# Maximum $k$-sum n-free sets of the 2-dimensional integer lattice

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**Abstract**

For a positive integer $n$, let $[n]$ denote $\{1,\ldots,n\}$. For a 2-dimensional integer lattice point $\mathbf{b}$ and positive integers $k \geqslant 2$ and $n$, a *$k$-sum $\mathbf{b}$-free set* of $[n]\times[n]$ is a subset $S$ of $[n]\times[n]$ such that there are no elements $\mathbf{a}_1,\ldots,\mathbf{a}_k$ in $S$ satisfying $\mathbf{a}_1+\cdots+\mathbf{a}_k=\mathbf{b}$. For a 2-dimensional integer lattice point $\mathbf{b}$ and positive integers $k \geqslant 2$ and $n$, we determine the maximum density of a $k$-sum $\mathbf{b}$-free set of $[n]\times[n]$. This is the first investigation of the non-homogeneous sum-free set problem in higher dimensions.

**Mathematics Subject Classifications:** 11B75, 11B30, 05D05

## 1 Introduction

Let $\mathbb{Z}_{>0}$ and $\mathbb{R}_{>0}$ denote the sets of positive integers and positive real numbers, respectively. For a positive integer $n$, let $[n]=\{1,\ldots,n\}$. Throughout this paper, a bold letter

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such as $\mathbf{n}, \mathbf{x}$, and $\mathbf{y}$ stands for a single vector in $\mathbb{R}^d_{>0}$ for some integer $d \geqslant 2$. For a positive integer $d$ and a $d$-dimensional integer lattice point $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d_{>0}$, let $[\mathbf{n}]$ denote the set $[n_1] \times \cdots \times [n_d]$ and let $|\mathbf{n}| = n_1 \cdots n_d$.

For an abelian group $(G, +)$, a set $S \subseteq G$ is *sum-free* if there are no elements $x, y, z$ in $S$ satisfying $x+y=z$. Sum-free sets were investigated by Schur [20] in 1917 as an attempt to prove Fermat's Last Theorem. Ever since, sum-free sets received a significant amount of attention over the years, aiding the growth of the field of additive combinatorics. In particular, understanding sum-free subsets of the additive group on the positive integers has been considered an important topic in the area. Given a set $S$, two natural questions arise: the maximum size of a sum-free subset of $S$ and the number of sum-free subsets of $S$. It is easy to see that a sum-free subset of $[n]$ has size at most $\lceil \frac{n}{2} \rceil$, which is tight as demonstrated by taking all integers of $[n]$ that are either odd or greater than $\lfloor \frac{n}{2} \rfloor$. Conjectures by Cameron and Erdős [4, 5] concerning the number of sum-free subsets or maximal sum-free subsets of $[n]$ were settled in [1, 11, 21]. Other structural aspects of a sum-free subset of $[n]$ were also studied in [6, 10, 22].

There is a vast literature on generalizations and variations of sum-free subsets of $[n]$. Among them, we emphasize the following two directions. The first is by Ruzsa [18, 19], who generalized the above classical problem to linear equations. For a positive integer $k \geqslant 2$ and integers $a_1, \ldots, a_k, b$, let $\mathcal{L} : a_1x_1 + \cdots + a_kx_k = b$ be a linear equation. An *$\mathcal{L}$-solution-free set* (or *$\mathcal{L}$-free set* for short) is a subset $S$ of $[n]$ such that no elements $x_1, \ldots, x_k$ in $S$ satisfy the equation $\mathcal{L}$. The case when $b = 0$, which is also referred to as "$\mathcal{L}$ is homogeneous", was actively studied due to its close ties to other subjects such as Sidon sets, progression-free sets, and Rado's boundedness conjecture. See [12–14] for recent results on $\mathcal{L}$-free sets where $\mathcal{L}$ is a homogeneous linear equation, and see [9, 17] for details regarding Rado's boundedness conjecture. Also, the complexity of finding a maximum $\mathcal{L}$-free set is known to be NP-complete in almost all cases, see [7, 16] for recent results.

The second is a direction in [3], which generalizes the problem to finding a sum-free subset of the $d$-dimensional integer lattice $\mathbb{Z}^d_{>0}$. To be precise, for a $d$-dimensional integer lattice point $\mathbf{n} \in \mathbb{Z}^d_{>0}$, a *sum-free set* of $[\mathbf{n}]$ is a subset $S$ of $[\mathbf{n}]$ such that there are no elements $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $S$ satisfying $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$. Regarding the question of the maximum density of a sum-free subset of $[\mathbf{n}]$, Cameron [2] and Katz [15] provided some partial results, and Elsholtz and Rackham [8] resolved the 2-dimensional case as follows.

