Generalizing tropical Kontsevich’s formula to multiple cross-ratios

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Abstract

Kontsevich’s formula is a recursion that calculates the number of rational degree d curves in \( \mathbb{P}^2_\mathbb{C} \) passing through \( 3d - 1 \) points in general position. Kontsevich proved it by considering curves that satisfy extra conditions besides the given point conditions. These crucial extra conditions are two line conditions and a condition called cross-ratio.

This paper addresses the question whether there is a general Kontsevich’s formula which holds for more than one cross-ratio. Using tropical geometry, we obtain such a recursive formula. For that, we use a correspondence theorem of Tyomkin that relates the algebro-geometric numbers in question to tropical ones. It turns out that the general tropical Kontsevich’s formula we obtain is capable of not only computing the algebro-geometric numbers we are looking for, but also of computing further tropical numbers for which there is no correspondence theorem yet.

We show that our recursive general Kontsevich’s formula implies the original Kontsevich’s formula and that the initial values are the numbers Kontsevich’s formula provides and purely combinatorial numbers, so-called cross-ratio multiplicities.

Mathematics Subject Classifications: 14N10, 14T90, 14H50

Introduction

Consider the following enumerative problem: Determine the number \( N_d \) of rational degree d curves in \( \mathbb{P}^2_\mathbb{C} \) passing through \( 3d - 1 \) points in general position. For small d, this question can be answered using methods from classical algebraic geometry. It took until ’94 when Kontsevich, inspired from developments in physics, presented a recursive formula to calculate the numbers \( N_d \) for all degrees.

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Theorem (Kontsevich’s formula, [KM94]). The numbers $N_d$ are determined by the recursion

$$
N_d = \sum_{d_1+d_2=d} \left( d_1^2 d_2^2 \cdot \frac{(3d-4)}{(3d_1-2)} - d_1^3 d_2 \cdot \frac{(3d-4)}{(3d_1-1)} \right) N_{d_1} N_{d_2}
$$

with initial value $N_1 = 1$.

This recursion is known as Kontsevich’s formula. The only initial value it needs is $N_1 = 1$, i.e. the fact that there is exactly one line passing through two different points.

A cross-ratio is an element of the ground field associated to four ordered collinear points. It encodes the relative position of these four points to each other. It is invariant under projective transformations and can therefore be used as a constraint that four points on $\mathbb{P}^1$ should satisfy. So a cross-ratio can be viewed as a condition on elements of the moduli space of $n$-pointed rational stable maps to a toric variety.

A crucial idea in the proof of Kontsevich’s formula is to consider curves that satisfy extra conditions besides the given point conditions. These extra conditions are two line conditions and a cross-ratio condition. In fact, the original proof of Kontsevich’s formula yields a formula to determine the number of rational plane curves satisfying an appropriate number of general positioned point conditions, two line conditions and one cross-ratio condition. Hence the following question naturally comes up:

Is there a general version of Kontsevich’s formula that recursively calculates the number of rational plane degree $d$ curves that satisfy general positioned point, curve and cross-ratio conditions?

We remark that Kontsevich’s formula was generalized in different ways before, e.g. Ernström and Kennedy took tangency conditions into account [EK98, EK99] and Di Francesco and Itzykson [DFI95] generalized it among others to $\mathbb{P}^1 \times \mathbb{P}^1$. We are not aware of any generalization that includes multiple cross-ratios.

Tropical geometry proved to be an effective tool to answer enumerative questions. To successfully apply tropical geometry to an enumerative problem, a so-called correspondence theorem is required. The first celebrated correspondence theorem was proved by Mikhalkin [Mik05]. It states that the numbers $N_d$ equal its tropical counterpart, i.e. they can be obtained from the weighted $^1$ count of rational tropical degree $d$ curves in $\mathbb{R}^2$ passing through $3d-1$ points in general position. Hence Kontsevich’s formula translates into a recursion on the tropical side called tropical Kontsevich’s formula and vice versa. Gathmann and Markwig demonstrated the efficiency of tropical methods by giving a purely tropical proof of tropical Kontsevich’s formula [GM08]. Applying Mikhalkin’s correspondence theorem then yields Kontsevich’s formula.

In the tropical proof — as in the classical case — rational tropical degree $d$ curves that satisfy point conditions, two line conditions and one tropical cross-ratio condition are considered. Roughly speaking, a tropical cross-ratio fixes the sum of lengths of a collection of bounded edges of a rational tropical curve.

$^1$Tropical curves are always counted with multiplicity.
**Example 1.** Figure 1 shows a plane rational tropical degree 2 curve $C$ such that $C$ satisfies four point conditions with its contracted ends labeled by $1, 2, 4, 5$, and such that $C$ satisfies one curve condition (which is a line that is indicated by dots) with its contracted end labeled with 3. Moreover, $C$ satisfies the tropical cross-ratio $\lambda' = (12|34)$ which determines the bold red length. Here, the notation $(12|34)$ indicates that 1, 2 are grouped together and 3, 4 are grouped together behind different vertices of the bold red edge on the right side of Figure 1.

![Figure 1](image)

Figure 1: On the left there is the curve $C$ of Example 1 with its bounded edges that contribute to the tropical cross-ratio $\lambda'$ colored bold red (the lengths of these edges are $l_1, l_2$). On the right there is the image of $C$ under a so-called forgetful map $ft_{\{1,2,3,4\}}$ that records the labels and the length $l_1 + l_2$ which appear in the tropical cross-ratio $\lambda'$.

