The bipartite $K_{2,2}$-free process
and bipartite Ramsey number $b(2,t)$

Deepak Bal

Department of Mathematics
Montclair State University
Montclair, NJ, U.S.A.

deepak.bal@montclair.edu

Patrick Bennett*

Department of Mathematics
Western Michigan University
Kalamazoo, MI, U.S.A.

patrick.bennett@wmich.edu

Abstract

The bipartite Ramsey number $b(s,t)$ is the smallest integer n such that every blue-red edge coloring of $K_{n,n}$ contains either a blue $K_{s,s}$ or a red $K_{t,t}$. In the bipartite $K_{2,2}$-free process, we begin with an empty graph on vertex set $X \cup Y$, $|X| = |Y| = n$. At each step, a random edge from $X \times Y$ is added under the restriction that no $K_{2,2}$ is formed. This step is repeated until no more edges can be added. In this note, we analyze this process and prove that the resulting graph shows that $b(2,t) = \Omega\left(t^{3/2}/\log t\right)$, thereby improving the best known lower bound.

Mathematics Subject Classifications: 05C55, 05D40, 05C80

1 Introduction

The *bipartite Ramsey number $b(s,t)$* is the smallest integer n such that every blue-red edge coloring of $K_{n,n}$ contains either a blue $K_{s,s}$ or a red $K_{t,t}$. This definition was first introduced by Beineke and Schwenk [2] in 1976. We will find it convenient to define the *bipartite independence number* of a graph $G \subseteq K_{n,n}$ as the largest value of t such that there are sets of vertices A and B on opposite sides of the bipartition of $K_{n,n}$ such that $|A| = |B| = t$ and G has no edges in $A \times B$. Thus, $b(s,t) > n$ if and only if there exists a $K_{s,s}$-free graph $G \subseteq K_{n,n}$ that has bipartite independence number less than t. The best known lower and upper bounds on the diagonal problem, due to Hattingh and Henning [13] and Conlon [9] respectively, are

$$\frac{\sqrt{2}}{e} t 2^{t/2} \leqslant b(t,t) \leqslant (1 + o(1)) 2^{t+1} \log_2 t.$$

As is the case for the ordinary Ramsey number, there is an exponential gap which remains to be closed.

In this note, we are concerned with the simplest nontrivial "off-diagonal" case, $b(2, t)$. The best known lower and upper bounds, both due to Caro and Rousseau [8], are

$$\Omega\left(\left(\frac{t}{\log t}\right)^{3/2}\right) \leqslant b(2, t) \leqslant O\left(\left(\frac{t}{\log t}\right)^2\right).$$

The upper bound follows directly from the well known upper bounds on the Zarankiewicz problem. $z(n, s)$ is the largest number of edges in a $K_{s,s}$-free subgraph of $K_{n,n}$. A theorem of Kővári, Sós and Turán [15] says that $z(n, s) = O(n^{2-1/s})$. By the pigeonhole principle, if the number of edges in $K_{n,n}$ exceeds $z(n, 2) + z(n, t)$, then in any blue-red coloring of the edges, one color class must exceed its respective Zarankiewicz bound. Thus solving the inequality $n^2 > c_1 n^{3/2} + c_2 n^{2-1/t}$ for n provides the upper bound. The proof of the lower bound in [8] makes use of the Lovász Local Lemma. In this paper we improve their lower bound by a logarithmic factor:

Theorem 1. *We have the following lower bound on $b(2, t)$:*

$$b(2, t) \geqslant \Omega\left(\frac{t^{3/2}}{\log t}\right).$$

Let $R(G_1, G_2)$ represent the ordinary two color Ramsey number, i.e., the smallest integer n such that every blue-red edge coloring of the edges of K_n contains a blue copy of G_1 or a red copy of G_2. When $G_1 = K_s, G_2 = K_t$, we write $R(s, t)$. The search for the asymptotics of the off diagonal ordinary Ramsey number $R(3, t)$ has a long and interesting history as laid out by Joel Spencer in [20]. Currently, the best known results differ only by a constant factor. Shearer [19], improving a result of Ajtai, Komlós and Szemerédi [1], proved that $R(3, t) \leqslant (1 + o(1))t^2/\log t$. Kim [14] first proved that $R(3, t) \geqslant \Omega(t^2/\log t)$, and Bohman [4] reproved the same bound using a random process first considered by Erdős, Suen and Winkler [10] and now commonly referred to as the triangle-free process. This is a stochastic process in which random edges are added to an empty graph one by one under the constraint that no triangles are formed. (Kim's original proof in [14] considered a "semi-random" version where we receive many random edges at a time instead of one at a time.) Recently, Bohman and Keevash [6], and independently Fiz Pontiveros, Griffiths and Morris [11], improved the analysis of the triangle-free process, and their analysis implies that $R(3, t) \geqslant (\frac{1}{4} - o(1))t^2/\log t$.

Of course one can also consider the H-free process for any H, where edges are randomly added one by one under the constraint that no copy of H is formed. Ruciński and Wormald [18] were among the first to consider such a process, analyzing the d-process, which is the H-free process where H is a star on $d + 1$ vertices. Further work has been done to analyze the H-free process for other families of graphs H, mostly when H is a clique or a cycle (see for example Picollelli [16, 17] and Warnke [21, 22]). Bohman and Keevash [5] have the most general results for the H-free process; they analyze the process and bound

the independence number of the resulting graph for a large class of graphs H including cycles of any length as well as cliques of any size (but also all strictly 2-balanced graphs), establishing new lower bounds on Ramsey numbers $R(H, K_t)$ where H is any fixed cycle or clique and $t \to \infty$. Bohman, Picollelli and Mubayi [7] studied the H-free process for certain hypergraphs H, resulting in new lower bounds for the corresponding hypergraph Ramsey numbers.

