Volumes of flow polytopes related to caracol graphs

Jihyeug Jang   Jang Soo Kim*

Department of Mathematics
Sungkyunkwan University (SKKU)
Suwon, Gyeonggi-do 16419, South Korea

{ab4242,jangsookim}@skku.edu

Abstract

Recently, Benedetti et al. introduced an Ehrhart-like polynomial associated to a graph. This polynomial is defined as the volume of a certain flow polytope related to a graph and has the property that the leading coefficient is the volume of the flow polytope of the original graph with net flow vector $(1, 1, \ldots, 1)$. Benedetti et al. conjectured a formula for the Ehrhart-like polynomial of what they call a caracol graph. In this paper their conjecture is proved using constant term identities, labeled Dyck paths, and a cyclic lemma.

Mathematics Subject Classifications: 05A15, 05C21, 52B20

1   Introduction

The main objects in this paper are flow polytopes, which are certain polytopes associated to acyclic directed graphs with net flow vectors. Flow polytopes have interesting connections with representation theory, geometry, analysis, and combinatorics. A well known flow polytope is the Chan–Robbins–Yuen polytope, which is the flow polytope of the complete graph K_{n+1} with net flow vector $(1, 0, \ldots, 0)$. Chan, Robbins, and Yuen [4] conjectured that the volume of this polytope is a product of Catalan numbers. Their conjecture was proved by Zeilberger [15] using the Morris constant term identity [11], which is equivalent to the famous Selberg integral [12].

Since the discovery of the Chan–Robbins–Yuen polytope, researchers have found many flow polytopes whose volumes have nice product formulas, see [3, 5, 6, 8, 9, 10, 14] and references therein. In this paper we add another flow polytope to this list by proving a product formula for the volume of the flow polytope coming from a caracol graph, which
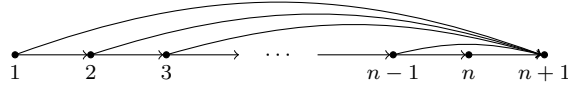

Figure 1: The Pitman-Stanley graph PS_{n+1}.

Figure 2: The caracol graph Car_{n+1}.

was recently conjectured by Benedetti et al. [3]. In order to state our results we introduce necessary definitions.

We let $[n] := \{1, 2, \ldots, n\}$. Throughout this paper, we only consider connected directed graphs in which every vertex is an integer and every directed edge is of the form (i, j) with $i < j$.

Let G be a directed graph on vertex set $[n + 1]$ with m directed edges. We allow G to have multiple edges but no loops. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. An m-tuple $(b_{ij})_{(i,j)\in E} \in \mathbb{R}^m_{\geqslant 0}$ is called an $\mathbf{a}$-*flow of G* if

$$\sum_{(i,j)\in E} b_{ij}(\mathbf{e}_i - \mathbf{e}_j) = \left(a_1, \ldots, a_n, -\sum_{i=1}^n a_i \right),$$

where $\mathbf{e}_i$ is the standard basis vector in $\mathbb{R}^{n+1}$ with a one in the ith entry and zeroes elsewhere. The *flow polytope $\mathcal{F}_G(\mathbf{a})$ of G with net flow $\mathbf{a}$* is defined as the set of all $\mathbf{a}$-flows of G.

In this paper we consider the following two graphs, see Figures 1 and 2:

- The *Pitman-Stanley graph* PS_{n+1} is the graph with vertex set $[n + 1]$ and edge set

$$\{(i, i + 1) : i = 1, 2, \ldots, n\} \cup \{(i, n + 1) : i = 1, 2, \ldots, n - 1\}.$$

- The *caracol graph* Car_{n+1} is the graph with vertex set $[n + 1]$ and edge set

$$\{(i, i+1) : i = 1, 2, \ldots, n\} \cup \{(1, i) : i = 3, 4, \ldots, n\} \cup \{(i, n+1) : i = 2, 3, \ldots, n-1\}.$$

We note that the flow polytope $\mathcal{F}_{\mathrm{PS}_{n+1}}(a_1, \ldots, a_n)$ is affinely equivalent to the polytope

$$\Pi_{n-1}(a_1, \ldots, a_{n-1}) := \{(x_1, \ldots, x_{n-1}) : x_i \geqslant 0, x_1 + \cdots + x_i \leqslant a_1 + \cdots + a_i, 1 \leqslant i \leqslant n-1\},$$

considered in [13]. Pitman and Stanley [13] found volume formulas for certain polytopes,

which can be restated as normalized volumes of flow polytopes as follows:

$$\text{vol}\,\mathcal{F}_{\text{PS}_{n+1}}(a, b^{n-2}, d) = a(a + (n-1)b)^{n-2}, \tag{1}$$

$$\text{vol}\,\mathcal{F}_{\text{PS}_{n+1}}(a, b^{n-3}, c, d) = a(a + (n-1)b)^{n-2} + (n-1)a(c-b)(a + (n-2)b)^{n-3}, \tag{2}$$

$$\text{vol}\,\mathcal{F}_{\text{PS}_{n+1}}(a, b^{n-m-2}, c, 0^{m-1}, d) = a\sum_{j=0}^{m} \binom{n}{j}(c - (m+1-j)b)^j (a + (n-1-j)b)^{n-j-2}, \tag{3}$$

where b^k means the sequence $b, b, \ldots, b$ of k b's. We note that $\text{vol}\,\mathcal{F}_{\text{PS}_{n+1}}(a_1, \ldots, a_n)$ is independent of a_n.

In [3], Benedetti et al. introduced combinatorial models called gravity diagrams and unified diagrams to compute volumes of flow polytopes. Using these models they showed

$$\text{vol}\,\mathcal{F}_{\text{Car}_{n+1}}(a^n) = C_{n-2}a^n n^{n-2}, \tag{4}$$

$$\text{vol}\,\mathcal{F}_{\text{Car}_{n+1}}(a, b^{n-1}) = C_{n-2}a^{n-2}(a + (n-1)b)^{n-2}, \tag{5}$$

where $C_k := \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number.

For a positive integer k and a directed graph G on $[n+1]$, let $\widehat{G}(k)$ be the directed graph obtained from G by adding a new vertex 0 and k multiple edges $(0, i)$ for each $1 \leqslant i \leqslant n$. Then we define

$$E_G(k) = \text{vol}\,\mathcal{F}_{\widehat{G}(k)}(1, 0^n). \tag{6}$$

In [3], Benedetti et al. showed that $E_G(k)$ is a polynomial function in k. Therefore we can consider the polynomial $E_G(x)$. They also showed that these polynomials $E_G(x)$ have similar properties as Ehrhart polynomials. For example, the leading coefficient of $E_G(x)$ is the normalized volume of $\mathcal{F}_G(1^n)$. For this reason, they called $E_G(x)$ an Ehrhart-like polynomial. In the same paper they proved the following theorem.

Theorem 1. *We have*

$$E_{\text{PS}_{n+1}}(k) = \frac{1}{kn-1}\binom{(k+1)n-2}{n}.$$

Our main result is the following theorem, which was conjectured in [3].

Theorem 2. *We have*

$$E_{\text{Car}_{n+1}}(k) = \frac{1}{kn+n-3}\binom{kn+2n-5}{n-1}\binom{n+k-3}{k-1}.$$

In this paper we prove Theorem 2 and give an alternative proof of Theorem 1.

The remainder of this paper is organized as follows. In Section 2 we use the Lidskii volume formula to interpret $E_{\text{Car}_{n+1}}(k)$ as a Kostant partition function, which is equal to the constant term of a Laurent series. In Section 3 we introduce labeled Dyck paths

and show that the constant term is equal to the number of certain labeled Dyck paths. In Section 4 we enumerate these labeled Dyck paths using a cyclic lemma. In Section 5 using our combinatorial models we show the following volume formulas:

$$\text{vol}\,\mathcal{F}_{\text{PS}_{n+1}}(a,b,c^{n-2}) = (a+b-c)(a+b+(n-2)c)^{n-2} + (-b+c)(b+(n-2)c)^{n-2},$$
$$\tag{7}$$

$$\begin{aligned}
\text{vol}\,\mathcal{F}_{\text{PS}_{n+1}}(a,b,c,d^{n-3}) &= (a+b+c-2d)(a+b+c+(n-3)d)^{n-2} \tag{8}\\
&\quad - (b+c-2d)(b+c+(n-3)d)^{n-2}\\
&\quad - (n-1)a(c-d)(c+(n-3)d)^{n-3},
\end{aligned}$$
$$\text{vol}\,\mathcal{F}_{\text{Car}_{n+1}}(a,b,c^{n-2}) = C_{n-2}a^{n-1}(a+b(n-1))(a+b+c(n-2))^{n-3}, \tag{9}$$

where (9) was conjectured by Benedetti et al. in [3].

2 Constant term identities

In this section we review the Lidskii volume formula and restate Theorems 1 and 2 as constant term identities.