Tropical cross-ratios are the tropical counterpart to classical cross-ratios. In [Mik07] Mikhalkin introduced a tropical version of cross-ratios under the name “tropical double ratio” to embed the moduli space of $n$-marked abstract rational tropical curves $\mathcal{M}_{0,n}$ into $\mathbb{R}^N$ in order to give it the structure of a balanced fan. Tyomkin proved a correspondence theorem [Tyo17] that involves cross-ratios, where the length of a tropical cross-ratio is related to a given classical cross-ratio via the valuation map. More precisely, Tyomkin’s correspondence theorem states that the number of rational plane degree $d$ curves satisfying point and cross-ratio conditions equals its tropical counterpart. Hence a general tropical Kontsevich’s formula that recursively computes the weighted number of rational plane tropical curves of degree $d$ that satisfy point and tropical cross-ratio conditions simultaneously computes the classical numbers as well.

Our approach to a general Kontsevich’s formula is inspired by the one of Gathmann and Markwig. Let us sum up the (for our purposes) most relevant ideas and techniques used in [GM08]:

1 Splitting curves

An important observation is that a count of tropical curves satisfying a tropical cross-ratio condition $\lambda'$ is independent of the length of the tropical cross-ratio. In particular, one can choose a large length for $\lambda'$. An even more important observation, which, at the end of the day, gives rise to a recursion is the following: If the length of $\lambda'$ is large enough, then all tropical curves satisfying $\lambda'$ have a contracted bounded
edge. Hence they can be split into two curves. See Section 2 and, in particular, propositions 29, 53.

2 Splitting multiplicities
Tropical curves are counted with multiplicities. So splitting curves using a large length for a tropical cross-ratio only yields a recursion if the multiplicities of such tropical curves split accordingly. See sections 3, 4 and, in particular, Proposition 64 and Theorem 68.

3 Using rational equivalence
A tropical cross-ratio appears as a pull-back of a point of $\mathcal{M}_{0,4}$ and pull-backs of different point of $\mathcal{M}_{0,4}$ are rationally equivalent [AR10]. Hence the number of tropical curves satisfying a tropical cross-ratio $\lambda' = (\beta_1\beta_2|\beta_3\beta_4)$ does not depend on how the labels $\beta_1, \ldots, \beta_4$ are grouped together — we could also consider the cross-ratio $\tilde{\lambda}' = (\beta_1\beta_3|\beta_2\beta_4)$ and obtain the same number. This yields an equation, see Corollary 71.

As a result, we obtain a general tropical Kontsevich’s formula (Theorem 68) that recursively calculates the weighted number of rational plane tropical curves of degree $d$ that satisfy point conditions, curve conditions and tropical cross-ratio conditions. In order to obtain a classical general Kontsevich’s formula (Corollary 69), we apply Tyomkin’s correspondence theorem [Tyo17]. Notice that Tyomkin’s correspondence theorem only holds for point and cross-ratio conditions. There is no correspondence theorem that relates the tropical numbers that also involve curve conditions to their classical counterparts yet.

The general Kontsevich’s formula we derive this way allows us to recover Kontsevich’s formula, see Corollary 71. The initial values of the general Kontsevich’s formula are the numbers provided by the original Kontsevich’s formula and so-called cross-ratio multiplicities, which are purely combinatorial [Gol20].

Organization of the paper
We use the framework provided by steps 1 to 3 described above to obtain a general Kontsevich’s formula. Although this general framework follows the outline of the tropical proof of Kontsevich’s formula in [GM08], new methods for steps 1 and 2 are required, which we elaborate right after the preliminary section. The preliminary section collects background on tropical moduli spaces and tropical intersection theory. For step 1, a new and general concept of moving parts of a tropical curve is established. Splitting the multiplicities in the 2nd step is done via a novel approach that considers “artificial” line conditions. Putting everything together to deduce our recursion is done in the last section. To complete the paper, we conclude the tropical and hence the classical Kontsevich’s formula from our general version.
1 Preliminaries

We recall some standard notations and definitions from tropical geometry [Mik07, GM08, GKM09] and give a very brief overview of the necessary tropical intersection theory [FS97, Rau09, All10, AR10, Kat12, Sha13, AHR16, Rau16]. After that, tropical cross-ratios and degenerated tropical cross-ratios are defined [Gol20].

Besides this, we try to make notations used as clear as possible by introducing notations in separate blocks to which we refer later.

Notation 2. We write \([m] := \{1, \ldots, m\}\) if \(0 \neq m \in \mathbb{N}\), and if \(m = 0\), then define \([m] := \emptyset\). Underlined symbols indicate a set of symbols, e.g. \(\underline{n} \subset [m]\) is a subset \(\{1, \ldots, m\}\). We may also use sets \(S\) of symbols as an index, e.g. \(p_S\), to refer to the set of all symbols \(p\) with indices taken from \(S\), i.e. \(p_S := \{p_i \mid i \in S\}\). The \#-symbol is used to indicate the number of elements in a set, for example \(\#[m] = m\).

Tropical moduli spaces

This subsection collects background from [Mik07, GM08, GKM09].

