ induces a graph containing a monochromatic $K_{1, k}$. Those $K_{1, k} \mathrm{~s}$ form a monochromatic radius-two


Figure 7
$k$-ary tree of color 1 and rooted at $x$, which we denote by $T_{x}$. The $k$ four-cycles $x a_{i, 1} a_{i, 2} a_{i, 3} x$ for $i \in[k]$ form a monochromatic $F_{k}$. Now $F_{k} \cup T_{x}$ is a monochromatic $J_{k}$.

Let $u:=a_{i, 1}$. By (1), there exists an edge $v w \in \operatorname{Tr}(78 k)$ such that $u v w u$ bounds an inner face of $\operatorname{Tr}(78 k)$ and $\sigma(u y)=0$ for any $y \in N(u)$ in the closed disc bounded by uvwu.

Let $G$ be the subgraph of $\operatorname{Tr}(78 k)$ contained in the closed disc bounded by uvwu (see Figure 5). Clearly $G$ is isomorphic to a copy of $\operatorname{Tr}(22 k)$. In the rest of the proof, we should only discuss the graph $G$ and all $\operatorname{Tr}(i)$ will be referred to this copy of $\operatorname{Tr}(22 k)$. Let $x_{0}:=w$ and for $i \in[4 k]$, let $x_{i} \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1))$ such that $x_{i}$ is inside $u x_{i-1} v u$.
(2) $G$ contains a monochromatic copy of $F_{k}$, say $F^{\prime}$, which has color 0 and center $u$ and is disjoint from the union of closed regions bounded by $u x_{i} x_{i+1} u$ over all $0 \leqslant i \leqslant 2 k-1$.

If for each $i \in\{k, k+1, \ldots, 2 k-1\}$ there exists a monochromatic $C_{4}$ inside $u x_{2 i} x_{2 i+1} u$ and containing $u$, then these $k$ monochromatic $C_{4} \mathrm{~S}$ of color 0 form a desired monochromatic $F_{k}$ centered at $u$ and thus (2) holds. Otherwise, since $u x_{2 i} x_{2 i+1} u$ bounds the outer face of a $\operatorname{Tr}(9 k+2)$, it follows from Lemma 4.4 that there exists a monochromatic $J_{k}$ in $G$.

For $j \in\{0,1, \ldots, 2 k-1\}$, let $B_{j}$ be the family of all vertices $x \in N\left(x_{j}\right)$ inside $u x_{j} x_{j+1} u$ and satisfying $\sigma\left(x x_{j}\right)=0$.
(3) There exists some $j \in\{0,1, \ldots, k-1\}$ such that $\left|B_{j}\right|<k$.

Suppose to the contrary that there exist subsets $Z_{j} \subseteq B_{j}$ of size $k$ for all $j \in\{0,1, \ldots, k-$ $1\}$. Then each $Z_{j} \cup\left\{u, x_{j}\right\}$ induces a graph containing a monochromatic $K_{1, k}$ which is
centered at $x_{j}$ and has color 0 . These $K_{1, k} \mathrm{~s}$ together with $F^{\prime}$ form a monochromatic $J_{k}$ of color 0 . This proves (3).

Let $p_{0}:=x_{j+1}$ and for $1 \leqslant \ell \leqslant 4 k$, let $p_{\ell} \in V(\operatorname{Tr}(j+\ell+1)) \backslash V(\operatorname{Tr}(j+\ell))$ such that $p_{\ell}$ is inside $u x_{j} p_{\ell-1} u$. By (3), there exists some $0 \leqslant \ell \leqslant 4 k-1$ such that $\sigma\left(z x_{j}\right)=1$ for any $z \in N\left(x_{j}\right)$ in the closed disc bounded by $x_{j} p_{\ell} p_{\ell+1} x_{j}$.
(4) There is a monochromatic $F_{k}$ inside $x_{j} p_{\ell} p_{\ell+1} x_{j}$, say $F^{\prime \prime}$, with color 1 and center $x_{j}$.