Let G be a directed graph on $[n+1]$ and define $\mathbf{a}' = (a_1,\ldots,a_n,-\sum_{i=1}^n a_i)$ for $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{Z}^n$. The *Kostant partition function* $K_G(\mathbf{a}')$ of G at $\mathbf{a}'$ is the number of integer points of $\mathcal{F}_G(\mathbf{a})$, *i.e.*, if G has m edges,

$$K_G(\mathbf{a}') = |\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^m|.$$

We denote by $G|_n$ the restriction of G to the vertices in $[n]$. Let $\text{outdeg}_G(i)$ denote the out-degree of vertex i in G. The following formula, known as the Lidskii volume formula, allows us to express the (normalized) volume of the flow polytope $\mathcal{F}_G(\mathbf{a})$ in terms of Kostant partition functions, see [2, Theorem 38].

Theorem 3 (Lidskii volume formula). *Let G be a connected directed graph on $[n+1]$ with m directed edges, where every directed edge is of the form (i,j) with $i < j$ and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. Denoting $\mathbf{t} = (t_1,\ldots,t_n) := (\text{outdeg}_G(1) - 1, \ldots, \text{outdeg}_G(n) - 1)$, we have*

$$\text{vol}\,\mathcal{F}_G(\mathbf{a}) = \sum_{\substack{|\mathbf{s}|=m-n \\ \mathbf{s} \geqslant \mathbf{t}}} \binom{m-n}{s_1, s_2, \ldots, s_n} a_1^{s_1} \ldots a_n^{s_n} K_{G|_n}(\mathbf{s} - \mathbf{t}),$$

where the sum is over all sequences $\mathbf{s} = (s_1,\ldots,s_n)$ of nonnegative integers such that $|\mathbf{s}| = s_1 + \cdots + s_n = m - n$ and $\mathbf{s} \geqslant \mathbf{t}$ in dominance order, i.e., $\sum_{i=1}^k s_i \geqslant \sum_{i=1}^k t_i$ for $k = 1, 2, \ldots, n$.

In this paper, we use both expressions $K_{G|_n}(a_1,\ldots,a_n)$ and $K_G(a_1,\ldots,a_n,0)$ together. Note that if $\mathbf{a} = (1,0^{n-1})$ in Theorem 3 there is only one nonzero term in the sum giving the following corollary, see [2], [13], or [7, Corollary 1.4].

Corollary 4. *For a directed graph G on $[n+1]$, we have*

$$\operatorname{vol} \mathcal{F}_G(1, 0^{n-1}) = K_G(p, 1 - \operatorname{outdeg}_G(2), 1 - \operatorname{outdeg}_G(3), \ldots, 1 - \operatorname{outdeg}_G(n), 0), \quad (10)$$

where $p = \operatorname{outdeg}_G(2) + \operatorname{outdeg}_G(3) + \cdots + \operatorname{outdeg}_G(n) - n + 1$.

For a multivariate rational function $f(x_1, x_2, \ldots, x_n)$ we denote by $\operatorname{CT}_{x_i} f$ the constant term of the Laurant series expansion of f with respect to x_i by considering other variables as constants. Since $\operatorname{CT}_{x_1} f$ is a rational function in $x_2, \ldots, x_n$, we can apply CT_{x_2} to it. Repeating in this way the constant term $\operatorname{CT}_{x_n} \ldots \operatorname{CT}_{x_1} f$ is defined. We also define $[x_n^{a_n} \ldots x_1^{a_1}]f$ to be the coefficient of the monomial $x_n^{a_n} \ldots x_1^{a_1}$ in the Laurent expansion of f when expanded in the variables $x_1, x_2, \ldots, x_n$ in this order. Note that we have

$$[x_n^{a_n} \ldots x_1^{a_1}]f = \operatorname{CT}_{x_n} \ldots \operatorname{CT}_{x_1} \left(x_n^{-a_n} \ldots x_1^{-a_1} f \right). \quad (11)$$

Let G be a directed graph on $[n+1]$. Then for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and $a_{n+1} = -(a_1 + \cdots + a_n)$, the Kostant partition function $K_G(\mathbf{a}')$ can be computed by

$$K_G(\mathbf{a}') = [x_{n+1}^{a_{n+1}} \cdots x_1^{a_1}] \prod_{(i,j) \in E(G)} \left(1 - \frac{x_i}{x_j} \right)^{-1}. \quad (12)$$

Now we are ready to express $E_{\mathrm{PS}_{n+1}}(k)$ and $E_{\mathrm{Car}_{n+1}}(k)$ as constant terms of Laurent series. Throughout this paper the factor $(x_j - x_i)^{-1}$, where $i < j$, means the Laurent expansion

$$(x_j - x_i)^{-1} = \frac{1}{x_j} \left(1 - \frac{x_i}{x_j} \right)^{-1} = \frac{1}{x_j} \sum_{l \geqslant 0} \left(\frac{x_i}{x_j} \right)^l.$$

Proposition 5. *We have*

$$E_{\mathrm{PS}_{n+1}}(k) = \operatorname{CT}_{x_n} \ldots \operatorname{CT}_{x_1} \prod_{i=1}^{n} (1 - x_i)^{-k} \prod_{i=1}^{n-1} (x_{i+1} - x_i)^{-1}, \quad (13)$$

$$E_{\mathrm{Car}_{n+2}}(k) = \operatorname{CT}_{x_n} \ldots \operatorname{CT}_{x_1} \frac{1}{x_1} \prod_{i=1}^{n} (1 - x_i)^{-k} \prod_{i=1}^{n-1} (x_n - x_i)^{-1} \prod_{i=1}^{n-2} (x_{i+1} - x_i)^{-1}. \quad (14)$$

Proof. We will only prove (14) since (13) can be proved similarly. Let $G = \mathrm{Car}_{n+1}$ and $H = \widehat{G}(k)$. Then H is a graph with vertices $0, 1, 2, \ldots, n+1$, and by (6) and (10),

$$E_{\mathrm{Car}_{n+1}}(k) = K_H(p, 1 - \operatorname{outdeg}_H(1), 1 - \operatorname{outdeg}_H(2), \ldots, 1 - \operatorname{outdeg}_H(n), 0),$$

where $p = \operatorname{outdeg}_H(1) + \operatorname{outdeg}_H(2) + \cdots + \operatorname{outdeg}_H(n) - n$. Since $\operatorname{outdeg}_H(1) = n - 1$, $\operatorname{outdeg}_H(n) = 1$, and $\operatorname{outdeg}_H(i) = 2$ for $2 \leqslant i \leqslant n - 1$, we can rewrite the above equation as

$$E_{\mathrm{Car}_{n+1}}(k) = K_H(2n - 4, 2 - n, (-1)^{n-2}, 0, 0).$$

Then, by (12), we obtain

$$E_{\mathrm{Car}_{n+1}}(k) = [x_{n-1}^{-1} \cdots x_2^{-1} x_1^{2-n} x_0^{2n-4}] \prod_{(i,j) \in E(H)} \left(1 - \frac{x_i}{x_j}\right)^{-1}. \tag{15}$$

Since every term in the expansion of

$$\prod_{(i,j) \in E(H)} \left(1 - \frac{x_i}{x_j}\right)^{-1} = \prod_{i=1}^{n} \left(1 - \frac{x_0}{x_i}\right)^{-k} \left(\prod_{i=2}^{n} \left(1 - \frac{x_1}{x_i}\right) \left(1 - \frac{x_i}{x_{n+1}}\right) \prod_{i=2}^{n-1} \left(1 - \frac{x_i}{x_{i+1}}\right)\right)^{-1}$$

is homogeneous of degree 0 in the variables $x_0, x_1, \ldots, x_{n+1}$, we can set $x_0 = 1$ in (15). Moreover, since every term in the expansion of $(1 - x_i/x_{n+1})^{-1}$ has a negative power of x_{n+1} except for the constant term 1, we can omit the factors involving x_{n+1} in (15). Then, by the same argument, we can also omit the factors involving x_n in (15) to obtain

$$E_{\mathrm{Car}_{n+1}}(k) = [x_1^{2-n} x_2^{-1} \cdots x_{n-1}^{-1}] \prod_{i=1}^{n-1} \left(1 - \frac{1}{x_i}\right)^{-k} \prod_{i=2}^{n-1} \left(1 - \frac{x_1}{x_i}\right)^{-1} \prod_{i=2}^{n-2} \left(1 - \frac{x_i}{x_{i+1}}\right)^{-1}.$$

By replacing x_j by x_{n-j}^{-1} for each $1 \leqslant j \leqslant n-1$ we have

$$E_{\mathrm{Car}_{n+1}}(k) = [x_{n-1}^{n-2} x_{n-2} \cdots x_1] \prod_{i=1}^{n-1} (1 - x_i)^{-k} \prod_{i=1}^{n-2} \left(1 - \frac{x_i}{x_{n-1}}\right)^{-1} \prod_{i=1}^{n-3} \left(1 - \frac{x_i}{x_{i+1}}\right)^{-1},$$

which is equivalent to (14) by (11). □

By Proposition 5, we can restate Theorems 1 and 2 as follows.