Definition 3 (Moduli space of abstract rational tropical curves). We use Notation 2. An abstract rational tropical curve is a metric tree \(\Gamma\) with unbounded edges called ends and with \(\text{val}(v) \geq 3\) for all vertices \(v \in \Gamma\). It is called \(N\)-marked abstract tropical curve \((\Gamma, x_{[N]}))\) if \(\Gamma\) has exactly \(N\) ends that are labeled with pairwise different \(x_1, \ldots, x_N \in \mathbb{N}\). Two \(N\)-marked tropical curves \((\Gamma, x_{[N]}))\) and \((\tilde{\Gamma}, \tilde{x}_{[N]}))\) are isomorphic if there is a homeomorphism \(\Gamma \to \tilde{\Gamma}\) mapping \(x_i\) to \(\tilde{x}_i\) for all \(i\) and each edge of \(\Gamma\) is mapped onto an edge of \(\tilde{\Gamma}\) by an affine linear map of slope \(\pm 1\). The set \(\mathcal{M}_{0,N}\) of all \(N\)-marked tropical curves up to isomorphism is called moduli space of \(N\)-marked abstract tropical curves. Forgetting all lengths of an \(N\)-marked tropical curve gives us its combinatorial type.

Theorem 4 (\(\mathcal{M}_{0,N}\) is a tropical fan, [SS06, Mik07, GKM09, GM10]). The moduli space \(\mathcal{M}_{0,N}\) can explicitly be embedded into a \(\mathbb{R}^t\) such that \(\mathcal{M}_{0,N}\) is a fan of pure dimension \(N - 3\) with its fan structure given by combinatorial types. Equip \(\mathbb{R}^t\) with a lattice which arises from considering integer edge lengths of abstract tropical curves in \(\mathcal{M}_{0,N}\) and let all weights of \(\mathcal{M}_{0,N}\) be one. Then \(\mathcal{M}_{0,N} \subset \mathbb{R}^t\) is a tropical fan, i.e. \(\mathcal{M}_{0,N}\) represents an affine tropical cycle in \(\mathbb{R}^t\) in the sense of [AR10]. This allows us to use tropical intersection theory on \(\mathcal{M}_{0,N}\). For an example, see Figure 2.

Definition 5 (Moduli space of rational tropical stable maps to \(\mathbb{R}^2\)). Let \(m, d \in \mathbb{N}\). A rational tropical stable map of degree \(d\) to \(\mathbb{R}^2\) with \(m\) contracted ends is a tuple \((\Gamma, x_{[N]}, h)\) with \(N \in \mathbb{N}_{>0}\), where \((\Gamma, x_{[N]}))\) is an \(N\)-marked abstract tropical curve with \(N = 3d + m\), \(x_{[N]} = [N]\) and a map \(h : \Gamma \to \mathbb{R}^2\) that satisfies the following:

(a) Let \(e \in \Gamma\) be an edge with length \(l(e) \in [0, \infty]\), identify \(e\) with \([0, l(e)]\) and denote the vertex of \(e\) that is identified with \(0 \in [0, l(e)]\) by \(V\). The map \(h\) is integer affine linear, i.e. \(h \mid_{e}: t \mapsto tv + a\) with \(a \in \mathbb{R}^2\) and \(v(e, V) := v \in \mathbb{Z}^2\), where \(v(e, V)\)
is called direction vector of $e$ at $V$ and the weight of an edge (denoted by $\omega(e)$) is the gcd of the entries of $v(e, V)$ if $v(e, V) \neq 0$ and zero otherwise. The vector $\frac{1}{\omega(e)} \cdot v(e, V)$ is called the primitive direction vector of $e$ at $V$. If $e = x_i \in \Gamma$ is an end, then $v(x_i)$ denotes the direction vector of $x_i$ pointing away from its one vertex it is adjacent to.

(b) The direction vector $v(x_i)$ of an end labeled with $x_i$ is given by

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$0, \ldots, m$</th>
<th>$m + 1, \ldots, m + d$</th>
<th>$m + d + 1, \ldots, m + 2d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(x_i)$</td>
<td>$(0)$</td>
<td>$(-1)$</td>
<td>$(0)$</td>
</tr>
<tr>
<td>$v(x_i)$</td>
<td></td>
<td></td>
<td>$(-1)$</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$m + 2d + 1, \ldots, m + 3d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(x_i)$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

Ends with direction vector zero are called contracted ends.

(c) The balancing condition

$$
\sum_{\substack{e \in \Gamma \text{ an edge, } \\forall \text{ vertex of } e}} v(e, V) = 0
$$

holds for every vertex $V \in \Gamma$.

Two rational tropical stable maps of degree $d$ with $m$ contracted ends, namely $(\Gamma, x_{[N]}, h)$ and $(\Gamma', x'_{[N]}, h')$, are isomorphic if there is an isomorphism $\varphi$ of their underlying $N$-marked tropical curves such that $h' \circ \varphi = h$. The set $\mathcal{M}_{0,m}(\mathbb{R}^2, d)$ of all (rational) tropical stable maps of degree $\Delta$ to $\mathbb{R}^2$ with $m$ contracted ends up to isomorphism is called the moduli space of (rational) tropical stable maps of degree $d$ to $\mathbb{R}^2$ (with $m$ contracted ends).
Theorem 6 \((\mathcal{M}_{0,m}(\mathbb{R}^2, \Delta))\) is a fan, [GKM09]). The map
\[
\mathcal{M}_{0,m}(\mathbb{R}^2, d) \to \mathcal{M}_{0,N} \times \mathbb{R}^2,
\]
\[
(\Gamma, x_{[N]}, h) \mapsto (\Gamma, x_{[N]}, h(x_1))
\]
with \(N = 3d + m\) is bijective and \(\mathcal{M}_{0,m}(\mathbb{R}^2, d)\) is a tropical fan of dimension \(3d + m - 1\) (notice that \(h(x_1)\) is an arbitrary choice of a so-called base point). Hence \(\mathcal{M}_{0,n}(\mathbb{R}^2, d)\) represents an affine tropical cycle in \(\mathbb{R}^t\). This allows us to use tropical intersection theory on \(\mathcal{M}_{0,n}(\mathbb{R}^2, d)\).