Inspired by the previous work on H-free processes, we study the *bipartite $K_{2,2}$-free process*, a version of the H-free process in a large balanced bipartite host graph (as opposed to the standard H-free process which uses K_n as a host graph). The process begins with an empty graph G_0 on vertex set $X \cup Y$ where $|X| = |Y| = n$. We form the graph G_i by adding to G_{i-1} an edge e_i chosen uniformly at random from all pairs of vertices in $X \times Y$ which do not already appear in G_{i-1} and which do not create a copy of $K_{2,2}$. Let M be the random variable representing the number of edges in the final graph produced at the end of the process. Then G_M is $K_{2,2}$-free by construction. The main contribution of this paper is to prove that with high probability, G_M has bipartite independence number at most $Cn^{2/3} \log^{2/3} n$ for some constant C.

Theorem 2. *With high probability, the graph produced at the end of the bipartite $K_{2,2}$-free process has bipartite independence number $O(n^{2/3} \log^{2/3} n)$.*

In the next section we prove Theorem 2, from which Theorem 1 follows trivially.

2 Proof of Theorem 2

Bennett and Bohman [3] proved a general result which we will find useful. We build our proof "on top of" the proof of Theorem 1.1 in [3], in the sense that we will use not only the statement of that theorem but also some other facts that are established in its proof. In particular, Theorem 1.1 in [3] is proved by establishing dynamic concentration of a family of random variables. In our proof, we will use (without further justification) the fact that these variables are dynamically concentrated. One of the goals of this paper is to demonstrate the utility of [3] as a "black box" that takes care of a lot of the work of analyzing these processes, making for shorter proofs.

In Section 2.1 we will summarize the relevant results from [3]. In Section 2.2 we prove a few lemmas which do not follow directly from the results in [3], namely, bounds on the maximum degree and on the maximum density of subsets. Finally, in Section 2.3 we bound the bipartite independence number of the graph produced by the process by using similar proof techniques to those used in [5] for analyzing the H-free process when H is a cycle.

2.1 The Black Box

In this section, we summarize the results from Bennett and Bohman [3] which we will utilize. Let $\mathcal{H}$ be a hypergraph on vertex set V (i.e. $\mathcal{H}$ is a collection of subsets of V and the sets in this collection are the *edges* of $\mathcal{H}$). An *independent set* in $\mathcal{H}$ is a set $I \subseteq V$

such that I contains no edge of $\mathcal{H}$. The *random greedy independent set process* (or just the *independent process*) forms a maximal independent set in $\mathcal{H}$ by iteratively choosing vertices at random to be in the independent set. To be precise, we begin with $\mathcal{H}(0) = \mathcal{H}$, $V(0) = V$ and $I(0) = \emptyset$. Given independent set $I(i)$ and hypergraph $\mathcal{H}(i)$ on vertex set $V(i)$, a vertex $v \in V(i)$ is chosen uniformly at random and added to $I(i)$ to form $I(i+1)$. The new vertex set $V(i+1)$ and new hypergraph $\mathcal{H}(i+1)$ are formed by

1. removing v from every edge in $\mathcal{H}(i)$ that contains v (so these edges become smaller edges),

2. deleting v from $V(i)$, and

3. deleting from $V(i)$ every vertex that is in a singleton edge (and any such edge containing a deleted vertex is removed).

Define the *degree* of a set $A \subseteq V$ to be the number of edges of $\mathcal{H}$ that contain A. For $a = 2, \ldots, r-1$ we define $\Delta_a(\mathcal{H})$ to be the maximum degree of A over $A \in \binom{V}{a}$. We also define the *b-codegree* of a pair of distinct vertices v, v' to be the number of pairs of edges $e, e' \in \mathcal{H}$ such that $v \in e \setminus e', v' \in e' \setminus e$ and $|e \cap e'| = b$. We let $\Gamma_b(\mathcal{H})$ be the maximum b-codegree of $\mathcal{H}$.

Theorem 3 (Theorem 1.1 in [3]). *Let r and $\varepsilon > 0$ be fixed. Let $\mathcal{H}$ be a r-uniform, D-regular hypergraph on N vertices such that $D > N^\varepsilon$. If*

$$\Delta_\ell(\mathcal{H}) < D^{\frac{r-\ell}{r-1} - \varepsilon} \quad for \ \ell = 2, \ldots, r-1 \tag{1}$$

and $\Gamma_{r-1}(\mathcal{H}) < D^{1-\varepsilon}$ then the random greedy independent set algorithm produces an independent set I in $\mathcal{H}$ with

$$|I| = \Omega_{r,\varepsilon}\left(N \cdot \left(\frac{\log N}{D} \right)^{\frac{1}{r-1}} \right) \tag{2}$$

with probability $1 - \exp\left\{ -N^{\Omega_{r,\varepsilon}(1)} \right\}$.