**Theorem 1** ( [8]). *As $n$ goes to infinity, the density of a sum-free subset of $[n] \times [n]$ is at most $\frac{3}{5} + O\left(\frac{1}{n}\right)$.*

We initiate an investigation that lies at the intersection of the two aforementioned research directions. Namely, we consider the following problem: given a positive integer $n$ and a linear equation $\mathcal{L}$, find the maximum size of a subset of the integer lattice $[n]^d$ that does not contain a solution to $\mathcal{L}$. This is the first investigation of the non-homogeneous sum-free set problem in higher dimensions. To this extent, we make the following definition: for a $d$-dimensional integer lattice point $\mathbf{b}$ and positive integers $k > 1$ and $n$, a *$k$-sum $\mathbf{b}$-free set* is a subset $S$ of $[n]^d$ such that there are no elements $\mathbf{a}_1, \ldots, \mathbf{a}_k$

in $S$ satisfying $\mathbf{a}_1 + \cdots + \mathbf{a}_k = \mathbf{b}$. For simplicity, let $\mathbf{n}$ denote the $d$-dimensional vector $(n, \dots, n)$, and recall that $[\mathbf{n}] = [n]^d$. Let $\mu_{k,\mathbf{b}}(\mathbf{n})$ denote the maximum size of a $k$-sum $\mathbf{b}$-free set of $[\mathbf{n}]$. We are interested in finding the value of $\mu_{k,\mathbf{b}}(\mathbf{n})$ where each coordinate of $\mathbf{n}$ is a positive integer. Note that we may further assume that each coordinate of $\mathbf{b}$ is also a positive integer, as otherwise $\mu_{k,\mathbf{b}}(\mathbf{n}) = |\mathbf{n}| = n^d$.

It turns out that our problem boils down to finding the value of $\mu_{k,\mathbf{n}}(\mathbf{n})$. This is because each coordinate of a point of $[\mathbf{n}]$ is positive, and hence if $n$ is sufficiently large so that $\mathbf{b} \in [\mathbf{n}]$, then

$$\mu_{k,\mathbf{b}}(\mathbf{n}) = n^d - |\mathbf{b}| + \mu_{k,\mathbf{b}}(\mathbf{b})$$

as one can see by taking all elements $\mathbf{x} = (x_1, \dots, x_d) \in [\mathbf{n}]$ such that $x_i$ is greater than the $i$th coordinate of $\mathbf{b}$ for every $i$, and all elements of a maximum $k$-sum $\mathbf{b}$-free subset of $[\mathbf{b}]$. Furthermore, the problem is easy when $k = 2$, as we know

$$\mu_{2,\mathbf{n}}(\mathbf{n}) = n^d - \left\lceil \frac{(n-1)^d}{2} \right\rceil$$

by the following simple argument: vectors $\mathbf{x}$ and $\mathbf{n} - \mathbf{x}$ cannot both be in a 2-sum $\mathbf{n}$-free set for some $\mathbf{x} \in [\mathbf{n}]$, and equality can be obtained by taking all elements of $\{(x_1, \dots, x_d) \in [\mathbf{n}] \mid x_1 + \cdots + x_d > \frac{dn}{2}\}$.

When $d = 2$, we succeed in finding the maximum density of a $k$-sum $\mathbf{n}$-free set of $[\mathbf{n}]$ for every positive integer $k \geqslant 2$. For brevity, let $\mu_k(\mathbf{n})$ denote $\mu_{k,\mathbf{n}}(\mathbf{n})$, and define

$$\nu_k(\mathbf{n}) := \frac{\mu_k(\mathbf{n})}{|\mathbf{n}|}.$$

**Theorem 2.** *Let $k \geqslant 2$ be a positive integer and let $\mathbf{n} = (n, n)$. As $n$ goes to infinity,*

$$\nu_k(\mathbf{n}) = \frac{k^2 - 2}{k^2} + O\left(\frac{1}{n}\right).$$

Theorem 2 is tight, as explained in Remark 5. We suspect that the 1-dimensional version of Theorem 2 is already known, yet, we could not find any references. As we use some ideas of the 1-dimensional case in the proof of the 2-dimensional case, we include the proof of the 1-dimensional case in Section 2 for completeness. We actually prove a stronger statement (Theorem 4) that implies Theorem 2, whose proof is in Section 3. We end the paper with some remarks and open questions in Section 4.

## 2 The 1-dimensional case

In this section, we provide the 1-dimensional analogue of Theorem 2. As mentioned before, we suspect this result is known, yet, we include a proof for completeness.