Definition 7 (Evaluation maps). For \(i \in [m]\), the map
\[
ev_i : \mathcal{M}_{0,m}(\mathbb{R}^2, d) \to \mathbb{R}^2,
\]
\[
(\Gamma, x_{[N]}, h) \mapsto h(x_i)
\]
is called \(i\)-th evaluation map. Under the identification from Theorem 6 the \(i\)-th evaluation map is a morphism of fans \(\ev_i : \mathcal{M}_{0,N} \times \mathbb{R}^2 \to \mathbb{R}^2\) [GKM09, Proposition 4.8]. This allows us to pull-back cycles via the evaluation map [AR10, Proposition 4.7].

Definition 8 (Forgetful maps). For \(N \geq 4\) the map
\[
\text{ft}_{x_{[N-1]} : \mathcal{M}_{0,N} \to \mathcal{M}_{0,N-1}},
\]
\[
(\Gamma, x_{[N]}) \mapsto (\Gamma', x_{[N-1]}),
\]
where \(\Gamma'\) is the stabilization (straighten 2-valent vertices) of \(\Gamma\) after removing its end marked by \(x_N\) is called the \(N\)-th forgetful map. Applied recursively, it can be used to forget several ends with markings in \(I^C \subset x_{[N]}\), denoted by \(\text{ft}_I\), where \(I^C\) is the complement of \(I \subset x_{[N]}\). With the identification from Theorem 6, and additionally forgetting the map \(h\) to the plane, we can also consider
\[
\text{ft}_I : \mathcal{M}_{0,m}(\mathbb{R}^2, d) \to \mathcal{M}_{0,|I|},
\]
\[
(\Gamma, x_{[N]}, h) \mapsto \text{ft}_I(\Gamma, x_{[N]}|i \in I).
\]
Any forgetful map is a morphism of fans [GKM09, Proposition 3.12, Remark 4.10]. This allows us to pull-back cycles via the forgetful map [AR10, Proposition 4.7].

Definition 9 (Tropical curves and multi-lines). A plane tropical curve \(C\) of degree \(d\) is the abstract 1-dimensional cycle a rational tropical stable map of degree \(d\) gives rise to, i.e. \(C\) is a weighted embedded 1-dimensional polyhedral complex in \(\mathbb{R}^2\). A (tropical) multi-line \(L\) is a tropical rational curve in \(\mathbb{R}^2\) with 3 ends such that the primitive direction of each of this ends is one of the standard directions \((-1, 0), (0, -1)\) or \((1, 1)\) \(\in \mathbb{R}^2\). The weight with which an end of \(L\) appears is denoted by \(\omega(L)\).
Tropical intersection products

As indicated in the last section, tropical intersection theory can be applied to the tropical moduli spaces that are interesting for us. For a short and — for our purposes — sufficient introduction to tropical intersection theory have a look at the preliminary section of [Gol20]. For more background of tropical intersection theory see [FS97, Rau09, All10, AR10, Kat12, Sha13, AHR16, Rau16]. In the present paper tropical intersection theory provides the overall framework in which we work but all we need from this machinery is the following:

Remark 10 (Enumerative meaning of our tropical intersection products). Throughout this paper, we consider intersection products of the form \( \varphi^*_1(Z_1) \cdots \varphi^*_r(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \), where \( \varphi_i \) is either an evaluation \( \text{ev}_i \) map from Definition 7 or a forgetful map \( \text{ft}_I \) to \( \mathcal{M}_{0,4} \) from Definition 8, and \( Z_i \) is a cycle we want to pull-back via \( \varphi_i \) for \( i \in [r] \). Notice that \( \text{ev}_i \) is a map to \( \mathbb{R}^2 \) while \( \text{ft}_I \) is a map to \( \mathcal{M}_{0,4} \). Using a projection \( \tilde{\pi} : \mathcal{M}_{0,4} \to \mathbb{R}^2 \) as in Remark 2.2 of [Gol20] and considering \( \tilde{\pi} \circ \text{ft}_I \) instead of \( \text{ft}_I \) does not affect \( \varphi^*_1(Z_1) \cdots \varphi^*_r(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) since by definition (see also [AR10, Definition 3.4])

\[
\begin{align*}
(\tilde{\pi} \circ \text{ft}_I)^*(\tilde{Z}_i) &= \text{ft}_I^*(\tilde{\pi}^*(\tilde{Z}_i)) \\
&= \text{ft}_I^*(Z_i)
\end{align*}
\]

holds for a suitable cycle \( \tilde{Z}_i \). Thus all our maps can be treated as maps to either \( \mathbb{R}^2 \) or \( \mathbb{R}^1 \). Hence Proposition 1.15 of [Rau16] can be applied, and together with Proposition 1.12 of [Rau16] and Lemma 2.11 of [Gol20] it follows that the support of the intersection product \( \varphi^*_1(Z_1) \cdots \varphi^*_r(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) equals \( \varphi_1^{-1}(Z_1) \cap \cdots \cap \varphi_r^{-1}(Z_r) \). Hence this intersection product gains an enumerative meaning if it is 0-dimensional. More precisely, each point in such an intersection product corresponds to a tropical stable map that satisfies certain conditions that are given by the cycles \( Z_i \) for \( i \in [r] \).

The weights of such intersection products \( \varphi^*_1(Z_1) \cdots \varphi^*_r(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) are discussed within the next section. Before proceeding with the next section, we want to briefly recall the concept of rational equivalence that is frequently used throughout this paper.