Theorem 6. *We have*

$$\mathrm{CT}_{x_n} \ldots \mathrm{CT}_{x_1} \prod_{i=1}^{n} (1 - x_i)^{-k} \prod_{i=1}^{n-1} (x_{i+1} - x_i)^{-1} = \frac{1}{kn-1} \binom{(k+1)n - 2}{n}.$$

Theorem 7. *We have*

$$\mathrm{CT}_{x_n} \ldots \mathrm{CT}_{x_1} \frac{1}{x_1} \prod_{i=1}^{n} (1 - x_i)^{-k} \prod_{i=1}^{n-1} (x_n - x_i)^{-1} \prod_{i=1}^{n-2} (x_{i+1} - x_i)^{-1}$$

$$= \frac{1}{k(n+1) + n - 2} \binom{kn + k + 2n - 3}{n} \binom{n + k - 2}{k - 1}.$$
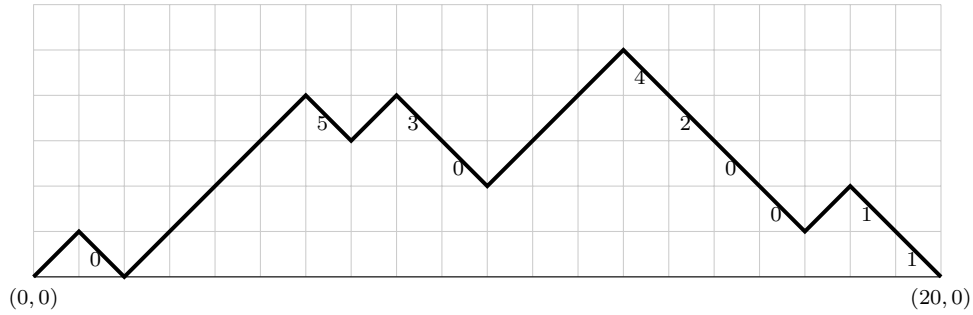
Figure 3: A 5-labeled Dyck path of length 20. The labels of every consecutive down-steps must be in weakly decreasing order.
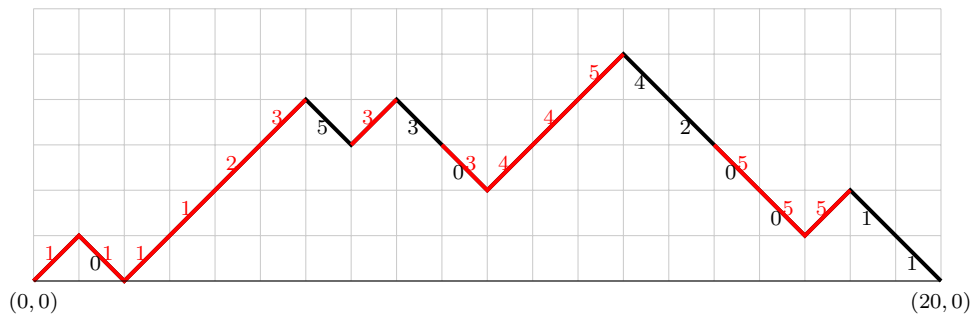

Figure 4: A doubly 5-labeled Dyck path of length 20. The down-steps labeled 0 and the up-steps are the red steps and their additional labels are written in red. The red labels must be in weakly increasing order.

3   Labeled Dyck Paths

In this section we give combinatorial meanings to the constant terms in Theorems 6 and 7 using labeled Dyck paths.

A *Dyck path of length* $2n$ is a lattice path from $(0,0)$ to $(2n, 0)$ consisting of *up-steps* $(1, 1)$ and *down-steps* $(1, -1)$ lying on or above the line $y = 0$. The set of Dyck paths of length $2n$ is denoted by Dyck_n.

Let k be a positive integer. A *k-labeled Dyck path* is a Dyck path with a labeling on the down-steps such that the label of each down-step is an integer $0 \leqslant i \leqslant k$ and the labels of any consecutive down-steps are in weakly decreasing order, see Figure 3. A *doubly k-labeled Dyck path* is a k-labeled Dyck path together with an additional labeling on the down-steps labeled 0 and the up-steps with integers from $\{1, 2, \ldots, k\}$ such that the additional labels on these steps are weakly increasing, see Figure 4.

We denote by $\mathrm{Dyck}_n(k)$ (resp. $\mathrm{Dyck}_n^{(2)}(k)$) the set of k-labeled Dyck paths (resp. doubly k-labeled Dyck paths) of length $2n$. We also denote by $\mathrm{Dyck}_n(k, d)$ the set of k-labeled Dyck paths of length $2n$ with exactly d down-steps labeled 0.

A *multiset* is a set with repetitions allowed. Let $\left(\!\binom{n}{m}\!\right) := \binom{n+m-1}{m}$. Then $\left(\!\binom{n}{m}\!\right)$ is the number of multisets with m elements taken from $[n]$. Equivalently, $\left(\!\binom{n}{m}\!\right)$ is the number of

nonnegative integer solutions $(a_1, a_2, \ldots, a_n)$ to $a_1 + a_2 + \cdots + a_n = m$ and also the number of m-tuples $(i_1, i_2, \ldots, i_m)$ of nonnegative integers satisfying $1 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_m \leqslant n$. The following proposition is immediate from the definitions of $\mathrm{Dyck}_n^{(2)}(k)$ and $\mathrm{Dyck}_n(k,d)$.

Proposition 8. *We have*

$$|\mathrm{Dyck}_n^{(2)}(k)| = \sum_{d=0}^{n} |\mathrm{Dyck}_n(k,d)| \left(\!\!\binom{k}{n+d}\!\!\right).$$

We now show that the constant terms in Theorems 6 and 7 have the following combinatorial interpretations.

Theorem 9. *We have*

$$\mathrm{CT}_{x_n} \ldots \mathrm{CT}_{x_1} \prod_{i=1}^{n}(1-x_i)^{-k} \prod_{i=1}^{n-1}(x_{i+1}-x_i)^{-1} = |\mathrm{Dyck}_{n-1}(k,0)|.$$

Proof. Consider that we choose $x_i^{a_{i1}} x_i^{a_{i2}} \cdots x_i^{a_{ik}}$ in $(1-x_i)^{-k} = (1 + x_i + x_i^2 + \cdots) \cdots (1 + x_i + x_i^2 + \cdots)$ for $i = 1, 2, \ldots, n$ and we choose $x_i^{b_i}/x_{i+1}^{b_i+1}$ in $(x_{i+1}-x_i)^{-1} = 1/x_{i+1} + x_i/x_{i+1}^2 + x_i^2/x_{i+1}^3 + \cdots$ for $i = 1, 2, \ldots, n-1$. Then $\prod_{i=1}^{n}(1-x_i)^{-k} \prod_{i=1}^{n-1}(x_{i+1}-x_i)^{-1} = \sum \prod_{i=1}^{n} (x_i^{a_{i1}+\cdots+a_{ik}+b_i-b_{i-1}-1})$, where the sum is over all nonnegative integers a_{ij}, b_i for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$ with $b_0 = -1$ and $b_n = 0$. Hence the left-hand side is the number of the nonnegative integer solutions to the equations $a_{i1} + \cdots + a_{ik} = b_{i-1} - b_i + 1$ for $i = 1, 2, \ldots, n$ with $b_0 = -1, b_n = 0$. If we set $r_i = b_{i-1} - b_i + 1$, so that $r_1 + r_2 + \cdots + r_j = j + b_0 - b_j \leqslant j - 1$, then the number of solutions is $\sum \prod_{i=1}^{n} \left(\!\!\binom{k}{r_i}\!\!\right)$, where the sum is over all nonnegative integers $r_1, \ldots, r_n$ with $r_1 + \cdots + r_j \leqslant j - 1$ for $j = 1, 2, \ldots, n-1$ and $r_1 + \cdots + r_n = n - 1$. For such an n-tuple $(r_1, \ldots, r_n)$, let D be the Dyck path of length $2(n-1)$ such that the number of consecutive down-steps after the ith up-step is r_{i+1} for $i = 1, \ldots, n-1$. The map $(r_1, \ldots, r_n) \mapsto D$ is a bijection from the set of n-tuples satisfying the above conditions to Dyck_{n-1}. Under this correspondence, $\prod_{i=1}^{n} \left(\!\!\binom{k}{r_i}\!\!\right)$ is the number of k-labeled Dyck paths in $\mathrm{Dyck}_{n-1}(k,0)$ whose underlying Dyck path is D. Therefore we obtain the result. $\square$

Theorem 10. *We have*

$$\mathrm{CT}_{x_n} \ldots \mathrm{CT}_{x_1} \frac{1}{x_1} \prod_{i=1}^{n}(1-x_i)^{-k} \prod_{i=1}^{n-1}(x_n-x_i)^{-1} \prod_{i=1}^{n-2}(x_{i+1}-x_i)^{-1} = |\mathrm{Dyck}_{n-1}^{(2)}(k)|.$$