**Proposition 3.** *Let $k \geqslant 2$ be a positive integer and let $\mathbf{n} = (n)$. If $n$ is a positive integer, then*

$$1 - \frac{1}{k} \leqslant \nu_k(\mathbf{n}) \leqslant 1 - \frac{1}{k} + \frac{1}{n}.$$

*Proof.* As $\mathbf{n}$ is a 1-dimensional vector, we will use $n$ to denote $\mathbf{n}$. As $\{x \in [n] \mid x > \frac{n}{k}\}$ is a $k$-sum $n$-free set of $[n]$, we know $\nu_k(n) \geqslant \frac{n-\lfloor \frac{n}{k} \rfloor}{n} \geqslant \frac{n-\frac{n}{k}}{n} = 1 - \frac{1}{k}$. We prove the other inequality by induction on $k$. When $k = 2$, since $x$ and $n - x$ cannot both be in a 2-sum $n$-free set for some $x \in [n]$, we know $\mu_2(n) = \lceil \frac{n}{2} \rceil$. (Furthermore, this is tight as demonstrated by taking all integers of $[n]$ that are either odd or greater than $\lfloor \frac{n}{2} \rfloor$.) Note that this implies $\nu_2(n) \leqslant \frac{1}{2} + \frac{1}{n}$.

Suppose $k \geqslant 3$. Let $S$ be a $k$-sum $n$-free set and let $m$ be the minimum element of $S$. If $m > \frac{n}{k}$, then $|S| \leqslant n - \lfloor \frac{n}{k} \rfloor \leqslant n - \frac{n}{k} + 1$, which implies the conclusion we seek. So let us assume $m \leqslant \frac{n}{k}$. Since $m \in S$, we know $S$ is also a $(k-1)$-sum $(n-m)$-free set of $[n]$. This further implies $S' := S \cap [n-m]$ is a $(k-1)$-sum $(n-m)$-free set of $[n-m]$. By the induction hypothesis, $\nu_{k-1}(n') \leqslant 1 - \frac{1}{k-1} + \frac{1}{n'}$ for every positive integer $n'$, hence

$$\frac{|S'|}{n-m} \leqslant 1 - \frac{1}{k-1} + \frac{1}{n-m}.$$

Since $|S| \leqslant |S'| + m$, we have

$$\frac{|S|}{n} \leqslant \frac{n - m - \frac{n-m}{k-1} + 1 + m}{n} = 1 - \frac{n-m}{n(k-1)} + \frac{1}{n} \leqslant 1 - \frac{n - \frac{n}{k}}{n(k-1)} + \frac{1}{n} = 1 - \frac{1}{k} + \frac{1}{n},$$

where the second inequality follows from the fact that $m \leqslant \frac{n}{k}$. Hence,

$$\frac{|S|}{n} \leqslant 1 - \frac{1}{k} + \frac{1}{n}.$$

□

## 3 The 2-dimensional case

In this section, we will prove the following statement, which is a stronger statement implying Theorem 2.

**Theorem 4.** *Let $k \geqslant 2$ be a positive integer and let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2_{>0}$. As both $n_1$ and $n_2$ go to infinity,*

$$\nu_k(\mathbf{n}) = \frac{k^2 - 2}{k^2} + O\left(\frac{1}{\min\{n_1, n_2\}}\right).$$

We first provide an example demonstrating the sharpness of Theorem 4. In Subsection 3.1 we show Theorem 4, whose proof is by induction on $k$, except the case when $k = 3$, which we deal with in Subsection 3.2.

*Remark* 5. Let $k \geqslant 2$ be a positive integer and let $\mathbf{n} = (n_1, n_2)$ be a 2-dimensional integer lattice point where both $n_1$ and $n_2$ are sufficiently large. The inequality $\nu_k(\mathbf{n}) \geqslant \frac{k^2-2}{k^2}$ can be verified by considering the following set:

$$S = \left\{\mathbf{x} = (x_1, x_2) \in [\mathbf{n}] \mid n_2 x_1 + n_1 x_2 > \frac{2n_1 n_2}{k}\right\}.$$

See Figure 1 for an illustration of $S$.

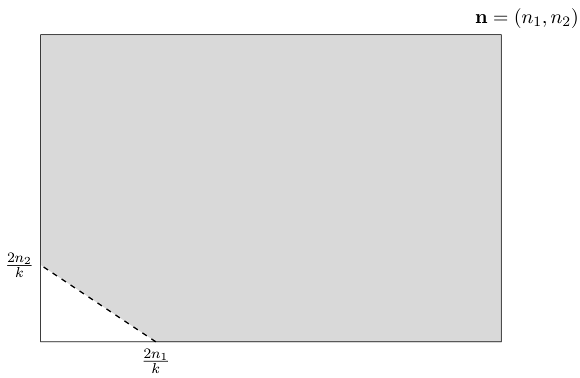


Figure 1: The shaded region corresponds to a $k$-sum $\mathbf{n}$-free set.