Remark 11 (Rational equivalence). When considering cycles \( Z_i \) as in Remark 10 that are conditions we impose on tropical stable maps, we usually want to ensure that a 0-dimensional cycle \( \varphi^*_1(Z_1) \cdots \varphi^*_r(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) is independent of the exact positions of the conditions \( Z_i \) for \( i \in [r] \). This is where rational equivalence comes into play. We usually consider cycles like \( Z_i \) up to a rational equivalence relation. The most important facts about this relation are the following:

(a) Two cycles \( Z, Z' \) in \( \mathbb{R}^n \) that only differ by a translation are rationally equivalent [MR09, Lemma 2.1].

(b) Pull-backs \( \varphi^*(Z), \varphi^*(Z') \) of rationally equivalent cycles \( Z, Z' \) are rationally equivalent [AR10, Lemma 8.5].
(c) If two 0-dimensional intersection products of the form \( \varphi_1^*(Z_1) \cdots \varphi_r^*(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) and \( \varphi_1^*(Z'_1) \cdots \varphi_r^*(Z'_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) are rationally equivalent, then

\[
\deg (\varphi_1^*(Z_1) \cdots \varphi_r^*(Z_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d)) = \deg (\varphi_1^*(Z'_1) \cdots \varphi_r^*(Z'_r) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d))
\]

holds [All10, Theorem 1.9.10], where \( \deg \) is the degree of a 0-dimensional cycle which is defined as the sum of all weights of all points in this cycle.

Notice that (a)-(c) allows us to “move” all conditions we consider slightly without affecting a count of tropical stable maps we are interested in.

Another fact about rational equivalence is the following:

**Theorem 12** (Recession fan, [AHR16]). Each tropical curve \( C \) of degree \( d \) in \( \mathbb{R}^2 \) is rationally equivalent to a multi-line \( L_C \) with weights \( \omega(L_C) = d \). Hence pull-backs of \( C \) and \( L_C \) along the evaluation maps are rationally equivalent. The multi-line \( L_C \) is also called recession fan of \( C \).

### Tropical cross-ratios and the numbers we want to determine

Mikhalkin introduced so-called tropical double ratios in [Mik07] as a tropical analogue of classical cross-ratios. The author gave an intersection theoretic version of Mikhalkin’s definition in [Gol20]:

**Definition 13.** A (tropical) cross-ratio \( \lambda' \) is an unordered pair of unordered numbers \( (\beta_1, \beta_2, \beta_3, \beta_4) \) together with an element in \( \mathbb{R}_{>0} \) denoted by \(|\lambda'|\), where \( \beta_1, \ldots, \beta_4 \) are labels of pairwise distinct ends of a tropical stable map of \( \mathcal{M}_{0,m}(\mathbb{R}^2, d) \). We say that \( C \in \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) satisfies the cross-ratio constraint \( \lambda' \) if \( C \in \text{ft}_1^*(\{\lambda'\}) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \), where \(|\lambda'|\) is the canonical local coordinate of the ray \((\beta_1, \beta_2, \beta_3, \beta_4)\) in \( \mathcal{M}_{0,4} \). Figure 1 of Example 1 in the introduction provides an example of a tropical stable map satisfying a non-degenerated cross-ratio \( \lambda' \) with length \(|\lambda'| = l_1 + l_2 \).

A degenerated (tropical) cross-ratio \( \lambda \) is defined as a set \( \{\beta_1, \ldots, \beta_4\} \), where \( \beta_1, \ldots, \beta_4 \) are pairwise distinct labels of ends of a tropical stable map \( \mathcal{M}_{0,m}(\mathbb{R}^2, d) \). We say that \( C \in \mathcal{M}_{0,m}(\mathbb{R}^2, d) \) satisfies the degenerated cross-ratio constraint \( \lambda \) if \( C \in \text{ft}_1^*(0) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \). A degenerated cross-ratio arises from a non-degenerated cross-ratio by taking \(|\lambda'| \to 0 \) (see [Gol20] for more details). We refer to \( \lambda \) as degeneration of \( \lambda' \) in this case.

Throughout the paper, we stick to the convention to denote a non-degenerated cross-ratio by \( \lambda' \) and a degenerated one by \( \lambda \).

**Definition 14.** Let \( m \in \mathbb{N}_{>0} \). Let \( \{n, \kappa, f\} \) be a partition of the set \([m]\), i.e. \( n, \kappa, f \subset [m] \) and \( n \cup \kappa \cup f = [m] \). Consider a degree \( d \in \mathbb{N} \), \( l \in \mathbb{N} \) degenerated cross-ratios \( \lambda_{[l]} \), \( \lambda' \in \mathbb{N} \) non-degenerated cross-ratios \( \mu'_{[\nu]} \), points \( p_\kappa \in \mathbb{R}^2 \) and tropical multi-lines \( L_{\kappa} \). Define the cycle

\[
Z_d \left( p_\kappa, L_{\kappa}, \lambda_{[l]}, \mu'_{[\nu]} \right) := \prod_{k \in \kappa} \text{ev}_k^*(L_k) \cdot \prod_{i \in \kappa} \text{ev}_i^*(p_i) \cdot \prod_{j=1}^l \text{ft}_{\mu'_{[\nu]}} \left( \{\mu'_{[\nu]}\} \right) \cdot \prod_{j=1}^l \text{ft}_{\lambda_{[l]}} \left( 0 \right) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) .
\]
Each point \( p_i \in p_\mathcal{L} \) is a 2-dimensional condition. Each multi-line \( L_k \in \mathcal{L} \) and each cross-ratio \( \mu'_j \in \mu'_{[v]} \), \( \lambda_j \in \lambda_{[l]} \) is a 1-dimensional condition. Hence the dimension of 
\[ Z_d\left(p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]}\right) \] 
is \( (3d-1+m) - (2 \cdot \#n + \tilde{l} + l' + \#f) \), where \( 3d-1+m \) is the dimension of \( \mathcal{M}_{0,m}(\mathbb{R}^2, d) \).