The $K_{2,2}$-free process in the host graph $K_{n,n}$ is an instance of the independent process in a hypergraph $\mathcal{H}_{K_{2,2}} = \mathcal{H}_{K_{2,2}}(n)$, where each vertex in $\mathcal{H}_{K_{2,2}}$ corresponds to an edge in $K_{n,n}$ and edges in $\mathcal{H}_{K_{2,2}}$ correspond to sets of edges in $K_{n,n}$ that form copies of $K_{2,2}$. Thus $\mathcal{H}_{K_{2,2}}$ is a 4-uniform, $(n-1)^2$-regular hypergraph on n^2 vertices. In the notation of Theorem 3, we have $r = 4$, $D = (n-1)^2$, $N = n^2$. Also, we have $\Delta_2(\mathcal{H}_{K_{2,2}}) = O(n)$, $\Delta_3(\mathcal{H}_{K_{2,2}}) = O(1)$ and $\Gamma_3(\mathcal{H}_{K_{2,2}}) = 0$, so the conditions of Theorem 3 are met with $r = 4$ and any $0 < \varepsilon < 1/3$. Thus, Theorem 3 tells us that the bipartite $K_{2,2}$-free process admits $\Omega\left(n^{4/3} \log^{1/3} n \right)$ many edges. For the rest of the paper, we will almost exclusively use language referring to the $K_{2,2}$-free process as opposed to language referring to the independent process on $\mathcal{H}_{K_{2,2}}$.

Now we state the dynamic concentration results that are established in [3] as part of the proof of Theorem 3. As in [3] we define the *scaled time parameter*

$$t = t(i) := \frac{D^{\frac{1}{r-1}}}{N} \cdot i = \frac{(n-1)^{2/3}}{n^2} \cdot i = \frac{i}{n^{4/3}}\left(1 + O\left(\frac{1}{n}\right)\right),$$

and

$$q = q(t) := e^{-t^3}.$$

We let $Q = Q(i)$ be the set of *open* edges at step i, that is, the edges that could be chosen without creating a $K_{2,2}$. For each open edge $e \in Q(i)$ and we define $d_2(e)$ to be the number of copies of $K_{2,2}$ that contain e, contain one additional open edge, and two chosen edges. Roughly speaking $d_2(e)$ is the number of open edges that, if chosen, would close e.

In the proof of their theorem, Bennett and Bohman establish dynamic concentration of the random variables Q and $d_2(e)$ (see equations (8) and (9) in [3]). In particular, their proof implies that there exists a positive constant ε such that w.h.p.

$$|Q| \in \left(1 \pm n^{-10\varepsilon^3}\right) n^2 q \tag{3}$$

$$d_2(e) \in \left(1 \pm n^{-10\varepsilon^3}\right) 3n^{2/3}t^2 q \quad \text{for all } e \in Q \tag{4}$$

for all $i \leqslant i_{max} := \varepsilon n^{4/3} \log^{1/3} n$. Note that lines (3) and (4) significantly simplify the bounds in [3] which actually depend on several constants. However it is not difficult to see that the bounds given in [3] imply that (3) and (4) hold for some sufficiently small $\varepsilon > 0$. In this paper we will always assume that n is sufficiently large and $\varepsilon > 0$ is sufficiently small, and will write some inequalities that only hold under those assumptions.

2.2 High degrees and dense subgraphs

Let G_i be the $K_{2,2}$-free graph at step i. In this section we show that w.h.p. G_i does not have any vertices of degree too high, nor subgraphs that are too dense. More specifically, G_i resembles (at least in these aspects) a binomial random bipartite graph with edge probability $\frac{i}{n^2}$. Let $\mathcal{E}_i$ be the event that (3), (4) hold for all steps up to and including step i, and recall that $\mathbb{P}(\mathcal{E}_{i_{max}}) = 1 - o(1)$.

Lemma 4. *For any set of edges $F \subseteq E(K_{n,n})$, we have*

$$\mathbb{P}\left[\mathcal{E}_{i_{max}} \text{ and } F \subseteq E(G_{i_{max}})\right] \leqslant n^{(-2/3+2\varepsilon^3)|F|}.$$

Proof. The probability that all edges of F are chosen is at most the number of ways to specify which steps these edges will be chosen multiplied by the probability of choosing the prescribed edges in the specified steps. If we know that $\mathcal{E}_{i_{max}}$ holds, then for any $i \leqslant i_{max}$

$$Q(i) \geqslant (1 - n^{-10\varepsilon^3})n^2 q\left(\frac{i_{max}}{n^{4/3}}\right) = (1 - n^{-10\varepsilon^3})n^2 q(\varepsilon \log^{1/3} n) \geqslant (1 + o(1))n^{2-\varepsilon^3}$$

and so the probability of choosing a particular edge on a particular step is at most $\frac{1}{Q} < \frac{1+o(1)}{n^{2-\varepsilon^3}}$ (conditional on the history of the process). So we have that the probability that all edges of F are chosen is at most

$$(\varepsilon n^{4/3} \log^{1/3} n)_{|F|} \cdot \left(\frac{1+o(1)}{n^{2-\varepsilon^3}} \right)^{|F|} < n^{(-2/3+2\varepsilon^3)|F|}. \qquad \square$$

Lemma 5. *W.h.p. we have*

$$deg(v) \leqslant n^{1/3+3\varepsilon^3} \quad \text{for all vertices } v. \tag{5}$$