Suppose there are elements $\mathbf{a}_1, \dots, \mathbf{a}_k$ in $S$ satisfying $\mathbf{a}_1 + \cdots + \mathbf{a}_k = \mathbf{n}$. Let $\mathbf{a}_i = (a_{i1}, a_{i2})$ for each $i \in [k]$. Then $a_{11} + \cdots + a_{k1} = n_1$ and $a_{12} + \cdots + a_{k2} = n_2$. Moreover, by the definition of $S$, we have $n_2 a_{i1} + n_1 a_{i2} > \frac{2n_1 n_2}{k}$ for each $i \in [k]$. By adding up the $k$ inequalities, each corresponding to one $\mathbf{a}_i$, we obtain

$$n_2(a_{11} + \cdots + a_{k1}) + n_1(a_{12} + \cdots + a_{k2}) > 2n_1 n_2,$$

which is a contradiction since the left side is also $2n_1 n_2$. Hence,

$$\nu_k(\mathbf{n}) \geqslant \frac{|S|}{|\mathbf{n}|} \geqslant \frac{|\mathbf{n}| - \frac{2n_1 n_2}{k^2}}{|\mathbf{n}|} = \frac{k^2 - 2}{k^2}.$$

Before starting the proof, we introduce some notation that will be used throughout the remaining two subsections. For $\mathbf{r} = (r_1, r_2) \in \mathbb{R}^2$, let $m(\mathbf{r}) := \min\{r_1, r_2\}$ and $M(\mathbf{r}) := \max\{r_1, r_2\}$, and for a real number $\alpha$, let $\alpha\mathbf{r} = (\alpha r_1, \alpha r_2)$. Note that $|\alpha\mathbf{r}| = \alpha^2 |\mathbf{r}|$. Also, let $\lfloor \mathbf{r} \rfloor$ and $\lceil \mathbf{r} \rceil$ denote the integer points $(\lfloor r_1 \rfloor, \lfloor r_2 \rfloor)$ and $(\lceil r_1 \rceil, \lceil r_2 \rceil)$, respectively. For $\mathbf{r} = (r_1, r_2)$ and $\mathbf{r}' = (r_1', r_2')$ in $\mathbb{R}^2$, let $\mathbf{r} \leqslant \mathbf{r}'$ and $\mathbf{r} < \mathbf{r}'$ denote $r_i \leqslant r_i'$ and $r_i < r_i'$, respectively, for each $i \in [2]$.

### 3.1 Proof of Theorem 4

In this subsection, we prove Theorem 4, except the case when $k = 3$, whose proof is in Subsection 3.2. To prove Theorem 4, it is sufficient to prove that for every $k$-sum $\mathbf{n}$-free subset $S$ of $[\mathbf{n}]$, the following:

$$|S| \leqslant \left( \frac{k^2 - 2}{k^2} \right) |\mathbf{n}| + O(M(\mathbf{n})). \tag{1}$$

To see why, suppose that $|S| \leqslant \alpha |\mathbf{n}| + cM(\mathbf{n})$ for a $k$-sum $\mathbf{n}$-free set $S$ of $[\mathbf{n}]$ and some constants $\alpha$ and $c$. Since $|\mathbf{n}| = M(\mathbf{n}) m(\mathbf{n})$,

$$\frac{|S|}{|\mathbf{n}|} \leqslant \alpha + \frac{cM(\mathbf{n})}{M(\mathbf{n}) m(\mathbf{n})} = \alpha + \frac{c}{m(\mathbf{n})},$$

which implies that $\nu_k(\mathbf{n}) \leqslant \frac{k^2-2}{k^2} + O\left(\frac{1}{m(\mathbf{n})}\right)$. Tightness is shown by the example in Remark 5.

In the following, let $S$ be a maximum $k$-sum $\mathbf{n}$-free set of $[\mathbf{n}]$. We prove (1) by induction on $k$, with two base cases, $k = 2$ and $k = 3$. When $k = 2$, since both integer lattice points $\mathbf{x}$ and $\mathbf{n} - \mathbf{x}$ cannot both be in $S$, the following holds:

$$\mu_{2,\mathbf{n}}(\mathbf{n}) = \left\lfloor \frac{|\mathbf{n}| + n_1 + n_2 - 1}{2} \right\rfloor \leqslant \frac{|\mathbf{n}|}{2} + M(\mathbf{n}). \tag{2}$$

Thus, (1) is true when $k = 2$. When $k = 3$, Theorem 6, whose proof is postponed to Subsection 3.2, implies that (1) is true when $k = 3$.