Proof. Similarly to the previous theorem, considering $x_i^{a_{i1}} x_i^{a_{i2}} \cdots x_i^{a_{ik}}$ in $(1-x_i)^{-k}$ and $x_i^{b_i}/x_{i+1}^{b_i+1}$ in $(x_{i+1}-x_i)^{-1}$ and $x_i^{c_i}/x_n^{c_i+1}$ in $(x_n-x_i)^{-1}$, we get that the left-hand side is the number of the nonnegative integer solutions to the equations $a_{i1} + \cdots + a_{ik} + b_i + c_i = 1 + b_{i-1}$ for $i = 1, 2, \ldots, n-1$ and $a_{n1} + \cdots + a_{nk} + b_n + c_n = n - 1 + c_1 + \cdots + c_{n-1}$ with $b_0 = b_{n-1} = b_n = c_n = 0$. If we set $r_i = b_{i-1} - b_i + 1$ for $i = 1, 2, \ldots, n-1$, then the number of solutions is $\sum_{r_i, c_i} \prod_{i=1}^{n-1} \left(\!\!\binom{k}{r_i-c_i}\!\!\right) \left(\!\!\binom{k}{n-1+c_1+\cdots+c_{n-1}}\!\!\right)$ where the sum is over all nonnegative integers r_i, c_i for $i = 1, 2, \ldots, n-1$ with $r_1 + \cdots + r_j \leqslant j$ for $j = 1, 2, \ldots, n-2$

and $r_1 + \cdots + r_{n-1} = n-1$. For such an n-tuple $(r_1, \ldots, r_n)$, let D be the Dyck path of length $2(n-1)$ such that the number of consecutive down-steps after the ith up-step is r_i for $i = 1, \ldots, n-1$. The map $(r_1, \ldots, r_n) \mapsto D$ is a bijection from the set of n-tuples satisfying the above conditions to Dyck_{n-1}. Regard $\left(\!\binom{k}{r-c}\!\right)$ as the number of r-tuples $(i_1, \ldots, i_r)$ of integers with $k \geqslant i_1 \geqslant \ldots \geqslant i_{r-c} \geqslant 1$ and $i_{r-c+1} = \cdots = i_r = 0$. Then $\sum_{c_i} \prod_{i=1}^{n-1} \left(\!\binom{k}{r_i - c_i}\!\right)\left(\!\binom{k}{n-1+c_1+\cdots+c_{n-1}}\!\right)$, where the sum is over all nonnegative integers c_i for $i = 1, 2, \ldots, n-1$ with $c_i \leqslant r_i$, is the number of doubly k-labeled Dyck paths whose underlying Dyck path is D. Therefore we obtain the result. $\qquad\square$

Note that by Proposition 8, we can compute the constant terms in Theorems 9 and 10 if we have a formula for the cardinality $|\mathrm{Dyck}_n(k,d)|$. Therefore our next step is to find this number.

4   A cyclic lemma

Let $\mathrm{Dyck}_n(k; a_0, a_1, \ldots, a_k)$ denote the set of k-labeled Dyck paths of length $2n$ such that the number of down-steps with label i is a_i for $0 \leqslant i \leqslant k$. In this section we prove the following theorem using a cyclic lemma.

Theorem 11. *We have*

$$|\mathrm{Dyck}_n(k; a_0, a_1, \ldots, a_k)| = \frac{1}{n+1} \prod_{i=0}^{k} \left(\!\!\binom{n+1}{a_i}\!\!\right).$$

Remark 12. A *parking function* of length n is a tuple $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}_{>0}^n$ with a condition that $q_i \leqslant i$ for $i = 1, 2, \ldots, n$ where $(q_1, q_2, \ldots, q_n)$ is the rearrangement of $(p_1, p_2, \ldots, p_n)$ in weakly increasing order. Let PF_n be the set of parking function of length n. There is a well-known bijection between PF_n and n-labeled Dyck paths of length $2n$ which the number of each label from 1 to n equals 1. Thus, using Theorem 11, we have

$$|PF_n| = |\mathrm{Dyck}_n(n; 0, 1, 1, \ldots, 1)| = (n+1)^{n-1}.$$

Remark 13. Recently, Yip [14, Theorem 3.18] considered a set $\mathcal{T}_k(n,i)$ of certain labeled Dyck paths and found a simple formula for its cardinality using a cyclic lemma. Using our notation, this set can be written

$$\mathcal{T}_k(n,i) = \bigcup_{a_0 + \cdots + a_{k-1} = n-i} \mathrm{Dyck}_n(k+i-1; a_0, \ldots, a_{k-1}, 1^i).$$

The proof of Theorem 11 in this section is essentially the same as that in [14, Theorem 3.18].

A *k-labeled Dyck word* of length $2n$ is defined as a sequence $w = w_1 \ldots w_{2n}$ of letters in $\{U, D_0, D_1, \ldots, D_k\}$ satisfying the following conditions:

- The number of U's is equal to n.

- For any prefix $w_1 \ldots w_j$, the number of U's is greater than or equal to the total number of D_i's for $0 \leqslant i \leqslant k$.

- The labels of any consecutive D_i's are in weakly decreasing order, i.e., if $w_i = D_a$ and $w_{i+1} = D_b$, then $a \geqslant b$.

Replacing each up step by U and each down step labeled i by D_i is an obvious bijection from k-labeled Dyck paths to k-labeled Dyck words. For example, the k-labeled Dyck word corresponding to the k-labeled Dyck path in Figure 3 is

$$U D_0 U U U U D_5 U D_3 D_0 U U U D_4 D_2 D_0 D_0 U D_1 D_1. \tag{16}$$

From now on, we will identify k-labeled Dyck paths with k-labeled Dyck words using this bijection. Note that $\mathrm{Dyck}_n(k; a_0, a_1, \ldots, a_k)$ is then the set of k-labeled Dyck words of length $2n$ in which the number of D_i's is equal to a_i for $0 \leqslant i \leqslant k$. We can count such words by using a well-known cyclic argument. We first need another definition.

An *extended k-labeled word* of length $2n + 1$ is a sequence $w = w_1 \ldots w_{2n+1}$ of letters in $\{U, D_0, D_1, \ldots, D_k\}$ satisfying the following conditions:

- $w_1 = U$.

- The number of U's is equal to $n + 1$.

- The labels of any consecutive D_i's are in weakly decreasing order, i.e., if $w_i = D_a$ and $w_{i+1} = D_b$, then $a \geqslant b$.

Define $\mathrm{EW}_n(k)$ to be the set of all extended k-labeled words of length $2n + 1$. For $w = w_1 \ldots w_{2n+1} \in \mathrm{EW}_n(k)$, we define the integer $\mathrm{index}(w)$ using the following algorithm. Here, $w = w_1 \ldots w_{2n+1}$ is cyclically ordered, which means that w_1 is followed by w_2, w_2 is followed by w_3, and so on, and w_{2n+1} is followed by w_1.

- Find a letter U followed by a D_i for some $0 \leqslant i \leqslant k$ in cyclic order and delete this pair U and D_i from w. Repeat this until there is only one letter left, which must be U.

- If the remaining U is the jth U in the original word w then define $\mathrm{index}(w) = j$.

We also define the *shifting operator* $s : \mathrm{EW}_n(k) \to \mathrm{EW}_n(k)$ by

$$s(w) := w_i w_{i+1} \cdots w_{2n+1} w_1 \ldots w_{i-1},$$

where i is the largest integer with $w_i = U$.

Example 14. Let $w = U D_1 D_0 U U D_1 U \in \mathrm{EW}_3(1)$. Then by the algorithm,

$$\underline{U D_1} D_0 U U D_1 U \to D_0 U \underline{U D_1} U \to \underline{D_0 U} U \underline{U} \to U,$$

we get $\mathrm{index}(w) = 2$ since the remaining U is the second U in w.

Since the above algorithm treats the word w cyclically one can easily see that the following lemma holds.

Lemma 15. *For any element $w \in \mathrm{EW}_n(k)$, we have*

$$\mathrm{index}(s(w)) \equiv \mathrm{index}(w) + 1 \mod n + 1.$$

Observe that for $w = w_1 w_2 \ldots w_{2n+1} \in \mathrm{EW}_n(k)$ we have $w_1 \ldots w_{2n} \in \mathrm{Dyck}_n(k)$ if and only if $\mathrm{index}(w) = n + 1$. Therefore, by Lemma 15, for each $w \in \mathrm{EW}_n(k)$ there is a unique integer $0 \leqslant j \leqslant n$ such that $s^j(w) = w'U$ for some k-labeled Dyck word w' of length $2n$. This defines a map $p : \mathrm{EW}_n(k) \to \mathrm{Dyck}_n(k)$ sending w to $p(w) = w'$. Again, by Lemma 15, this is a $(n+1)$-to-1 map. Note that w and w' have the same the number of steps D_i for each $0 \leqslant i \leqslant k$. We have proved the following proposition.