Let $z_{0}:=p_{\ell+1}$ and for $s \in[2 k+1]$, let $z_{s} \in V(\operatorname{Tr}(j+\ell+s+2)) \backslash V(\operatorname{Tr}(j+\ell+s+1))$ such that $z_{s}$ is inside $x_{j} z_{s-1} p_{\ell} x_{j}$. Note that each $x_{j} z_{2 s} z_{2 s+1} x_{j}$ bounds a $\operatorname{Tr}(9 k+3)$. If for each $s \in[k]$ there exists a monochromatic $C_{4}$ of color 1 inside $x_{j} z_{2 s} z_{2 s+1} x_{j}$ and containing $x_{j}$, then these monochromatic copies of $C_{4}$ form the desired monochromatic $F_{k}$ centered at $x_{j}$. Otherwise, it follows from Lemma 4.4 that there exists a monochromatic $J_{k}$.

As $\left|B_{j}\right|<k$, there exists a subset $A \subseteq\left\{p_{1}, p_{2}, \ldots, p_{4 k}\right\}$ of size $2 k$ such that $\sigma\left(\alpha x_{j}\right)=1$ for each $\alpha \in A$ and moreover, there is no neighbors of $A$ belonging to $V\left(F^{\prime \prime}\right)$. Let $A:=\left\{\alpha_{1}, \ldots, \alpha_{2 k}\right\}$. Note that for each $h \in[2 k]$, we have $\sigma\left(\alpha_{h} u\right)=0$ and $\sigma\left(\alpha_{h} x_{j}\right)=1$.

It is easy to see that there exist pairwise disjoint sets $N_{h} \subseteq N\left(\alpha_{h}\right)$ of size $2 k$ for $h \in[2 k]$. Then there exists $M_{h} \subseteq N_{h}$ such that $\left|M_{h}\right|=k-1$ and $\sigma\left(x \alpha_{h}\right)$ is the same for all $x \in M_{h}$. This gives $2 k$ monochromatic copies of $K_{1, k-1}$ with centers $\alpha_{h}$ for $h \in[2 k]$. At least $k$ of them (say with centers $\alpha_{h}$ for $h \in[k]$ ) have the same color. If this color is 0 , these copies together with $\left\{u \alpha_{h}: h \in[k]\right\}$ and $F^{\prime}$ give a monochromatic $J_{k}$ with color 0 and center $u$. Otherwise, this color is 1 . Then these copies together with $\left\{x_{j} \alpha_{h}: h \in[k]\right\}$ and $F^{\prime \prime}$ give a monochromatic $J_{k}$ with color 1 and center $u$. This proves $\operatorname{Tr}(100 k) \rightarrow J_{k}$.

## 5 Monochromatic bistar

In this section we prove $\operatorname{Tr}(6 k+30) \rightarrow B_{k}$. We first establish the following lemma.
Lemma 5.1. Let uvwu be the triangle bounding the outer face of $\operatorname{Tr}(k+10)$. Let $\sigma$ : $E(\operatorname{Tr}(k+10)) \rightarrow\{0,1\}$ such that $|\{\sigma(u x): x \in V(\operatorname{Tr}(k+10))\}|=1$ and there is no monochromatic $C_{4}$ containing $u$. Then $\operatorname{Tr}(k+10)$ contains a monochromatic $B_{k}$.

Proof. Without loss of generality, let $\sigma(u v)=0$. Let $x_{0}:=w$ and, for $i \in$ [6], let $x_{i} \in V(\operatorname{Tr}(i)) \backslash V(\operatorname{Tr}(i-1))$ such that $x_{i}$ is inside $u v x_{i-1} u$.