Proof. By Lemma 4, we have that the probability a particular vertex has degree at least x at step i_{max} (and hence at any step in the process) is at most

$$\binom{n}{x} \cdot n^{(-2/3+2\varepsilon^3)x}.$$

Thus, if we let Z represent the number of vertices with degree at least $n^{1/3+3\varepsilon^3}$, then

$$
\begin{aligned}
\mathbb{E}[Z] &\leqslant 2n \cdot \binom{n}{n^{1/3+3\varepsilon^3}} \cdot n^{(-\frac{2}{3}+2\varepsilon^3)n^{1/3+3\varepsilon^3}} \\
&\leqslant 2n \cdot \exp \left\{ n^{1/3+3\varepsilon^3} \left(\log \left(\frac{ne}{n^{1/3+3\varepsilon^3}} \right) + \left(-\frac{2}{3} + 2\varepsilon^3 \right) \log n \right) \right\} \\
&= 2n \cdot \exp \left\{ n^{1/3+3\varepsilon^3} \left((2/3 - 3\varepsilon^3 - 2/3 + 2\varepsilon^3) \log n + 1 \right) \right\} \\
&= 2n \cdot \exp \left\{ -(1+o(1))\varepsilon^3 n^{1/3+3\varepsilon^3} \log n \right\} = o(1).
\end{aligned}
$$

The result follows from Markov's inequality. $\qquad \square$

Lemma 6. *W.h.p. we have*

$$e(A, B) \leqslant 2\varepsilon^{-3} \max\{a+b, abn^{-2/3+3\varepsilon^3}\} \text{ for all } A \subseteq X \text{ and } B \subseteq Y \text{ with } |A| = a, |B| = b. \tag{6}$$

Proof. Set $h = h(a, b) = 2\varepsilon^{-3} \max\{a+b, abn^{-2/3+3\varepsilon^3}\}$. Let Z represent the event that there exist sets $A \subseteq X, B \subseteq Y$ (of size a and b respectively) with $e(A, B) \geqslant h(a, b)$ at step i_{max} (and hence at any step). Then using Lemma 4, we have

$$
\begin{aligned}
\mathbb{P}[Z] &\leqslant \sum_{1 \leqslant a,b \leqslant n} n^{a+b} \binom{ab}{h} n^{(-2/3+2\varepsilon^3)h} \\
&\leqslant \sum_{1 \leqslant a,b \leqslant n} \exp \left\{ \left[a + b + \left(-\frac{2}{3} + 2\varepsilon^3 \right) h \right] \log n + h \log \left(\frac{abe}{h} \right) \right\} \\
&\leqslant \sum_{1 \leqslant a,b \leqslant n} \exp \left\{ \left(\frac{5}{2}\varepsilon^3 - \frac{2}{3} \right) h \log n + h \log \left(\frac{abe}{abn^{-2/3+3\varepsilon^3}} \right) \right\}
\end{aligned}
$$

$$\leqslant \sum_{1\leqslant a,b\leqslant n} \exp\left\{(1+o(1))\left(-\frac{1}{2}\varepsilon^3\right)h\log n\right\}$$

$$\leqslant \sum_{1\leqslant a,b\leqslant n} \exp\left\{-(1+o(1))(a+b)\log n\right\} = o(1).$$

On the second and fourth line we have used the fact that $h \geqslant 2\varepsilon^{-3}(a+b)$ and on the second line we used $h \geqslant 2\varepsilon^{-3}abn^{-2/3+3\varepsilon^3}$. $\qquad\square$

2.3 Bipartite Independence Number

Let $I_X \subseteq X$ and $I_Y \subseteq Y$ with

$$|I_X| = |I_Y| = \alpha := 2\varepsilon^{-1}n^{2/3}\log^{2/3}n.$$

We would like to show that the number of open pairs in $I := I_X \times I_Y$ remains significant throughout the process so that an edge will land in one of these open pairs with high probability. Define $Q_I = Q_I(i)$ to be the number of open pairs in I at step i. We would like to show that $Q_I \approx \alpha^2 q$ throughout the entire process, for every choice of I; i.e. that the density of open pairs in each I is approximately the same as the global density of open pairs.

We will track Q_I by writing $Q_I = \widetilde{Q_I} - A_I$, where A_I represents the effect of "large" one-step changes. Formally, define

$$A_I(i) = \sum_{j\leqslant i} |\Delta Q_I(j)| \cdot \mathbb{1}_{|\Delta Q_I(j)|>n^{2/3-12\varepsilon^3}}$$

and

$$\widetilde{Q_I} = Q_I + A_I.$$

Our motivation for defining A_I is to ensure that the one-step change of $\widetilde{Q_I}$ is not too large. We will then be able to apply a martingale inequality. We will establish dynamic concentration for $\widetilde{Q_I}$, and just a crude bound for A_I which can be regarded as an error term.

2.3.1 Bounding A_I

In this section, we prove the following lemma which provides an upper bound on A_I.

Lemma 7. *With high probability, for every $I = I_X \times I_Y$ (with $I_X \subseteq X$, $I_Y \subseteq Y$, $|I_X| = |I_Y| = \alpha$) and every $i \leqslant i_{max}$, we have $A_I(i) \leqslant n^{1+45\varepsilon^3}$.*

Proof. Assume that $\mathcal{E}_{i_{max}}$ and (5), (6) hold. Under these assumptions, we will show that for every I and every $i \leqslant i_{max}$, $A_I(i) \leqslant n^{1+45\varepsilon^3}$. Fix an $I = I_X \times I_Y$. The sets $I_X^{(1)}, I_Y^{(1)}, I_X^{(2)}, I_Y^{(2)}$ which we define below are all dependent on i and the graph G_i, but as usual, we omit this from our notation. Let $I_X^{(1)} \subseteq Y$ be the set of vertices with at least $n^{1/3-16\varepsilon^3}$ neighbors in I_X (in graph G_i), and similarly define $I_Y^{(1)} \subseteq X$. Let $I_X^{(2)} \subseteq X$ be

the set of vertices with at least $n^{1/3-16\varepsilon^3}$ neighbors in $I_X^{(1)}$ (in graph G_i), and similarly define $I_Y^{(2)} \subseteq Y$. We first claim that in $G_{i_{max}}$,

$$e(I_Y^{(1)}, I_X^{(1)}) + e(I_X, I_Y^{(2)}) + e(I_X^{(2)}, I_Y) = O\left(n^{1/3+37\varepsilon^3}\right). \tag{7}$$