Proposition 16. *There is an $(n + 1)$-to-1 map $p : \mathrm{EW}_n(k) \to \mathrm{Dyck}_n(k)$ preserving the number of D_i's for $0 \leqslant i \leqslant k$.*

We now can prove Theorem 11 easily.

Proof of Theorem 11. By Proposition 16, $(n + 1)|\mathrm{Dyck}_n(k; a_0, a_1, \ldots, a_k)|$ is the number of elements $w \in \mathrm{EW}_n(k)$ in which D_i appears a_i times for $0 \leqslant i \leqslant k$. Since consecutive D_i's are always ordered according to their subscripts, such elements w are obtained from the sequence $U \ldots U$ of $n+1$ U's by inserting a_i D_i's after U's in $\left(\!\binom{n+1}{a_i}\!\right)$ ways for $0 \leqslant i \leqslant k$ independently. Thus we have

$$(n + 1)|\mathrm{Dyck}_n(k; a_0, a_1, \ldots, a_k)| = \prod_{i=0}^{k} \left(\!\binom{n + 1}{a_i}\!\right),$$

which completes the proof. $\qquad\square$

As corollaries we obtain formulas for $|\mathrm{Dyck}_n(k, d)|$ and $|\mathrm{Dyck}^{(2)}_{n-1}(k)|$.

Corollary 17. *We have*

$$|\mathrm{Dyck}_n(k, d)| = \frac{1}{n + 1} \left(\!\binom{n + 1}{d}\!\right) \left(\!\binom{k(n + 1)}{n - d}\!\right).$$

Proof. By Theorem 11,

$$|\mathrm{Dyck}_n(k, d)| = \frac{1}{n + 1} \left(\!\binom{n + 1}{d}\!\right) \sum_{a_1 + \cdots + a_k = n - d} \prod_{i=1}^{k} \left(\!\binom{n + 1}{a_i}\!\right).$$

The above sum is equal to the number of k-tuples $(A_1, \ldots, A_k)$ of multisets such that each element $x \in A_i$ satisfies $(n + 1)(i - 1) + 1 \leqslant x \leqslant (n + 1)i$ and $\sum_{i=1}^{k} |A_i| = n - d$. Since such a k-tuple is completely determined by $A := A_1 \cup \cdots \cup A_k$, the sum is equal to $\left(\!\binom{k(n+1)}{n-d}\!\right)$, the number of multisets of size $n - d$ whose elements are in $[k(n + 1)]$. Thus we obtain the formula. $\qquad\square$

Corollary 18. *We have*

$$|\operatorname{Dyck}^{(2)}_{n-1}(k)| = \frac{1}{k(n+1)+n-2} \binom{kn+k+2n-3}{n} \binom{n+k-2}{k-1}.$$

Proof. We will use the following standard notation in hypergeometric series:

$$_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right) = \sum_{i \geqslant 0} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!},$$

where $(a)_i = a(a+1)\ldots(a+i-1)$. By Proposition 8 and Corollary 17,

$$\begin{aligned}
|\operatorname{Dyck}^{(2)}_{n-1}(k)| &= \sum_{d=0}^{n-1} |\operatorname{Dyck}_{n-1}(k,d)| \left(\!\!\binom{k}{d+n-1}\!\!\right) \\
&= \sum_{d=0}^{n-1} \frac{1}{n} \left(\!\!\binom{kn}{n-d-1}\!\!\right) \left(\!\!\binom{n}{d}\!\!\right) \left(\!\!\binom{k}{d+n-1}\!\!\right) \\
&= \frac{(kn+n-2)!(k+n-2)!}{n!(kn-1)!(n-1)!(k-1)!} {}_2F_1\left(\begin{matrix} -n+1, k+n-1 \\ -kn-n+2 \end{matrix}; 1\right). \quad (17)
\end{aligned}$$

By the Vandermonde summation formula [1, Corollary 2.2.3]

$$_2F_1\left(\begin{matrix} -n,b \\ c \end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n},$$

we have

$$\begin{aligned}
{}_2F_1\left(\begin{matrix} -n+1, k+n-1 \\ -kn-n+2 \end{matrix}; 1\right) &= \frac{(-kn-2n-k+3)_{n-1}}{(-kn-n+2)_{n-1}} \\
&= \frac{(kn+2n+k-3)!}{(kn+n+k-2)!} \frac{(kn-1)!}{(kn+n-2)!}.
\end{aligned}$$

By (17) and the above equation, we obtain the result. $\qquad\square$

The constant term identities in Theorems 6 and 7 follow immediately from Theorems 9, 10 and Corollary 18. This completes the proof of Theorems 1 and 2 in the introduction.

5 More Properties of Labeled Dyck Paths

In this section we find volumes of flow polytopes of Pitman–Stanley graph PS_{n+1} and caracol graph Car_{n+1} for certain flow vectors using Lidskii's formula and k-labeled Dyck prefixes.

A *k-labeled Dyck prefix* is the part of a k-labeled Dyck path from $(0,0)$ to (a,b) for some point (a,b) in the path. The set of k-labeled Dyck prefixes from $(0,0)$ to $(2n-i,i)$ is
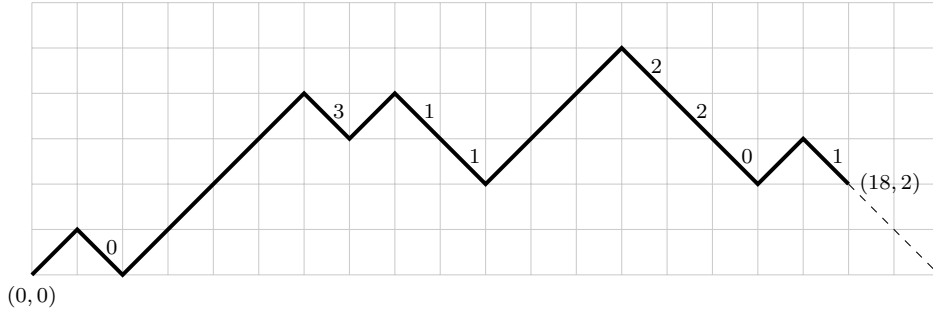
Figure 5: An element of $\mathrm{Dyck}_{20,2}(3; 2, 3, 2, 1)$ whose steps are drawn in solid segments. By appending it with the two dashed down-steps, this element can be considered as an element in $\mathrm{Dyck}_{20}(3)$. This labeled Dyck path can be expressed as $UD_0UUUUD_3UD_1D_1UUUD_2D_2D_0UD_1D_0D_0$, where the last two D_0 steps correspond to the dashed down-steps.

denoted by $\mathrm{Dyck}_{n,i}$. We also denote by $\mathrm{Dyck}_{n,i}(k; a_0, a_1, \ldots, a_k)$ the set of k-labeled Dyck prefixes in $\mathrm{Dyck}_{n,i}$ such that the number of down-steps labeled j is a_j for $0 \leqslant j \leqslant k$.

Recall that $\mathrm{Dyck}_n(k)$ is in bijection with the set of k-Dyck words of length $2n$. Therefore one can consider an element in $\mathrm{Dyck}_{n,i}$ as a k-Dyck word of length $2n$ whose last i letters are D_0's. See Figure 5.

Now we find the cardinality of $\mathrm{Dyck}_{n,i}(k; a_0, a_1, \ldots, a_k)$.

Lemma 19. *We have*

$$|\mathrm{Dyck}_{n,i}(k; a_0, a_1, \ldots, a_k)| = \frac{i+1}{n+1} \prod_{j=0}^{k} \left(\binom{n+1}{a_j} \right).$$

Proof. Let $\mathrm{EW}_{n,i}(k)$ be the set of words $w = w_1 \ldots w_{2n-i+1}$ of letters in $\{U, D_0, \ldots, D_k\}$ with $w_1 = U$ and exactly $n+1$ U's that satisfies the third condition of a k-labeled Dyck word: the labels of any consecutive D_i's are in weakly decreasing order, i.e., if $w_i = D_a$ and $w_{i+1} = D_b$, then $a \geqslant b$. For $w \in \mathrm{EW}_{n,i}(k)$, an *index candidate* of w is an integer j satisfying the following condition:

- Find a letter U followed by a D_i for some $0 \leqslant i \leqslant k$ in cyclic order and delete this pair U and D_i from w. Repeat this until there are $i+1$ letters left, which must be all U's. Then the jth U in the original word w is one of the remaining U's.

Note that there are $i+1$ index candidates for any $w \in \mathrm{EW}_{n,i}(k)$.