Since $\operatorname{Tr}(k+10)$ has no monochromatic $C_{4}$ containing $u$, we see that $\mid\{0 \leqslant i \leqslant 6$ : $\left.\sigma\left(v x_{i}\right)=0\right\} \mid \leqslant 1$. So there exists some $i \in\{0,1,2,3,4\}$ such that $\sigma\left(v x_{i}\right)=\sigma\left(v x_{i+1}\right)=$ $\sigma\left(v x_{i+2}\right)=1$. We have either $\sigma\left(x_{i} x_{i+1}\right)=1$ or $\sigma\left(x_{i+1} x_{i+2}\right)=1$; as otherwise $u x_{i} x_{i+1} x_{i+2} u$ is a monochromatic $C_{4}$ of color 0 and containing $u$, a contradiction. We consider two cases.

Case 1. $\sigma\left(x_{i} x_{i+1}\right)=\sigma\left(x_{i+1} x_{i+2}\right)$.
In this case, we have $\sigma\left(x_{i} x_{i+1}\right)=\sigma\left(x_{i+1} x_{i+2}\right)=1$. So $x_{i} x_{i+1} x_{i+2} v x_{i}$ is a monochromatic $C_{4}$ of color 1. Let $y_{0}:=x_{i+1}$ and for $\ell \in[k+1]$, let $y_{\ell} \in V(\operatorname{Tr}(i+1+\ell)) \backslash V(\operatorname{Tr}(i+\ell))$
such that $y_{\ell}$ is inside $u y_{\ell-1} x_{i} u$. Similarly let $z_{0}:=x_{i+1}$ and for $\ell \in[k+1]$, let $z_{\ell} \in$ $V(\operatorname{Tr}(i+2+\ell)) \backslash V(\operatorname{Tr}(i+1+\ell))$ such that $z_{\ell}$ is inside $u z_{\ell-1} x_{i+2} u$.

Since $\operatorname{Tr}(k+10)$ has no monochromatic $C_{4}$ containing $u$, this shows that $\mid\{\ell \in[k+1]$ : $\left.\sigma\left(x_{i} y_{\ell}\right)=0\right\} \mid \leqslant 1$ and $\left|\left\{\ell \in[k+1]: \sigma\left(x_{i+2} z_{\ell}\right)=0\right\}\right| \leqslant 1$. Therefore, there exist $Y \subseteq\left\{y_{\ell}: \ell \in[k+1]\right\}$ and $Z \subseteq\left\{z_{\ell}: \ell \in[k+1]\right\}$ such that $|Y|=|Z|=k, \sigma\left(y x_{i}\right)=1$ for each $y \in Y$ and $\sigma\left(z x_{i+2}\right)=1$ for each $z \in Z$. Hence, $\operatorname{Tr}(k+10)$ has two monochromatic $K_{1, k} \mathrm{~s}$ of color 1 with centers $x_{i}, x_{i+1}$ and leave sets $Y, Z$, respectively. These two $K_{1, k} \mathrm{~s}$ together with $v x_{i} x_{i+1} x_{i+2} v$ form a monochromatic $B_{k}$ of color 1 .

Case 2. $\sigma\left(x_{i} x_{i+1}\right) \neq \sigma\left(x_{i+1} x_{i+2}\right)$.
Without loss of generality, let $\sigma\left(x_{i} x_{i+1}\right)=0$ and $\sigma\left(x_{i+1} x_{i+2}\right)=1$. Let $y \in V(\operatorname{Tr}(i+2)) \backslash$ $V(\operatorname{Tr}(i+1))$ be inside $u x_{i} x_{i+1} u$. Because $\sigma(u y)=0$ and $\operatorname{Tr}(k+10)$ has no monochromatic $C_{4}$ containing $u, \sigma\left(y x_{i}\right)=\sigma\left(y x_{i+1}\right)=1$. Therefore, $y x_{i+1} v x_{i} y$ is a monochromatic $C_{4}$ of color 1. Let $y_{0}:=y$ and, for $\ell \in[k+1]$, let $y_{\ell} \in V(\operatorname{Tr}(i+2+\ell)) \backslash V(\operatorname{Tr}(i+1+\ell))$ such that $y_{\ell}$ is inside $u y_{\ell-1} x_{i} u$. Let $z_{0}:=y$ and, for $\ell \in[k+1]$, let $z_{\ell} \in V(\operatorname{Tr}(i+2+\ell)) \backslash V(\operatorname{Tr}(i+1+\ell))$ such that $z_{\ell}$ is inside $u z_{\ell-1} x_{i+1} u$.