Recall that $|I_X| = \alpha = 2\varepsilon^{-1}n^{2/3}\log^{2/3} n$. Then by (6) we have

$$|I_X^{(1)}| \cdot n^{1/3-16\varepsilon^3} \leqslant e(I_X^{(1)}, I_X) \leqslant 2\varepsilon^{-3}\max\left\{\alpha + |I_X^{(1)}|, \alpha \cdot |I_X^{(1)}| \cdot n^{-2/3+3\varepsilon^3}\right\}.$$

If the maximum above were achieved by the second argument, we would have a contradiction since $n^{3\varepsilon^3}\log^{2/3} n \ll n^{1/3-16\varepsilon^3}$. Thus we have $|I_X^{(1)}| \cdot n^{1/3-16\varepsilon^3} \leqslant 2\varepsilon^{-3}(\alpha + |I_X^{(1)}|)$ and so $|I_X^{(1)}| \leqslant 2\varepsilon^{-3}\alpha/(n^{1/3-16\varepsilon^3} - 2\varepsilon^{-3})$. Thus (since $I_Y^{(1)}$ is similar) we have

$$|I_X^{(1)}|, |I_Y^{(1)}| < n^{1/3+17\varepsilon^3}. \tag{8}$$

By (6), we have

$$|I_X^{(2)}| \cdot n^{1/3-16\varepsilon^3} \leqslant e(I_X^{(2)}, I_X^{(1)}) \leqslant 2\varepsilon^{-3}\max\left\{|I_X^{(2)}| + |I_X^{(1)}|, |I_X^{(2)}| \cdot |I_X^{(1)}| \cdot n^{-2/3+3\varepsilon^3}\right\}.$$

The first argument must be the maximum otherwise we get a contradiction. Thus using (8), we have $|I_X^{(2)}| \cdot n^{1/3-16\varepsilon^3} \leqslant 2\varepsilon^{-3}(n^{1/3+17\varepsilon^3} + |I_X^{(2)}|)$ and so rearranging (and since $I_Y^{(2)}$ is similar) we get

$$|I_X^{(2)}|, |I_Y^{(2)}| < n^{34\varepsilon^3}. \tag{9}$$

Note that by (6) and (8) we have w.h.p.,

$$e(I_Y^{(1)}, I_X^{(1)}) \leqslant 2\varepsilon^{-3}\max\left\{|I_Y^{(1)}| + |I_X^{(1)}|, |I_Y^{(1)}||I_X^{(1)}|n^{-2/3+3\varepsilon^3}\right\} = O(n^{1/3+17\varepsilon^3})$$

since the maximum is the first argument. By (5) and (9) we have $e(I_X, I_Y^{(2)}) + e(I_X^{(2)}, I_Y) \leqslant O\left(n^{34\varepsilon^3} \cdot n^{1/3+3\varepsilon^3}\right) = O\left(n^{1/3+37\varepsilon^3}\right)$ w.h.p.. Thus we have proved (7).

In order to have $|\Delta Q_I(i)| > n^{2/3-12\varepsilon^3}$ the edge (x_i, y_i) chosen at step i must be in one of the following three sets: $I_Y^{(1)} \times I_X^{(1)}$, $I_X \times I_Y^{(2)}$, or $I_X^{(2)} \times I_Y$. Indeed, suppose (x_i, y_i) is not in any of the three sets. For a vertex v and a set of vertices S, let $d_S(v)$ be the number of neighbors of v in S. Let $d_S^{(2)}(v)$ be the number of vertices in S that are distance 2 from v. Then

$$|\Delta Q_I| < d_I(x_i)d_I(y_i) + d_I^{(2)}(y_i)\mathbb{1}_{x_i \in I_X} + d_I^{(2)}(x_i)\mathbb{1}_{y_i \in I_Y} \tag{10}$$

and since $(x_i, y_i) \notin I_X \times I_Y^{(2)}$, we can bound $d_I^{(2)}(y_i)\mathbb{1}_{x_i \in I_X}$ as follows. Either $x_i \notin I_X$ or $y_i \notin I_Y^{(2)}$. If $x_i \notin I_X$ then of course $d_I^{(2)}(y_i)\mathbb{1}_{x_i \in I_X} = 0$. So assume $y_i \notin I_Y^{(2)}$. Then y_i has at most $n^{1/3-16\varepsilon^3}$ neighbors in $I_Y^{(1)}$ which each have at most $n^{1/3+3\varepsilon^3}$ neighbors in I_Y. y_i has at most $n^{1/3+3\varepsilon^3}$ additional neighbors that all have at most $n^{1/3-16\varepsilon^3}$ neighbors in I_Y. Thus

$$d_I^{(2)}(y_i)\mathbb{1}_{x_i \in I_X} \leqslant n^{1/3-16\varepsilon^3} \cdot n^{1/3+3\varepsilon^3} + n^{1/3+3\varepsilon^3} \cdot n^{1/3-16\varepsilon^3} = O(n^{2/3-13\varepsilon^3}).$$

We have the same bound on $d_I^{(2)}(x_i)\mathbb{1}_{y_i \in I_Y}$ by symmetry. To bound $d_I(x_i)d_I(y_i)$, note that at least one of $x_i \notin I_Y^{(1)}$ or $y_i \notin I_X^{(1)}$ holds. Thus $d_I(x_i)d_I(y_i) \leqslant n^{1/3+3\varepsilon^3} \cdot n^{1/3-16\varepsilon^3} = O(n^{2/3-13\varepsilon^3})$. Thus we have $|\Delta Q_I| < O(n^{2/3-13\varepsilon^3}) < n^{2/3-12\varepsilon^3}$.