Notice that each tropical stable map in 
\[ Z_d\left(p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]}\right) \] 
has 3 different kinds of contracted ends, namely contracted ends with labels in \( n \) that satisfy point conditions, contracted ends with labels in \( \mathcal{K} \) that satisfy multi-line conditions and contracted ends with labels in \( f \) that satisfy no point or multi-line conditions. Given \( n \) and \( \mathcal{K} \), we can calculate \( \#f \) using
\[ m = \#n + \#\mathcal{K} + \#f. \]

**Definition 15** (General position). Let \( p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]} \) be conditions as in Definition 14 such that
\[ 3d-1 = \#n + \tilde{l} + l' - \#f \]  
holds. These conditions are in *general position* if 
\[ Z_d\left(p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]}\right) \] 
is a zero-dimensional nonzero cycle that lies inside top-dimensional cells of \( \prod_{j=1}^{\tilde{l}} \mathbb{R} \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d) \).

**Remark 16.** “General position” usually requires some sort of continuity. But tropical cross-ratios partly consist of discrete data: the four labels of ends and how they are grouped together. The condition of the cycle of Definition 15 to be nonzero yields that this discrete data behaves “general”, i.e. we cannot fix the same tropical cross-ratios twice and we cannot fix a set of tropical cross-ratios such that the length of one of them is determined by the others.

**Definition 17.** For condition in general position as in Definition 15, where we additionally require from the cross-ratios that no label of a non-contracted end appears in any of the cross-ratios \( \lambda_{[l]}, \mu'_{[v]} \), we define
\[ N_d\left(p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]}\right) := \deg\left(Z_d\left(p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]}\right)\right), \]
where \( \deg \) is the degree function that sums up all multiplicities of the points in the intersection product. In other words, \( N_d\left(p_\mathcal{L}, L_{\mathcal{L}}, \lambda_{[l]}, \mu'_{[v]}\right) \) is the number of rational tropical stable maps to \( \mathbb{R}^2 \) (counted with multiplicity) of degree \( d \) satisfying the cross-ratios \( \lambda_{[l]}, \mu'_{[v]} \), the multi-line conditions \( L_{\mathcal{L}} \) and point conditions \( p_\mathcal{L} \).

**Remark 18.** Allowing only tropical multi-line conditions in Definition 17 instead of arbitrary rational tropical curves as conditions is not a restriction, since we can always pass to the recession fan of a rational tropical curve without effecting the count, see Theorem 12 and [All10].
Remark 19. The numbers $N_d\left(p_n, L_κ, λ[\tilde{l}], µ'[l']\right)$ are independent of the exact position of points $p_n$ and multi-lines $L_κ$ as long as the set of all conditions is in general position. Moreover, the numbers are also independent of the exact lengths $|µ'_1|, \ldots, |µ'_{l'}|$ of the non-degenerated cross-ratios. In particular,

$$N_d\left(p_n, L_κ, λ[\tilde{l}], µ'[l']\right) = N_d\left(p_n, L_κ, λ[\tilde{l}], µ[l']\right),$$

where $µ'_j$ is the degeneration of $µ'_j$.

Given a tropical stable map $C$ that satisfies a cross-ratio condition $λ'$, we can think of this condition as a path of fixed length $|λ'|$ inside $C$. Thus a degenerated cross-ratio condition $λ$ can be thought of as a path of length zero inside a tropical stable map, i.e. there is a vertex of valence $> 3$ in $C$ satisfying a degenerated cross-ratio. Or in other words, there is a vertex $v \in C$ such that the image of $v$ under $f_{t,λ}$ is 4-valent. We say that $λ$ is satisfied at $v$. Obviously, a tropical stable map $C$ satisfies a degenerated cross-ratio condition if and only if there is a vertex of $C$ that satisfies the degenerated cross-ratio.

We define the set $λ_v$ of cross-ratios associated to a vertex $v$ that consists of all given cross-ratios whose images of $v$ using the forgetful map are 4-valent.

Remark 20. An equivalent and more descriptive way of saying that a cross-ratio is satisfied at a vertex is the path criterion: Let $C$ be a tropical stable map and let $λ = \{β_1, \ldots, β_4\}$ be a degenerated cross-ratio, then a pair $(β_i, β_j)$ induces a unique path in $C$. If the paths associated to $(β_{i_1}, β_{i_2})$ and $(β_{i_3}, β_{i_4})$ intersect in exactly one vertex $v$ of $C$ for all pairwise different choices of $i_1, \ldots, i_4$ such that $\{i_1, \ldots, i_4\} = \{1, \ldots, 4\}$, then and only then the degenerated cross-ratio $λ$ is satisfied at $v$. Note that “for all choices” above is equivalent to “for one choice”.

Construction 21. Let $v$ be a vertex of an abstract tropical curve and $λ_j \in λ_v$. We say that $v$ is resolved according to $λ'_j$ (where $λ'_j$ is a cross-ratio that degenerates to $λ_j$) if the equality

$$\text{val}(v) = 3 + \#λ_v$$

holds, $v$ is replaced by two vertices $v_1, v_2$ that are connected by a new edge such that $λ'_j$ is satisfied,

$$λ_v = \{λ_j\} \cup λ_{v_1} \cup λ_{v_2}$$

is a union of pairwise disjoint sets and

$$\text{val}(v_k) = 3 + \#λ_{v_k}$$

holds for $k = 1, 2$.