The remaining proof is similar as in Case 1 . We observe that $\mid\left\{\ell \in[k+1]: \sigma\left(x_{i} y_{\ell}\right)=\right.$ $0\} \mid \leqslant 1$ and $\left|\left\{\ell \in[k+1]: \sigma\left(x_{i+1} z_{\ell}\right)=0\right\}\right| \leqslant 1$. Therefore, there exist $Y \subseteq\left\{y_{\ell}: \ell \in[k+1]\right\}$ and $Z \subseteq\left\{z_{\ell}: \ell \in[k+1]\right\}$ such that $|Y|=|Z|=k, \sigma\left(y x_{i}\right)=1$ for $y \in Y$, and $\sigma\left(z x_{i+1}\right)=1$ for $z \in Z$. Hence, $\operatorname{Tr}(k+10)$ has two monochromatic $K_{1, k}$ s of color 1 with centers $x_{i}, x_{i+1}$ and leave sets $Y, Z$, respectively. These two $K_{1, k}$ s together with $y x_{i+1} v x_{i} y$ form a monochromatic $B_{k}$ of color 1. This proves Lemma 5.1.

We are ready to prove $\operatorname{Tr}(6 k+30) \rightarrow B_{k}$. Let $\sigma: E(\operatorname{Tr}(6 k+30)) \rightarrow\{0,1\}$. By Lemma 4.1, each copy of $\operatorname{Tr}(16)$ with the same outer face as of $\operatorname{Tr}(6 k+30)$ contains a monochromatic $C_{4}$, say $u_{1} u_{2} u_{3} u_{4} u_{1}$ of color 1 (see Figure 6). For each $i \in\{1,3\}$, let $v_{i} w_{i}$ be an edge in $\operatorname{Tr}(18)$ such that $u_{i} v_{i} w_{i} u_{i}$ is a triangle inside $u_{1} u_{2} u_{3} u_{4} u_{1}$. Note that $u_{i} v_{i} w_{i} u_{i}$ bounds the outer face of a $\operatorname{Tr}(6 k+12)$. Let $A_{i}$ be the family of all vertices $x \in N\left(u_{i}\right)$ inside $u_{i} v_{i} w_{i} u_{i}$ and satisfying $\sigma\left(x u_{i}\right)=1$. If $\left|A_{1}\right| \geqslant k$ and $\left|A_{3}\right| \geqslant k$, then together with the monochromatic 4-cycle $u_{1} u_{2} u_{3} u_{4} u_{1}$, it is easy to form a monochromatic $B_{k}$ of color 1 .

Hence by symmetry, we may assume that $\left|A_{1}\right|<k$. Then there exists an edge vw in $\operatorname{Tr}(18+k)$ such that $u_{1} v w u_{1}$ bounds an inner face of $\operatorname{Tr}(18+k)$ and $\sigma\left(u_{1} x\right)=0$ for all $x \in N\left(u_{1}\right)$ in the closed disc bounded by $u_{1} v w u_{1}$. We may assume that the induced subgraph contained in the closed disc bounded by $u_{1} v w u_{1}$ has a monochromatic $C_{4}$ say $u_{1} x y z u_{1}$ (as otherwise, it contains a $B_{k}$ by Lemma 5.1). Furthermore, we have $\{x, y, z\} \subseteq V(\operatorname{Tr}(2 k+28))$.