Let $\mathrm{EW}'_{n,i}(k)$ be the set of words obtained from a word $w \in \mathrm{EW}_{n,i}(k)$ by adding i D_0's to the left of the jth U in w for an index candidate j of w. Note that $\mathrm{EW}'_{n,i}(k)$ is a subset of $\mathrm{EW}_n(k)$ which is defined in Section 4. Thus every $w' \in \mathrm{EW}'_{n,i}(k)$ has length $2n+1$ and the unique index $\mathrm{index}(w')$ exists. Then by the map p defined in Proposition 16, there is an $(n+1)$-to-1 map from $\mathrm{EW}'_{n,i}(k)$ to $\mathrm{Dyck}_{n,i}(k)$. Since there are $(i+1)$ ways to choose

an index candidate for $w \in \mathrm{EW}_{n,i}(k)$, we have

$$(i+1) \prod_{j=0}^{k} \left(\left(\binom{n+1}{a_j} \right) \right) = (n+1) | \mathrm{Dyck}_{n,i}(k; a_0, a_1, \ldots, a_k)|,$$

which completes the proof. $\qquad\qquad\square$

5.1 Volumes of flow polytopes for the Pitman–Stanley graph.

Recall that $\mathrm{vol}\,\mathcal{F}_{\mathrm{PS}_{n+1}}(a^n)$ and $\mathrm{vol}\,\mathcal{F}_{\mathrm{PS}_{n+1}}(a, b^{n-1})$ were computed in [3] and [13]. In this subsection using Lemma 19 we compute $\mathrm{vol}\,\mathcal{F}_{\mathrm{PS}_{n+1}}(a_1, \ldots, a_k, b^{n-k})$ for $0 \leqslant k \leqslant 3$. For simplicity, we will consider PS_{n+2} instead of PS_{n+1}.

Note that PS_{n+2} has $2n+1$ edges and $\mathbf{t} := (\mathrm{outdeg}(1) - 1, \ldots, \mathrm{outdeg}(n+1) - 1) = (1, 1, \ldots, 1, 0)$. Since $\mathrm{PS}_{n+2}|_{n+1}$ is the path graph on $[n+1]$ with edges $(i, i+1)$ for $1 \leqslant i \leqslant n$, one can easily see that $K_{\mathrm{PS}_{n+2}|_{n+1}}(\mathbf{s} - \mathbf{t}) = 1$ for any sequence $\mathbf{s} \geqslant \mathbf{t}$. Moreover, if $\mathbf{s} = (s_1, \ldots, s_{n+1}) \geqslant \mathbf{t}$, then $s_{n+1} = 0$. Thus Lidskii's formula (Theorem 3) implies

$$\mathrm{vol}\,\mathcal{F}_{\mathrm{PS}_{n+2}}(a_1, \ldots, a_{n+1}) = \sum_{\substack{s_1 + \cdots + s_n = n \\ (s_1, \ldots, s_n) \geqslant (1^n)}} \binom{n}{s_1, s_2, \ldots, s_n} a_1^{s_1} \ldots a_n^{s_n}.$$

Thus, we have

$$\mathrm{vol}\,\mathcal{F}_{\mathrm{PS}_{n+2}}(a_1, \ldots, a_k, b^{n-k+1})$$

$$= \sum_{\substack{s_1 + \cdots + s_n = n \\ (s_1, \ldots, s_n) \geqslant (1^n)}} \binom{n}{s_1, s_2, \ldots, s_n} a_1^{s_1} \ldots a_k^{s_k} b^{s_{k+1} + \cdots + s_n}$$

$$= \sum_{m=0}^{n} \sum_{\substack{s_1 + \cdots + s_k = m \\ (s_1, \ldots, s_k) \geqslant (1^k)}} \sum_{\substack{s_{k+1} + \cdots + s_n = n-m \\ (m, s_{k+1}, \ldots, s_n) \geqslant (k, 1^{n-k})}} \binom{n}{s_1, s_2, \ldots, s_n} a_1^{s_1} \ldots a_k^{s_k} b^{n-m}$$

$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} A_{k,m}(a_1, \ldots, a_k) B_{n,k,m}, \tag{18}$$

where

$$A_{k,m}(a_1, \ldots, a_k) = \sum_{\substack{s_1 + \cdots + s_k = m \\ (s_1, \ldots, s_k) \geqslant (1^k)}} \binom{m}{s_1, s_2, \ldots, s_k} a_1^{s_1} \ldots a_k^{s_k},$$

$$B_{n,k,m} = \sum_{\substack{s_{k+1} + \cdots + s_n = n-m \\ (m, s_{k+1}, \ldots, s_n) \geqslant (k, 1^{n-k})}} \binom{n-m}{s_{k+1}, \ldots, s_n}.$$

The following lemma shows that $B_{n,k,m}$ has a simple formula.

Lemma 20. *We have*

$$B_{n,k,m} = (m - k + 1)(n - k + 1)^{n-m-1}.$$

Proof. For a sequence $(s_{k+1}, \ldots, s_n)$ of nonnegative integers, we have $s_{k+1} + \cdots + s_n = n - m$ and $(m, s_{k+1}, \ldots, s_n) \geqslant (k, 1^{n-k})$ if and only if $UD^{s_n}UD^{s_{n-1}} \ldots UD^{s_{k+1}}U^k D^m$ is a Dyck path from $(0,0)$ to $(2n, 0)$, or equivalently, $UD^{s_n}UD^{s_{n-1}} \ldots UD^{s_{k+1}}$ is a Dyck prefix from $(0,0)$ to $(2n - m - k, m - k)$. Moreover, if such a sequence $(s_{k+1}, \ldots, s_n)$ is given, $\binom{n-m}{s_{k+1}, \ldots, s_n}$ is the number of ways to label the down steps of this Dyck prefix with labels from $\{0, 1, \ldots, n - m - 1\}$ such that there is exactly one down step labeled j for each $0 \leqslant j \leqslant n - m - 1$ and the labels of consecutive down steps are in decreasing order. Thus

$$B_{n,k,m} = |\operatorname{Dyck}_{n-k,m-k}(n - m - 1; 1^{n-m})|.$$

By Lemma 19 we obtain the formula. $\qquad \square$

By (18) and Lemma 20, we obtain the following proposition.

Proposition 21. *We have*

$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a_1, \ldots, a_k, b^{n-k+1})$$
$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m}(m - k + 1)(n - k + 1)^{n-m-1} A_{k,m}(a_1, \ldots, a_k),$$

where

$$A_{k,m}(a_1, \ldots, a_k) = \sum_{\substack{s_1 + \cdots + s_k = m \\ (s_1, \ldots, s_k) \geqslant (1^k)}} \binom{m}{s_1, s_2, \ldots, s_k} a_1^{s_1} \ldots a_k^{s_k}.$$

By Proposition 21, in order to compute $\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a_1, \ldots, a_k, b^{n-k+1})$, it is enough to find $A_{k,m}(a_1, \ldots, a_k)$. For $k = 0, 1$, using this method we can easily recover the following formulas in [3, 13]:

$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a^{n+1}) = a^n(n + 1)^{n-1},$$
$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a, b^n) = a(a + nb)^{n-1}.$$

We now find a formula for this volume for $\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a_1, \ldots, a_k, b^{n-k+1})$ for $k = 2, 3$.

Proposition 22. *For positive integers a, b, and c, we have*

$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a, b, c^{n-1}) = (a + b - c)(a + b + (n - 1)c)^{n-1} - (b - c)(b + (n - 1)c)^{n-1}.$$

Proof. By Proposition 21,

$$\operatorname{vol} \mathcal{F}_{\mathrm{PS}_{n+2}}(a, b, c^{n-1}) = \sum_{m=0}^{n} \binom{n}{m} c^{n-m}(m - 1)(n - 1)^{n-m-1} A_{2,m}(a, b),$$

where $A_{2,0}(a, b) = A_{2,1}(a, b) = 0$ and for $m > 2$,

$$A_{2,m}(a, b) = \sum_{\substack{i+j=m \\ (i,j) \geqslant (1,1)}} \binom{m}{i, j} a^i b^j = (a + b)^m - b^m.$$

Thus

$$\text{vol}\, \mathcal{F}_{\text{PS}_{n+2}}(a, b, c^{n-1}) = \sum_{m=2}^{n} \binom{n}{m} c^{n-m} ((a + b)^m - b^m) (m - 1)(n - 1)^{n-m-1}$$

$$= \frac{1}{n-1} \Big(g_n(a + b, c(n - 1)) - g_n(b, c(n - 1)) $$
$$- f_n(a + b, c(n - 1)) + f_n(b, c(n - 1)) \Big), \qquad (19)$$

where

$$f_n(x, y) = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m} = (x + y)^n,$$

$$g_n(x, y) = \sum_{m=0}^{n} m \binom{n}{m} x^m y^{n-m} = nx(x + y)^{n-1}.$$

Simplifying (19) we obtain the result. $\qquad \square$

In a similar way one can check $A_{3,m}(a, b, c) = (a + b + c)^m - (b + c)^m - ac^{m-1}$ and obtain the following proposition. We omit the details.