Thus using (7), we see that the number of steps with $|\Delta Q_I| \geqslant n^{2/3-12\varepsilon^3}$ can be bounded by $e(I_Y^{(1)}, I_X^{(1)}) + e(I_X, I_Y^{(2)}) + e(I_X^{(2)}, I_Y) = O\left(n^{1/3+37\varepsilon^3}\right)$. So for all $i \leqslant \varepsilon n^{4/3}\log^{1/3} n$ and all I, we have

$$A_I(i) \leqslant O\left(n^{2/3+6\varepsilon^3} n^{1/3+37\varepsilon^3}\right) < n^{1+45\varepsilon^3}$$

where we use the fact that the largest possible value of $|\Delta Q_I(i)|$ is the square of the maximum degree, $n^{1/3+3\varepsilon^3}$. $\qquad\square$

2.3.2 Dynamic concentration of $\widetilde{Q_I}(i)$

Define $\mathcal{E}_i' \subseteq \mathcal{E}_i$ as the event that $\mathcal{E}_i$ holds, that (5) and (6) hold, and that for all $j \leqslant i$ we have

$$\widetilde{Q_I}(j) \in \alpha^2 q\left(t(j)\right) \pm f(t(j)) \tag{11}$$

where

$$f(t) = n^{4/3-5\varepsilon^3} e^{t^3+t}.$$

By Lemmas 5 and 6 we have that (5) and (6) hold w.h.p. and in the last section we proved that in the event that (5) and (6) hold we (deterministically) have for all I and for all $j \leqslant i$ that

$$A_I(j) < n^{1+45\varepsilon^3}.$$

Note that since $t \leqslant \varepsilon \log^{1/3} n$ we have $f(t) \leqslant n^{4/3-3\varepsilon^3}$. We now define $\widetilde{Q_I}^+$ and $\widetilde{Q_I}^-$ as

$$\widetilde{Q_I}^{\pm}(i) := \begin{cases} \widetilde{Q_I}(i) - \alpha^2 q(t) \mp f(t) & \text{if } \mathcal{E}_{i-1}' \text{ holds} \\ \widetilde{Q_I}^{\pm}(i-1) & \text{otherwise.} \end{cases}$$

We will show that $\widetilde{Q_I}^+$ is a supermartingale. Since $\Delta\widetilde{Q_I}^+(i) = 0$ if $\mathcal{E}_i'$ fails to hold, we will assume $\mathcal{E}_i'$ holds.

Recall that $d_2(uv)$ is the number of potential copies of $K_{2,2}$ containing the edge uv in which two edges of the $K_{2,2}$ other than uv are in $G(i)$ and the last pair is open. By (3) and (4), we know that $Q \in (1 \pm n^{-10\varepsilon^3})n^2 q$ and $d_2(u,v) \in (1 \pm n^{-10\varepsilon^3})3n^{2/3}t^2 q$ where we recall that $q = q(t) = e^{-t^3}$. We first calculate

$$\mathbb{E}\left[\Delta Q_I | \mathcal{F}_i\right] = -\frac{1}{Q}\sum_{uv \in Q_I} d_2(uv),$$

and since by (8) and (9) we have

$$\mathbb{E}\left[\Delta A_I | \mathcal{F}_i\right] \leqslant n^{2/3+6\varepsilon^3} \cdot \frac{|I_Y^{(1)} \times I_X^{(1)}| + |I_X \times I_Y^{(2)}| + |I_X^{(2)} \times I_Y|}{Q}$$

$$\leqslant n^{2/3+6\varepsilon^3} \cdot \frac{O\left(n^{2/3+35\varepsilon^3}\right)}{(1+o(1))n^{2-\varepsilon^3}} \leqslant n^{-2/3+43\varepsilon^3}$$

we see that

$$\mathbb{E}\left[\Delta\widetilde{Q_I}|\mathcal{F}_i\right] = -\frac{1}{Q}\sum_{uv \in Q_I} d_2(uv) + O\left(n^{-2/3+43\varepsilon^3}\right). \tag{12}$$

Now we use Taylor's theorem to bound the one-step change of $\alpha^2 q(t) + f(t)$, the deterministic terms in $\widetilde{Q_I}^+$. We have

$$\alpha^2 q\left(t + \frac{1}{n^{4/3}}\right) + f\left(t + \frac{1}{n^{4/3}}\right) - \alpha^2 q(t) - f(t)$$

$$= \left(\alpha^2 q'(t) + f'(t)\right)\frac{1}{n^{4/3}} + O\left(\left(\alpha^2 q''(t) + f''(t)\right)\frac{1}{n^{8/3}}\right)$$

$$= \left(-\alpha^2 3t^2 q + f'\right)n^{-4/3} + O\left(n^{-4/3+2\varepsilon^3}\right) \tag{13}$$

where on the second line when we write $O\left((\alpha^2 q''(t) + f''(t))\frac{1}{n^{8/3}}\right)$ we mean an absolute bound on the function inside the big-O that holds holds for all $t \leqslant \varepsilon \log^{1/3} n$. Considering the particular functions $q(t), f(t)$ we arrive at the big-O term on the last line.