Let $\left\{p_{0}, q_{0}\right\} \subseteq V(\operatorname{Tr}(2 k+29)) \backslash V(\operatorname{Tr}(2 k+28))$ such that both $x y p_{0} x$ and $y z q_{0} y$ bound two inner faces of $\operatorname{Tr}(2 k+29)$. For $\ell \in[3 k]$, let $p_{\ell} \in V(\operatorname{Tr}(2 k+29+\ell)) \backslash V(\operatorname{Tr}(2 k+28+\ell))$ such that $p_{\ell}$ is inside $x p_{\ell-1} y x$. Similarly, for $\ell \in[3 k]$, let $q_{\ell} \in V(\operatorname{Tr}(2 k+29+\ell)) \backslash V(\operatorname{Tr}(2 k+$ $28+\ell)$ ) such that $q_{\ell}$ is inside $y q_{\ell-1} z y$. Moreover, let

$$
\begin{aligned}
B_{1} & :=\left\{p \in N(x): p \text { is inside } x p_{0} y x \text { and } \sigma(x p)=0\right\}, \\
B_{2} & :=\left\{q \in N(z): q \text { is inside } y q_{0} z y \text { and } \sigma(z q)=0\right\} .
\end{aligned}
$$



Figure 8

If $\left|B_{1}\right| \geqslant k$ and $\left|B_{2}\right| \geqslant k$, we can find two monochromatic $K_{1, k}$ S of color 0 , one inside $x p_{0} y x$ rooted at $x$ and one inside $y q_{0} z y$ rooted at $z$; these two $K_{1, k}$ S and $u_{1} x y z u_{1}$ form a monochromatic $B_{k}$ of color 0 . So we may assume, without loss of generality, that $\left|B_{1}\right|<k$.

Let $C:=\left\{\ell \in[3 k]: \sigma\left(y p_{\ell}\right)=0\right\}$. We claim $|C|<k$. Suppose to the contrary that $|C| \geqslant k$. Then there is a monochromatic $K_{1, k}$ with root $y$ and leaves in $C$ of color 0 . Since $\sigma\left(u_{1} p\right)=0$ for all $p \in N\left(u_{1}\right)$ inside $u_{1} v w$, there is also a monochromatic $K_{1, k}$ with root $u_{1}$ and leaves inside $u_{1} x y z u_{1}$ of color 0 . Now these two $K_{1, k} \mathrm{~S}$ and $u_{1} x y z u_{1}$ form a monochromatic $B_{k}$ of color 0 .

So $\left|B_{1}\right|<k$ and $|C|<k$. Then there exist $p_{h}, p_{s}$ with $h, s \in[3 k]$ such that $\sigma\left(p_{h} x\right)=$ $\sigma\left(p_{h} y\right)=\sigma\left(p_{s} x\right)=\sigma\left(p_{s} y\right)=1$. Because $x p_{0} p_{1} x$ bounds an inner face of $\operatorname{Tr}(2 k+30)$, it also bounds the outer face of a $\operatorname{Tr}(4 k)$. As $\left|B_{1}\right|<k$, there exists a monochromatic $K_{1, k}$ of color 1 with the root $x$ and $k$ leaves inside $x p_{0} p_{1} x$. Similarly, as $|C|<k$, there exists a monochromatic $K_{1, k}$ of color 1 with root $y$ and $k$ leavers inside $y p_{0} p_{1} y$. Now these two $K_{1, k} \mathrm{~s}$ and the 4 -cycle $x p_{h} y p_{s} x$ form a monochromatic $B_{k}$ of color 1 . This proves that $\operatorname{Tr}(6 k+30) \rightarrow B_{k}$ and thus completes the proof of Theorem 1.2.

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[^1]:    ${ }^{1}$ This is fair because $\operatorname{Tr}(n+1) \nrightarrow C_{k}$ implies $\operatorname{Tr}(n) \nrightarrow C_{k}$.