Definition 22 (Cross-ratio multiplicity). Let $v$ be a $(3 + \#λ_v)$-valent vertex of an abstract tropical curve with $λ_v = \{λ_{j_1}, \ldots, λ_{j_r}\}$ and let $λ'_j$ be cross-ratios that degenerate to $λ_{j_k}$ for
Figure 3: Let $\lambda_1 := \{1, 2, 3, 4\}$ and $\lambda_2 := \{1, 2, 3, 5\}$ be two degenerated cross-ratios. On the right there is a 5-valent vertex $v$ with $\lambda_v = \{\lambda_1, \lambda_2\}$. On the left $v$ is resolved according to $\lambda'_1 := (12|34)$. Notice that the resolution is unique in this case.

$t = 1, \ldots, r$ such that $|\lambda'_{j_1}| > \cdots > |\lambda'_{j_r}|$. A total resolution of $v$ is a 3-valent labeled abstract tropical curve on $r$ vertices that arises from $v$ by resolving $v$ according to the following recursion. First, resolve $v$ according to $\lambda'_{j_1}$. The two new vertices are denoted by $v_1, v_2$. Choose $v_k$ with $\lambda_{j_2} \in \lambda_{v_k}$ and resolve it according to $\lambda'_{j_2}$ (this may not be unique, pick one resolution). Now we have 3 vertices $v_1, v_2, v_3$ from which we pick the one with $\lambda_{j_3} \in \lambda_{v_k}$, resolve it and so on. We define the cross-ratio multiplicity $\text{mult}_{cr}(v)$ of $v$ to be the number of total resolution of $v$. This number does not depend on the choice of non-degenerated cross-ratios $\lambda'_{j_1}, \ldots, \lambda'_{j_r}$, in particular, it does not depend on the order $|\lambda'_{j_1}| > \cdots > |\lambda'_{j_r}|$, see [Gol20]. In the special case of $\#\lambda_v = 0$, we set $\text{mult}_{cr}(v) = 1$.

**Example 23.** Let $v$ be a 6-valent vertex such that $\lambda_v = \{\lambda_1, \lambda_2, \lambda_3\}$ and the degenerated cross-ratios are given by $\lambda'_1 := (12|56), \lambda'_2 := (34|56), \lambda'_3 := (12|34)$. The following two 3-valent trees schematically show all total resolutions of $v$ with respect to $|\lambda'_1| > |\lambda'_2| > |\lambda'_3|$.

**Open problem 24.** The numbers $\text{mult}_{cr}(v)$ are not well understood. Of course, one can calculate them by considering all trees with an appropriate number of labeled ends and pick the ones that are total resolutions of $v$ with respect to the given cross-ratios. This approach is neither fast nor pleasing. So a question naturally comes up: is there another, more efficient way to calculate the cross-ratio multiplicity $\text{mult}_{cr}(v)$ of a vertex $v$ satisfying degenerated cross-ratios?

**Definition 25 (Evaluation multiplicity).** Let $C$ be a tropical stable map that contributes to $N_d(p_m, L_{\infty}, \lambda_{[t]})$. Consider the ev-matrix $M(C)$ of $C$, which is given by the map

$$\bigotimes_{t \in \text{ev}_v} \text{ev}_t : \mathcal{M}_{0,m} \left( \mathbb{R}^2, d \right) \to \mathbb{R}^{2\#v+\#_v}$$

that is locally linear around $C \in \mathcal{M}_{0,m} \left( \mathbb{R}^2, d \right)$, where the coordinates on the moduli space $\mathcal{M}_{0,m} \left( \mathbb{R}^2, d \right)$ are the bounded edges’ lengths. The evaluation multiplicity $\text{mult}_v(C)$ of $C$ is defined by

$$\text{mult}_v(C) := |\det(M(C))|.$$
The matrix in Example 63 provides an example of an ev-matrix.

**Proposition 26 ([Gol20]).** If \( C \) is a tropical stable map that contributes to the number \( N_d(p_n, L_\kappa, \lambda_{[l]}) \), then the multiplicity \( \text{mult}(C) \) with which \( C \) contributes to this intersection product is given by

\[
\text{mult}(C) = \text{mult}_{ev}(C) \prod_{v \text{ vertex of } C} \text{mult}_{cr}(v),
\]

where \( \text{mult}_{ev}(C) \) is the absolute value of the determinant of the ev-matrix associated to \( C \), see [Rau09, Gol20].

**Corollary 27 ([Gol20]).** Let \( C \) be a tropical stable map that contributes to the number \( N_d(p_n, L_\kappa, \lambda_{[l]}) \). Let \( v \in C \) be a vertex of \( C \) such that \( \text{val}(v) > 3 \). Then for every edge \( e \) adjacent to \( v \) in \( C \) there is an entry \( \beta_i \) in some \( \lambda_j \in \lambda_v \) such that \( e \) is in the shortest path from \( v \) to the end labeled with \( \beta_i \).

The following correspondence theorem allows us to obtain algebro-geometric results from our tropical ones in case of no multi-line conditions.

**Theorem 28** (Correspondence Theorem 5.1 of [Tyo17]). Let \( N_{\text{class}}^d(p_n, \mu_{[l]}) \) denote the number of plane rational degree \( d \) curves over an algebraically closed field of characteristic zero that satisfy point conditions and classical cross-ratios \( \mu_1, \ldots, \mu_l \) such that all conditions are in general position. Then

\[
N_{\text{class}}^d(p_n, \mu_{[l]}) = N_d(p_n, \lambda_{[1]}')
\]

holds, where \( \lambda_j' \) is the tropical cross-ratio associated to \( \mu_j \) for \( j \in [l] \) in the sense of [Tyo17].