Now we will do the supermartingale calculation for $\widetilde{Q_I}^+$. Throughout the following, keep in mind that $\alpha^2 3t^2 q n^{-4/3} \leqslant n^{2\varepsilon^3}$, and that

$$n^{-5\varepsilon^3} \leqslant \frac{f}{\alpha^2 q} \leqslant n^{-2\varepsilon^3}.$$

Thus, using (12), (13), and the fact that we have dynamic concentration in the event $\mathcal{E}'_i$ we have that

$$\mathbb{E}\left[\Delta\widetilde{Q_I}^+|\mathcal{F}_i\right] \leqslant -\frac{1}{(1+n^{-10\varepsilon^3})n^2 q}\left(\alpha^2 q - n^{1+45\varepsilon^3} - f\right)\left((1-n^{-10\varepsilon^3})3n^{2/3}t^2 q\right)$$

$$+ \alpha^2 3t^2 q n^{-4/3} - n^{-4/3}f' + O\left(n^{-2/3+43\varepsilon^3} + n^{-4/3+2\varepsilon^3}\right)$$

$$= \alpha^2 3t^2 q n^{-4/3}\left[-\frac{\left(1 - \frac{n^{1+45\varepsilon^3}+f}{\alpha^2 q}\right)\left(1 - n^{-10\varepsilon^3}\right)}{(1+n^{-10\varepsilon^3})} + 1\right]$$

$$- n^{-4/3}f' + O\left(n^{-2/3+43\varepsilon^3}\right)$$

$$\leqslant \alpha^2 3t^2 q n^{-4/3}\left[\frac{f}{\alpha^2 q} + 2n^{-10\varepsilon^3}\right] - n^{-4/3}f' + O\left(n^{-10\varepsilon^3}\right)$$

$$\leqslant n^{-4/3}\left[3t^2 f - f'\right] + O\left(n^{-8\varepsilon^3}\right) \leqslant 0.$$

In the third line, we have used the geometric series expansion $1/(1+n^{-10\varepsilon^3}) = 1 - n^{-10\varepsilon^3} + O(n^{-20\varepsilon^3})$ to calculate

$$\frac{\left(1 - \frac{n^{1+45\varepsilon^3}+f}{\alpha^2 q}\right)\left(1 - n^{-10\varepsilon^3}\right)}{(1 + n^{-10\varepsilon^3})} = 1 - \frac{n^{1+45\varepsilon^3}+f}{\alpha^2 q} - n^{-10\varepsilon^3} - n^{-10\varepsilon^3}$$

$$+ O\left(n^{-20\varepsilon^3} + \frac{f}{\alpha^2 q}n^{-10\varepsilon^3}\right)$$

$$= 1 - \frac{f}{\alpha^2 q} - 2n^{-10\varepsilon^3} + O\left(n^{-12\varepsilon^3}\right).$$

where we used that $\frac{n^{1+45\varepsilon^3}}{\alpha^2 q} \ll n^{-20\varepsilon^3}$. Thus $\widetilde{Q_I}^+$ is a supermartingale.

We will use the following martingale inequality found in [12].

Lemma 8 (Freedman). *Let $Y(i)$ be a supermartingale, with $\Delta Y(i) \leqslant C$ for all i, and $V(i) := \sum_{k \leqslant i} Var[\Delta Y(k)|\mathcal{F}_k]$. Then*

$$\mathbb{P}\left[\exists i : V(i) \leqslant v, Y(i) - Y(0) \geqslant \lambda\right] \leqslant \exp\left(-\frac{\lambda^2}{2(v + C\lambda)}\right).$$

In order to apply Lemma 8, we must bound the variance.

$$\mathbf{Var}\left[\Delta\widetilde{Q_I}|\mathcal{F}_i\right] \leqslant \mathbb{E}\left[\left(\Delta\widetilde{Q_I}\right)^2|\mathcal{F}_i\right] \leqslant n^{2/3-12\varepsilon^3} \cdot \mathbb{E}\left[\left|\Delta\widetilde{Q_I}\right| \,|\, \mathcal{F}_i\right] \leqslant n^{2/3-12\varepsilon^3}. \qquad (14)$$

So by Lemma 8 using $v = n^{2-12\varepsilon^3}\log n$, $C = n^{2/3-12\varepsilon^3}$ and $\lambda = -\widetilde{Q_I}^+(0) = f(0) = n^{4/3-5\varepsilon^3}$ we see that the probability that $\widetilde{Q_I}^+(i_{max}) > 0$ is at most

$$\mathbb{P}\left[\exists i : V(i) \leqslant v, \widetilde{Q_I}^+(i) - \widetilde{Q_I}^+(0) \geqslant f(0)\right] \leqslant \exp\left\{-\frac{f(0)^2}{2(n^{2-12\varepsilon^3}\log n + f(0)n^{2/3-12\varepsilon^3})}\right\}$$

$$\leqslant \exp\left\{-\Omega\left(n^{2/3+\varepsilon^3}\right)\right\}$$

which is small enough to overcome a union bound over all choices of I since the number of such choices is

$$\binom{n}{\alpha}^2 \leqslant \exp\left\{2\alpha\log\left(\frac{ne}{\alpha}\right)\right\} < \exp\left\{2\varepsilon^{-1}n^{2/3}\log^{5/3}n\right\}. \qquad (15)$$

In a completely analogous fashion, we may prove that $\widetilde{Q_I}^-$ remains non-negative until time i_{max} for every I, w.h.p. Thus $\mathcal{E}'_{i_{max}}$ holds w.h.p..