**Theorem 6.** *Let $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2_{>0}$. As both $n_1$ and $n_2$ go to infinity,*

$$\mu_3(\mathbf{n}) \leqslant \frac{7}{9}|\mathbf{n}| + O(M(\mathbf{n})).$$

For the induction step, suppose $k \geqslant 4$. Let $\mathbf{a} = 2\left\lfloor \frac{\mathbf{n}}{k} \right\rfloor$. Suppose that $S \cap [\mathbf{a}]$ is a 2-sum $\mathbf{a}$-free set of $[\mathbf{a}]$. By (2), we have $|S \cap [\mathbf{a}]| \leqslant \frac{1}{2}|\mathbf{a}| + M(\mathbf{a})$. Then,

$$|S| \leqslant |\mathbf{n}| - |\mathbf{a}| + \frac{1}{2}|\mathbf{a}| + M(\mathbf{a}) = |\mathbf{n}| - \frac{1}{2}|\mathbf{a}| + M(\mathbf{a}) \leqslant |\mathbf{n}| - \frac{1}{2}|\mathbf{a}| + M(\mathbf{n}).$$

Also,

$$\begin{aligned} \frac{1}{4}|\mathbf{a}| = \left|\left\lfloor \frac{\mathbf{n}}{k} \right\rfloor\right| &\geqslant \frac{M(\mathbf{n}) - (k-1)}{k} \cdot \frac{m(\mathbf{n}) - (k-1)}{k} \\ &\geqslant \frac{M(\mathbf{n})m(\mathbf{n})}{k^2} - \frac{2(k-1)M(\mathbf{n})}{k^2} = \frac{|\mathbf{n}|}{k^2} - \frac{2(k-1)M(\mathbf{n})}{k^2}. \end{aligned}$$

Hence,

$$|S| \leqslant |\mathbf{n}| - \frac{2|\mathbf{n}|}{k^2} + \frac{4(k-1)}{k^2} \cdot M(\mathbf{n}) + M(\mathbf{n}) = \left(\frac{k^2-2}{k^2}\right)|\mathbf{n}| + \left(1 + \frac{4(k-1)}{k^2}\right)M(\mathbf{n}),$$

which implies that (1) holds.

Suppose that $S \cap [\mathbf{a}]$ is not a 2-sum $\mathbf{a}$-free set of $[\mathbf{a}]$. Then, there are two elements $\mathbf{x}$ and $\mathbf{y}$ in $S \cap [\mathbf{a}]$ such that $\mathbf{x} + \mathbf{y} = \mathbf{a}$. Let $\mathbf{b} = \mathbf{n} - \mathbf{a}$, and now we consider $S' = S \cap [\mathbf{b}]$. Now, $S'$ is a $(k-2)$-sum $\mathbf{b}$-free set. Since $k \geqslant 4$, by induction hypothesis, we know

$$|S'| \leqslant \frac{(k-2)^2 - 2}{(k-2)^2}|\mathbf{b}| + O\left(M(\mathbf{b})\right) \leqslant \frac{(k-2)^2 - 2}{(k-2)^2}|\mathbf{b}| + cM(\mathbf{b})$$

for some constant $c$ not depending on $\mathbf{b}$. Since $|S| \leqslant |\mathbf{n}| - |\mathbf{b}| + |S'|$, we obtain

$$|S| \leqslant |\mathbf{n}| - \frac{2}{(k-2)^2}|\mathbf{b}| + cM(\mathbf{b}).$$

By the definitions of $\mathbf{a}$ and $\mathbf{b}$, we have $|\mathbf{b}| = \left|\mathbf{n} - 2\left\lfloor \frac{\mathbf{n}}{k} \right\rfloor\right| \geqslant \frac{(k-2)^2}{k^2}|\mathbf{n}|$. It follows that

$$|S| \leqslant |\mathbf{n}| - \frac{2}{k^2}|\mathbf{n}| + O(M(\mathbf{n})).$$

### 3.2 Proof of Theorem 6

In this subsection, we prove Theorem 6, which is the crucial part of the proof.