Proposition 23. *For positive integers a, b, c, and d, we have*

$$\text{vol}\, \mathcal{F}_{\text{PS}_{n+2}}(a, b, c, d^{n-2}) = (a + b + c - 2d)(a + b + c + (n - 2)d)^{n-1}$$
$$- (b + c - 2d)(b + c + (n - 2)d)^{n-1} - na(c - d)(c + (n - 2)d)^{n-2}.$$

5.2 Volumes of flow polytopes for the caracol graph.

In [3], Benedetti et al. computed $\text{vol}\, \mathcal{F}_{\text{Car}_{n+1}}(a^n)$ and $\text{vol}\, \mathcal{F}_{\text{Car}_{n+1}}(a, b^{n-1})$ using unified diagrams and conjectured a formula for $\text{vol}\, \mathcal{F}_{\text{Car}_{n+1}}(a, b, c^{n-2})$, see Proposition 26 below. In this subsection we prove their conjecture. As before, for simplicity, we consider Car_{n+2} instead of Car_{n+1}.

The caracol graph Car_{n+2} has $3n - 1$ edges and $\mathbf{t}' := (\text{outdeg}(1) - 1, \dots, \text{outdeg}(n + 1) - 1) = (n - 1, 1, 1, \dots, 1, 0)$. Note that $\mathbf{s} = (s_1, \dots, s_{n+1}) \geqslant \mathbf{t}'$ implies $s_{n+1} = 0$. Thus, by Lidskii's formula,

$$\text{vol}\, \mathcal{F}_{\text{Car}_{n+2}}(a_1, \dots, a_{n+1})$$
$$= \sum_{\substack{s_1 + \cdots + s_n = 2n-2 \\ (s_1, \dots, s_n) \geqslant (n-1, 1^{n-1})}} \binom{2n - 2}{s_1, \dots, s_n} a_1^{s_1} \dots a_n^{s_n} K_{\text{Car}_{n+2} \mid n}((s_1, \dots, s_n) - (n - 1, 1^{n-1})).$$

Our goal is to find a formula for $X := \operatorname{vol} \mathcal{F}_{\mathrm{Car}_{n+2}}(a, b, c^{n-1})$. By the above equation,

$$X = \sum_{\substack{s_1 + \cdots + s_n = 2n-2 \\ (s_1, \ldots, s_n) \geqslant (n-1, 1^{n-1})}} \binom{2n-2}{s_1, s_2, \ldots, s_n} a^{s_1} b^{s_2} c^{s_3 + \cdots + s_n}$$

$$\times K_{\mathrm{Car}_{n+2}|_n}(s_1 - n + 1, s_2 - 1, \ldots, s_n - 1).$$

By replacing s_1 by $s_1 + n - 2$, we obtain

$$X = \sum_{\substack{s_1 + \cdots + s_n = n \\ (s_1, \ldots, s_n) \geqslant (1^n)}} \binom{2n-2}{s_1 + n - 2, s_2, \ldots, s_n} a^{s_1 + n - 2} b^{s_2} c^{s_3 + \cdots + s_n}$$

$$\times K_{\mathrm{Car}_{n+2}|_n}(s_1 - 1, \ldots, s_n - 1).$$

Considering $p = s_1$, $q = s_2$, and $r = s_3 + \cdots + s_n$ separately, we can rewrite the above equation as

$$X = \sum_{\substack{p+q+r=n \\ (p,q) \geqslant (1,1)}} \binom{2n-2}{p + n - 2, q, r} a^{p+n-2} b^q c^r A(p, q, r), \tag{20}$$

where

$$A(p, q, r) = \sum_{\substack{s_3 + \cdots + s_n = r \\ (p, q, s_3, \ldots, s_n) \geqslant (1^n)}} \binom{r}{s_3, \ldots, s_n} K_{\mathrm{Car}_{n+2}|_n}(p - 1, q - 1, s_3 - 1, \ldots, s_n - 1).$$

In the next two lemmas we find a formula for $A(p, q, r)$ using labeled Dyck paths.

Note that every Dyck path of length $2n$ can be expressed uniquely as a sequence $U D^{d_n} U D^{d_{n-1}} \ldots U D^{d_1}$ of up steps U and down steps D for some n-tuple $(d_1, \ldots, d_n) \in \mathbb{Z}_{\geqslant 0}^n$ such that $d_1 + \cdots + d_n = n$ and $(d_1, \ldots, d_n) \geqslant (1^n)$. For nonnegative integers $a_1, \ldots, a_n$ whose sum is at most n, let

$$D_n(a_1, \ldots, a_n) := \{U D^{d_n} U D^{d_{n-1}} \ldots U D^{d_1} \in \mathrm{Dyck}_n : d_i \geqslant a_i\}.$$

Lemma 24. Let $(s_1, \ldots, s_n) \in \mathbb{Z}_{\geqslant 0}^n$ with $\sum_{i=1}^n s_i = n$ and $(s_1, \ldots, s_n) \geqslant (1^n)$. Then

$$K_{\mathrm{Car}_{n+2}|_n}(s_1 - 1, \ldots, s_n - 1) = |D_{n-1}(s_2, \ldots, s_n)|.$$

Proof. Note that $\mathrm{Car}_{n+2}|_n$ is a directed graph on $[n]$ with edges $(1, i)$ for $2 \leqslant i \leqslant n$ and $(j, j+1)$ for $2 \leqslant j \leqslant n - 1$. By definition of Kostant partition function, $K_{\mathrm{Car}_{n+2}|_n}((s_1, \ldots, s_n) - (1^n))$ is the number of nonnegative integer solutions $\{b_{1,i}, b_{j,j+1} : 2 \leqslant i \leqslant n, 2 \leqslant j \leqslant n - 1\}$ satisfying

$$b_{1,2} + b_{1,3} + \cdots + b_{1,n} = s_1 - 1,$$
$$b_{2,3} - b_{1,2} = s_2 - 1,$$
$$b_{j,j+1} - b_{j-1,j} - b_{1,j} = s_j - 1, \qquad (3 \leqslant j \leqslant n),$$

where $b_{n,n+1} = 0$. The above equations are equivalent to

$$b_{1,2} + b_{1,3} + \cdots + b_{1,n} = s_1 - 1,$$
$$b_{j,j+1} = (s_2 + \cdots + s_j) + (b_{1,2} + \cdots + b_{1,j}) - (j - 1), \qquad (2 \leqslant j \leqslant n).$$

Thus the integers $b_{j,j+1}$ for $2 \leqslant j \leqslant n-1$ are completely determined by the integers $b_{1,i}$ for $2 \leqslant i \leqslant n$. Moreover, the condition $b_{j,j+1} \geqslant 0$ for $2 \leqslant j \leqslant n-1$ is equivalent to $(s_2, \ldots, s_n) + (b_{12}, \ldots, b_{1n}) \geqslant (1^{n-1})$ in dominance order. Hence $K_{\mathrm{Car}_{n+2}|n}((s_1, \ldots, s_n) - (1^n))$ is the number of $(n-1)$-tuples $(b_{12}, b_{13}, \ldots, b_{1n}) \in \mathbb{Z}_{\geqslant 0}^{n-1}$ such that $b_{12} + b_{13} + \cdots + b_{1n} = s_1 - 1$ and $(s_2, \ldots, s_n) + (b_{12}, \ldots, b_{1n}) \geqslant (1^{n-1})$.

Now let $d_i = s_{i+1} + b_{1,i+1}$ for $1 \leqslant i \leqslant n-1$. Then we can express $K_{\mathrm{Car}_{n+2}|n}((s_1, \ldots, s_n) - (1^n))$ as the number of $(n-1)$-tuples $(d_1, \ldots, d_{n-1}) \in \mathbb{Z}_{\geqslant 0}^{n-1}$ such that $d_1 + \cdots + d_{n-1} = n - 1$, $(d_1, \ldots, d_{n-1}) \geqslant (1^{n-1})$ and $d_i \geqslant s_{i+1}$ for $1 \leqslant i \leqslant n-1$. Since the condition $(d_1, \ldots, d_{n-1}) \geqslant (1^{n-1})$ is equivalent to the condition $UD^{d_{n-1}}UD^{d_{n-2}} \cdots UD^{d_1} \in \mathrm{Dyck}_{n-1}$, we obtain the desired result. $\square$

Lemma 25. *Let p, q and r be fixed nonnegative integers with $p + q + r = n$ and $(p, q) \geqslant (1, 1)$. Then*

$$A(p, q, r) = (p + q - 1)\binom{n + p - 2}{n - 1}(n - 1)^{r-1} - \binom{n + p - 2}{n}(n - 1)^r.$$

Proof. By Lemma 24,

$$A(p, q, r) = \sum_{\substack{s_3 + \cdots + s_n = r \\ (p,q,s_3,\ldots,s_n) \geqslant (1^n)}} \binom{r}{s_3, \ldots, s_n} |D_{n-1}(q, s_3, \ldots, s_n)|.$$

We will give a combinatorial interpretation of each summand in the above formula using labeled Dyck paths. Let $s_3, \ldots, s_n$ be nonnegative integers satisfying $s_3 + \cdots + s_n = r$ and $(p, q, s_3, \ldots, s_n) \geqslant (1^n)$. Consider a Dyck path $\pi = UD^{d_{n-1}}UD^{d_{n-2}} \ldots UD^{d_1} \in D_{n-1}(q, s_3, \ldots, s_n)$. Then $d_1 \geqslant q$ and $d_i \geqslant s_{i+1}$ for $2 \leqslant i \leqslant n-1$. Now we label the down steps of π except the last consecutive down steps D^{d_1} as follows:

- Distribute the r labels $1, 2, \ldots, r$, each label occurring exactly once, to the sequences $D^{d_{n-1}}, D^{d_{n-2}}, \ldots, D^{d_2}$ consecutive down steps of π so that the sequence D^{d_i} gets s_{i+1} labels. There are $\binom{r}{s_3, \ldots, s_n}$ ways to do this.