2.3.3 Final bound on bipartite independence number

Now for any fixed I we will show that it is very likely that some edge is chosen in I. We have already shown that the event $\mathcal{E}'_{i_{max}}$ holds w.h.p., so in particular we have that $Q = n^2 q(1 + o(1))$ and $Q_I = \alpha^2 q(1 + o(1))$ for all $i \leqslant i_{max}$. Thus the probability of choosing an edge in I at any given step is $Q_I/Q = (1 + o(1))\frac{\alpha^2 q}{n^2 q} \geqslant 3\varepsilon^{-2} n^{-2/3} \log^{4/3} n$. So the probability that no edge from I is chosen in the first i_{max} steps is at most

$$\left(1 - 3\varepsilon^{-2} n^{-2/3} \log^{4/3} n\right)^{\varepsilon n^{4/3} \log^{1/3} n} \leqslant \exp\left\{-3\varepsilon^{-1} n^{2/3} \log^{5/3} n\right\}.$$

which is small enough to overcome a union bound over (15) many choices for the set I.

3 Conclusion

In [8], Caro and Rousseau write, "our knowledge of $b(2,n)$ closely parallels that of $R(C_4, K_n)$." In this paper, we have furthered this parallel, but the most intriguing open question (as mentioned in [8]) is to prove or disprove that $b(2,t) = o(t^{2-\varepsilon})$ for some $\varepsilon > 0$. Erdős famously conjectured that $R(C_4, K_n) = o(n^{2-\varepsilon})$. It may be the case that improving the upper bound on $b(2,t)$ is a simpler task than improving that of $R(C_4, K_n)$. Another direction would be to extend our technique to find new lower bounds on $b(s,t)$ with s fixed and $t \to \infty$ or on the bipartite Ramsey number of a fixed cycle versus a large bipartite clique. The techniques used in this paper could be applied here, but we have opted not to pursue this for the sake of brevity. Most likely the techniques in [17, 21] can also be used to show that the bipartite $K_{2,2}$-free process terminates in $O(n^{4/3} \log^{1/3} n)$ steps, matching the lower bound we proved in this paper.

References

[1] Miklós Ajtai, János Komlós, and Endre Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980.

[2] Lowell W. Beineke and Allen J. Schwenk. On a bipartite form of the Ramsey problem. pages 17–22. Congressus Numerantium, No. XV, 1976.

[3] Patrick Bennett and Tom Bohman. A note on the random greedy independent set algorithm. *Random Structures Algorithms*, 49(3):479–502, 2016.

[4] Tom Bohman. The triangle-free process. *Adv. Math.*, 221 (5): 1653–1677, 2009.

[5] Tom Bohman and Peter Keevash. The early evolution of the H-free process. *Invent. Math.*, 181(2):291–336, 2010.

[6] Tom Bohman and Peter Keevash. Dynamic concentration of the triangle-free process. In *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, volume 16 of *CRM Series*, pages 489–495. Ed. Norm., Pisa, 2013.

[7] Tom Bohman, Dhruv Mubayi, and Michael Picollelli. The independent neighborhoods process. *Israel J. Math.*, 214(1):333–357, 2016.

[8] Yair Caro and Cecil Rousseau. Asymptotic bounds for bipartite Ramsey numbers. *Electron. J. Combin.*, 8(1):Research Paper 17, 5, 2001.

[9] David Conlon. A new upper bound for the bipartite Ramsey problem. *J. Graph Theory*, 58(4):351–356, 2008.

[10] Paul Erdős, Stephen Suen, and Peter Winkler. On the size of a random maximal graph. In *Proceedings of the Sixth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science, "Random Graphs '93" (Poznań, 1993)*, volume 6, pages 309–318, 1995.

[11] Gonzalo Fiz Pontiveros, Simon Griffiths, and Robert Morris. The triangle-free process and the Ramsey number $R(3, k)$. *Mem. Amer. Math. Soc.*, 263, no. 1274, v+125 pp., 2020.

[12] David A. Freedman. On tail probabilities for martingales. *Ann. Probability*, 3:100–118, 1975.

[13] Johannes H. Hattingh and Michael A. Henning. Bipartite Ramsey theory. *Util. Math.*, 53:217–230, 1998.

[14] Jeong Han Kim. The Ramsey Number $R(3, t)$ has order of magnitude $t^2 / \log t$. *Random Structures & Algorithms*, 7: 173–207, 1995.

[15] T. Kővari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.

[16] Michael E. Picollelli. The diamond-free process. *Random Structures Algorithms*, 45(3):513–551, 2014.

[17] Michael E. Picollelli. The final size of the C_ℓ-free process. *SIAM J. Discrete Math.*, 28(3):1276–1305, 2014.

[18] A. Ruciński and N. C. Wormald. Random graph processes with degree restrictions. *Combin. Probab. Comput.*, 1(2):169–180, 1992.

[19] James B. Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, 46(1):83–87, 1983.

[20] Joel Spencer. Eighty years of Ramsey $R(3, k)$... and counting! In *Ramsey theory*, volume 285 of *Progr. Math.*, pages 27–39. Birkhäuser/Springer, New York, 2011.

[21] Lutz Warnke. The C_ℓ-free process. *Random Structures Algorithms*, 44(4):490–526, 2014.

[22] Lutz Warnke. When does the K_4-free process stop? *Random Structures Algorithms*, 44(3):355–397, 2014.