Assume $S$ is a 3-sum $\mathbf{n}$-free set of $[\mathbf{n}]$. For simplicity, let

$$A = \left\{ (x_1, x_2) \in [\mathbf{n}] \mid n_2 x_1 + n_1 x_2 < \frac{2n_1 n_2}{3} \right\}.$$

As shown in Remark 5, if $A \cap S = \emptyset$, namely, $S$ belongs to the shaded region of Figure 1, then we have the desired conclusion. Thus, we may assume $A \cap S \neq \emptyset$ in the following.

For a 2-dimensional integer lattice point $\mathbf{x}$, let

$$S_{\mathbf{x}} = S \cap [\mathbf{n} - \mathbf{x}].$$

We often use the fact that if $\mathbf{x} \in S$, then $S_{\mathbf{x}}$ is a 2-sum $(\mathbf{n} - \mathbf{x})$-free set. By (2), we know $|S_{\mathbf{x}}| \leqslant \frac{|\mathbf{n}-\mathbf{x}|}{2} + M(\mathbf{n} - \mathbf{x})$. Since $M(\mathbf{n} - \mathbf{x}) \leqslant M(\mathbf{n})$, we obtain

$$|S_{\mathbf{x}}| \leqslant \frac{|\mathbf{n} - \mathbf{x}|}{2} + M(\mathbf{n}). \tag{3}$$

If $S$ contains an element $\mathbf{x}$ where $\mathbf{x} \leqslant \frac{\mathbf{n}}{3}$, which is equivalent to $\mathbf{n} - \mathbf{x} \geqslant \frac{2}{3}\mathbf{n}$, then we know $|\mathbf{n} - \mathbf{x}| \geqslant \frac{4}{9}|\mathbf{n}|$. Since $|S| \leqslant |\mathbf{n}| - |\mathbf{n} - \mathbf{x}| + |S_{\mathbf{x}}|$, by (3), we obtain

$$|S| \leqslant |\mathbf{n}| - \frac{|\mathbf{n} - \mathbf{x}|}{2} + M(\mathbf{n}) \leqslant \frac{7}{9}|\mathbf{n}| + M(\mathbf{n}),$$

which is the desired conclusion.

Now suppose $S$ has no element $\mathbf{x}$ where $\mathbf{x} \leqslant \frac{\mathbf{n}}{3}$. For convenience, let $\mathbf{a} = \left(\frac{n_1}{3}, \frac{2n_2}{3}\right)$, $\mathbf{b} = \left(\frac{n_1}{3}, \frac{n_2}{3}\right)$, and $\mathbf{c} = \left(\frac{2n_1}{3}, \frac{n_2}{3}\right)$. See Figure 2.

Since $A \cap S \neq \emptyset$, we know $S$ contains some point in $A \setminus \left\{\mathbf{x} \in [\mathbf{n}] \mid \mathbf{x} \leqslant \frac{\mathbf{n}}{3}\right\}$. By symmetry, we may assume that there exists $\mathbf{x} = (x_1, x_2) \in S \cap A$ where $x_2 > \frac{n_2}{3}$ and $0 < x_1 \leqslant \frac{n_1}{3}$. Let $\ell_{\mathbf{x}}$ be the line defined by the two points $\mathbf{x}$ and $\mathbf{b}$. We may further assume that $S$ does not contain a point of $A$ below $\ell_{\mathbf{x}}$ where the 2nd coordinate is greater than $\frac{n_2}{3}$; this is the hatched region of Figure 2. Let $p$ be the 2nd coordinate of the intercept of the line $\ell_{\mathbf{x}}$ and the vertical line passing through the origin, that is,

$$p = \frac{n_1 x_2 - n_2 x_1}{n_1 - 3x_1}.$$

We consider two cases, depending on the larger value of $p$ and the 2nd coordinate of $\mathbf{n} - \mathbf{x}$.

**Case (i)**: Suppose $p < n_2 - x_2$.

Since $x_1 < \frac{n_1}{3}$ is equivalent to $n_1 - 3x_1 > 0$, it follows that $p < n_2 - x_2$ is equivalent to

$$-3x_1 x_2 < n_1 n_2 - 2n_1 x_2 - 2n_2 x_1. \tag{4}$$

Now,

$$\begin{aligned} |\mathbf{n}| - \frac{|\mathbf{n}-\mathbf{x}|}{2} &= \frac{n_1n_2 + (n_2x_1 + n_1x_2) - x_1x_2}{2} \\ &< \frac{3n_1n_2 + 3(n_2x_1 + n_1x_2) + (n_1n_2 - 2n_1x_2 - 2n_2x_1)}{6} \\ &= \frac{4n_1n_2 + (n_2x_1 + n_1x_2)}{6} \\ &< \frac{4n_1n_2 + \frac{2n_1n_2}{3}}{6} = \frac{7}{9}|\mathbf{n}| \end{aligned}$$

where the first inequality comes from (4) and the second inequality follows from the fact that $\mathbf{x} \in A$. Thus, since $|S| \leqslant |\mathbf{n}| - |\mathbf{n}-\mathbf{x}| + |S_{\mathbf{x}}|$, by (3), we obtain

$$|S| \leqslant |\mathbf{n}| - \frac{|\mathbf{n}-\mathbf{x}|}{2} + M(\mathbf{n}) < \frac{7}{9}|\mathbf{n}| + M(\mathbf{n}),$$

which is the desired conclusion.