- Add $d_i - s_{i+1}$ zero labels to the sequence D^{d_i} and arrange the labels in weakly decreasing order.

By this process, we obtain that $\binom{r}{s_3, \ldots, s_n}|D_{n-1}(q, s_3, \ldots, s_n)|$ is the number of Dyck paths $\pi = UD^{d_{n-1}}UD^{d_{n-2}} \ldots UD^{d_1}$ together with a labeling on the down steps except the last consecutive down steps D^{d_1} satisfying the following conditions:

1. $d_i \geqslant s_{i+1}$ for $2 \leqslant i \leqslant n-1$.

2. $q \leqslant d_1 \leqslant n - 1 - r$.

3. The number of down steps labeled i is 1 for $1 \leqslant i \leqslant r$.

4. The number of down steps labeled 0 is $n - 1 - r - d_1$.

5. The labels of any consecutive down steps are weakly decreasing.

Summing over all possible $s_3, \ldots, s_n$ we obtain that $A(p, q, r)$ is the number of Dyck paths $\pi = UD^{d_{n-1}}UD^{d_{n-2}} \ldots UD^{d_1}$ together with a labeling on the down steps of its prefix $UD^{d_{n-1}}UD^{d_{n-2}} \ldots UD^{d_2}$ from $(0, 0)$ to $(2n - 3 - d_1, d_1 - 1)$ satisfying the above conditions except (1). This implies

$$A(p, q, r) = \sum_{d_1 = q}^{n-1-r} |\operatorname{Dyck}_{n-2, d_1 - 1}(r; n - 1 - r - d_1, 1^r)|.$$

By Lemma 19,

$$A(p, q, r) = \sum_{d_1 = q}^{n-1-r} \frac{d_1}{n-1} \left(\binom{n-1}{n-1-r-d_1} \right) (n-1)^r$$

$$= (n-1)^{r-1} \sum_{d_1 = q}^{n-1-r} d_1 \binom{2n - 3 - r - d_1}{n - 2}.$$

Replacing d_1 by $n - 1 - r - i$, we have

$$A(p, q, r) = (n-1)^{r-1} \sum_{i=0}^{p-1} (n - 1 - r - i) \binom{n - 2 + i}{n - 2}.$$

Since

$$(n - 1 - r - i) \binom{n - 2 + i}{n - 2} = ((2n - 2 - r) - (n - 1 + i)) \binom{n - 2 + i}{n - 2}$$

$$= (2n - 2 - r) \binom{n - 2 + i}{n - 2} - (n - 1) \binom{n - 1 + i}{n - 1}$$

$$= (n - 1 - r) \binom{n - 2 + i}{n - 2} - (n - 1) \binom{n - 2 + i}{n - 1},$$

we have

$$A(p, q, r) = (n-1)^{r-1} \left((p + q - 1) \sum_{i=0}^{p-1} \binom{n - 2 + i}{n - 2} - (n - 1) \sum_{i=0}^{p-2} \binom{n - 1 + i}{n - 1} \right).$$

Finally the identity $\sum_{i=0}^{k} \binom{m+i}{m} = \binom{m+k+1}{m+1}$ finishes the proof. $\qquad \square$

Now we are ready to compute $X = \text{vol}\,\mathcal{F}_{\text{Car}_{n+2}}(a, b, c^{n-1})$.

Proposition 26. *[3, Conjecture 6.16] For positive integers a, b, and c, we have*

$$\text{vol}\,\mathcal{F}_{\text{Car}_{n+2}}(a, b, c^{n-1}) = C_{n-1}a^{n-1}(a + nb)(a + b + (n-1)c)^{n-2}.$$

Proof. By (20) and Lemma 25, we have

$$\text{vol}\,\mathcal{F}_{\text{Car}_{n+2}}(a, b, c^{n-1}) = X = Y - Z,$$

where

$$Y = \sum_{\substack{p+q+r=n \\ (p,q) \geqslant (1,1)}} \binom{2n-2}{p+n-2, q, r} a^{p+n-2}b^q c^r (p+q-1)\binom{n+p-2}{n-1}(n-1)^{r-1},$$

$$Z = \sum_{\substack{p+q+r=n \\ (p,q) \geqslant (1,1)}} \binom{2n-2}{p+n-2, q, r} a^{p+n-2}b^q c^r \binom{n+p-2}{n}(n-1)^r.$$

Note that in the above two sums, the condition $(p, q) \geqslant (1, 1)$ can be omitted since the summand is zero if $p = 0$ or $(p, q) = (1, 0)$. Thus

$$Y = \frac{a^{n-1}}{n-1}\binom{2n-2}{n-1} \sum_{p+q+r=n} (p+q-1)\binom{n-1}{p-1, q, r} a^{p-1}b^q(c(n-1))^r,$$

$$Z = \sum_{p+q+r=n} \binom{2n-2}{p+n-2, q, r} a^{p+n-2}b^q c^r \binom{n+p-2}{n}(n-1)^r$$

$$= a^n \binom{2n-2}{n} \sum_{p+q+r=n} \binom{n-2}{p-2, q, r} a^{p-2}b^q(c(n-1))^r.$$

Using the multinomial theorem

$$\sum_{i+j+k=m} \binom{m}{i, j, k} x^i y^j z^k t^{i+j} = (xt + yt + z)^m,$$

and its derivative with respect to t, i.e,

$$\sum_{i+j+k=m} (i+j)\binom{m}{i, j, k} x^i y^j z^k t^{i+j-1} = m(x + y)(xt + yt + z)^{m-1},$$

we obtain

$$Y = C_{n-1}a^{n-1}n(a + b)(a + b + (n-1)c)^{n-2},$$
$$Z = C_{n-1}a^n(n-1)(a + b + (n-1)c)^{n-2},$$

and the proof follows. $\qquad\square$

Acknowledgments

References

[1] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[2] W. Baldoni and M. Vergne. Kostant partitions functions and flow polytopes. *Transform. Groups*, 13(3-4):447–469, 2008.

[3] C. Benedetti, R. S. González D'León, C. R. H. Hanusa, P. E. Harris, A. Khare, A. H. Morales, and M. Yip. A combinatorial model for computing volumes of flow polytopes. *Trans. Amer. Math. Soc.*, 372(5):3369–3404, 2019.

[4] C. S. Chan, D. P. Robbins, and D. S. Yuen. On the volume of a certain polytope. *Experiment. Math.*, 9(1):91–99, 2000.

[5] S. Corteel, J. S. Kim, and K. Mészáros. Flow polytopes with Catalan volumes. *C. R. Math. Acad. Sci. Paris*, 355(3):248–259, 2017.

[6] K. Mészáros. Product formulas for volumes of flow polytopes. *Proc. Amer. Math. Soc.*, 143(3):937–954, 2015.

[7] K. Mészáros and A. H. Morales. Volumes and Ehrhart polynomials of flow polytopes. arXiv:1710.00701.

[8] K. Mészáros, A. H. Morales, and B. Rhoades. The polytope of Tesler matrices. *Selecta Math. (N.S.)*, 23(1):425–454, 2017.

[9] K. Mészáros, A. H. Morales, and J. Striker. On flow polytopes, order polytopes, and certain faces of the alternating sign matrix polytope. *Discrete Comput. Geom.*, 62(1):128–163, 2019.

[10] K. Mészáros, C. Simpson, and Z. Wellner. Flow polytopes of partitions. *Electron. J. Combin.*, 26(1):#P1.47, 2019.

[11] W. G. Morris. *Constant term identities for finite and affine root systems: conjectures and theorems*. PhD thesis, The University of Wisconsin - Madison, 1982.

[12] A. Selberg. Remarks on a multiple integral. *Norsk Mat. Tidsskr.*, 26:71–78, 1944.

[13] R. P. Stanley and J. Pitman. A polytope related to empirical distributions, plane trees, parking functions, and the associahedron. *Discrete Comput. Geom.*, 27(4):603–634, 2002.

[14] M. Yip. A Fuss-Catalan variation of the caracol flow polytope. arXiv:1910.10060.

[15] D. Zeilberger. Proof of a conjecture of Chan, Robbins, and Yuen. *Electron. Trans. Numer. Anal.*, 9:147–148 (electronic), 1999. Orthogonal polynomials: numerical and symbolic algorithms (Leganés, 1998).