**Case (ii)**: Now suppose $p \geqslant n_2 - x_2$.

This means that $S$ contains no integer lattice points in the following set:

$$R := \{(z_1, z_2) \in [\mathbf{n}] \mid z_1 > 0, z_2 > n_2 - x_2, \text{ and } (z_1, z_2) \text{ is below the line } \ell_{\mathbf{x}}\}$$

See Figure 3 for an illustration. In other words, $R \cap S = \emptyset$, and so

$$|S| \leqslant |\mathbf{n}| - |\mathbf{n}-\mathbf{x}| - |R| + |S_{\mathbf{x}}|.$$

By (3), we obtain

$$|S| \leqslant |\mathbf{n}| - \frac{|\mathbf{n}-\mathbf{x}|}{2} - |R| + M(\mathbf{n}).$$

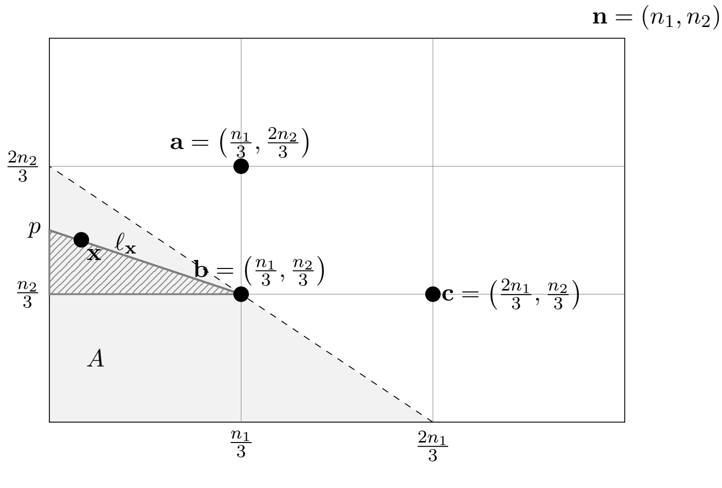


Figure 2: $A$ is the shaded region and no element of $S$ is in the hatched region.

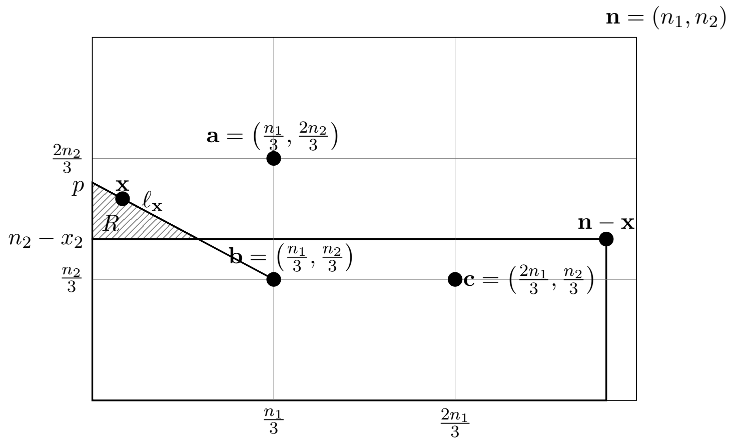


Figure 3: An illustration for Case (ii), when $p \geqslant n_2 - x_2$.

By Pick's Theorem, the number of integer lattice points in the interior of a triangular region $T$ is exactly $A - \frac{B}{2} + 1$ where $A$ is the area of $T$ and $B$ is the number of integer lattice points on the boundary of $T$. Let $R'$ denote the triangular region corresponding to $R$. Since the slope of $\ell_{\mathbf{x}}$ is $-\frac{3x_2 - n_2}{n_1 - 3x_1}$ and the height of $R'$ is $p - n_2 + x_2$, the length of the base of $R'$ is $\frac{(p-n_2+x_2)(n_1-3x_1)}{3x_2-n_2}$. Thus, the area of $R'$ is $\frac{(p-n_2+x_2)^2(n_1-3x_1)}{2(3x_2-n_2)}$. Note that both $\frac{1}{|\mathbf{n}|}(p - n_2 + x_2)$ and $\frac{1}{|\mathbf{n}|} \cdot \frac{(p-n_2+x_2)(n_1-3x_1)}{3x_2-n_2}$ go to 0 as $n_1, n_2$ go to infinity. Therefore, in order to prove our theorem, it suffices to show that

$$\frac{1}{|\mathbf{n}|}\left(|\mathbf{n}| - \frac{|\mathbf{n} - \mathbf{x}|}{2} - \frac{(p - n_2 + x_2)^2(n_1 - 3x_1)}{2(3x_2 - n_2)}\right) \leqslant \frac{7}{9}. \tag{5}$$

Let

$$\alpha = \frac{x_1}{n_1} \quad \text{and} \quad \beta = \frac{x_2}{n_2}.$$

Then, the left side of (5) is equal to

$$1 - \frac{(1-\alpha)(1-\beta)}{2} - \frac{(2\alpha + 2\beta - 1 - 3\alpha\beta)^2}{2(1-3\alpha)(3\beta - 1)}.$$

Suppose to the contrary that (5) does not hold, that is,

$$1 - \frac{(1-\alpha)(1-\beta)}{2} - \frac{(2\alpha + 2\beta - 1 - 3\alpha\beta)^2}{2(1-3\alpha)(3\beta - 1)} > \frac{7}{9},$$

or

$$\frac{2}{9} > \frac{(1-\alpha)(1-\beta)}{2} + \frac{(2\alpha + 2\beta - 1 - 3\alpha\beta)^2}{2(1-3\alpha)(3\beta - 1)}.$$

Note that $(1 - 3\alpha)(1 - 3\beta)$ is negative since the slope of $\ell_{\mathbf{x}}$ is negative. Now, by multiplying $2(1 - 3\alpha)(1 - 3\beta)$ to both sides, we obtain

$$\frac{4(1-3\alpha)(1-3\beta)}{9} < (1-\alpha)(1-\beta)(1-3\alpha)(1-3\beta) - (2\alpha + 2\beta - 1 - 3\alpha\beta)^2.$$

The right side of the above is equal to

$$\begin{aligned} & (1-\alpha-\beta+\alpha\beta)(1-3\alpha-3\beta+9\alpha\beta) \\ & -(4\alpha^2+4\beta^2+1+9\alpha^2\beta^2+8\alpha\beta-4\alpha-4\beta-12\alpha^2\beta-12\alpha\beta^2+6\alpha\beta) \\ = \; & -\alpha^2-\beta^2+2\alpha\beta. \end{aligned}$$

Thus,

$$\frac{4(1-3\alpha-3\beta+9\alpha\beta)}{9} < -\alpha^2-\beta^2+2\alpha\beta,$$

or

$$9\alpha^2+9\beta^2+18\alpha\beta-12\alpha-12\beta+4<0.$$

This is equivalent to $(3\alpha+3\beta-2)^2 < 0$, which is a contradiction. This completes the proof.

# 4 Remarks

We found the maximum density of a $k$-sum $\mathbf{n}$-free set in the 2-dimensional integer lattice for all positive integers $k$ and all 2-dimensional integer lattice points $\mathbf{n}$; this is equivalent to an $\mathcal{L}$-free set where $\mathcal{L}$ is an equation of the form $\mathbf{x}_1+\cdots+\mathbf{x}_k=\mathbf{n}$. Several fundamental questions remain unsolved regarding this topic, and we list a few.

**Problem 7.** Determine the minimum real number $\alpha$ such that for a $k$-sum $(n,n)$-free set $S$, $|S| \geqslant \alpha n^2$ is a subset of the extremal example in Remark 5.

**Problem 8.** What is the number of $k$-sum $(n,n)$-free sets in $[n]\times[n]$? Among them, how many are maximal?

Of course it would be interesting to obtain a higher dimension analogue to the question of $k$-sum $\mathbf{n}$-free sets.

**Problem 9.** For an integer $d \geqslant 3$, determine $\nu_k(\mathbf{n})$ for a $d$-dimensional integer lattice point $\mathbf{n}$ in $\mathbb{Z}^d_{>0}$.

In a slightly different avenue, it would be interesting to consider a more general linear equation $\mathcal{L}$. However, we do not have a complete answer even for the 1-dimensional case regarding this question. That is, determine the maximum size of an $\mathcal{L}$-free set of $[n]$, where $\mathcal{L} : a_1x_1+\cdots+a_kx_k=b$ for some integer coefficients $a_i$ and $b$. It was recently revealed that the problem is $\sharp$P-complete, see [7].


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