## 2 Splitting curves with cross-ratios

### Existence of contracted bounded edges

The aim of this subsection is to prove Propositions 29, 53, which are crucial for the recursion we aim for. They guarantee that the tropical stable maps we are dealing with have a contracted bounded edge at which we can split them. Proposition 29 covers the case where we have at least one point condition. Proposition 53 covers the case of no point conditions.

**Proposition 29.** Let \( n \geq 1 \) and let \( C \) be a tropical stable map that contributes to the number \( N_d(p_n, L_\kappa, \lambda_{[l-1]}, \lambda_{[l]}) \), where \( \lambda_j' \) is a non-degenerated tropical cross-ratio. If \( |\lambda_j'| \) is large, then there is exactly one contracted bounded edge in \( C \).

To keep track of the overall structure of the proof of Proposition 29, we briefly outline important steps:

- **(S1) Definition 30:** Forget \( \lambda_j' \), to obtain a 1-dimensional cycle \( Y \) in \( M_{0,m}(\mathbb{R}^2, d) \).
Definition 30, Remark 31, Example 32: Consider the 1-dimensional unbounded edges of $Y$. They correspond to tropical curves $\Gamma$ that satisfy $p_\kappa, L_\kappa, \lambda_{[i-1]}$ such that $\Gamma$ admits a movement which gives rise to an unbounded 1-dimensional family of curves of the same combinatorial type as $\Gamma$. Hence we should study tropical curves $\Gamma$ that have a movable component (i.e. a subgraph) $B$ which can be moved unboundedly without changing the combinatorial type of $\Gamma$.

Definition 37, Corollary 50: Show that $B$ contains a single vertex. For this, we define chains of vertices in $B$ and show that no chain has more than one element.

Proof of Proposition 29: Conclude that there must be a contracted bounded edge.

Let us start with step (S1):

Definition 30 (Movable component). Let $\Gamma$ be a tropical curve with no contracted bounded edge coming from a stable map in the 1-dimensional cycle (for notation, see Definition 15)

$$Y := \prod_{k \in \mathbb{Z}} \text{ev}^*_k(L_k) \cdot \prod_{i \in \mathbb{Z}} \text{ev}^*_i(p_i) \cdot \prod_{j=1}^{l-1} \text{ft}^*_\lambda_j(0) \cdot \mathcal{M}_{0,m}(\mathbb{R}^2, d)$$

such that $\Gamma$ gives rise to a 1-dimensional family of curves by moving some of its vertices. Since the family obtained by moving vertices of $\Gamma$ is 1-dimensional, no vertex can be moved freely, i.e. in each possible direction. Hence each vertex of $\Gamma$ is either fixed, i.e. it can not be moved at all, or movable in a direction given by a vector in $\mathbb{R}^2$ which we call direction of movement of $v$. Directions of movements of vertices are indicated in Figure 4 of Example 32. Since each movable vertex $v$ cannot move freely, its movement is restricted by a condition imposed to it via an edge adjacent to $v$. More precisely, $v$ either needs to be adjacent to a fixed vertex or to a contracted end which satisfies a multi-line condition. The connected component of $\Gamma$ which consists of all movable vertices of $\Gamma$ (and edges connecting movable vertices) is called the movable component $B$ of $\Gamma$. Notice that there is exactly one movable component since $\Gamma$ gives rise to a 1-dimensional family only. A connected component of $\Gamma$ that is obtained from $\Gamma$ by removing the movable component is called fixed component. We say that a movable component allows an unbounded movement, if the movement of the movable component gives rise to a family of curves of the same combinatorial type as $\Gamma$ that is unbounded.

Let us now take a closer look at unbounded movements as outlined in step (S2):

Remark 31. Consider a 1-dimensional family of curves of the same combinatorial type that is unbounded and the movable component within some curve of this family that allows an unbounded movement. Notice that the direction of movement $b$ of a vertex $v$ in this movable component might change as moving the component generates the family. Since $v$ is either adjacent to a fixed vertex or adjacent to an end satisfying a multi-line condition, $b$ can only change, when $v$ is adjacent to an end that satisfies a multi-line condition $L$. Thus $b$ can only change once if $v$ passes over the vertex of $L$, see Example 32. Hence
the direction of movement of a vertex in the movable component cannot change if we already moved the movable component (and in particular \( v \)) enough. In the following we focus on movable components that allow an unbounded movement and that already have been moved sufficiently such that we can assume that the direction of movement of each vertex therein does not change anymore when moving. In particular, we may assume that the direction of movement of a vertex satisfying a multi-line condition is parallel to \((-1, 0), (0, -1)\) or \((1, 1)\).

**Example 32.** Figure 4 provides an example of a curve \( C \) in \( \mathbb{R}^2 \) whose contracted ends labeled with 1, 2, 4, 5 satisfy point conditions and the contracted end labeled with 3 satisfies a multi-line condition (the dashed line). The vertex \( v \) adjacent to the end labeled with 3 is in the movable component of \( C \) and the direction of movement \( b \) (indicated by an arrow) of \( v \) might changes as \( v \) is moved. The movement shown in Figure 4 is bounded.

![Figure 4](image)

**Remark 33.** Showing that \( B \) contains a single vertex is non-trivial. However, the difficulties arise primarily due to the cross-ratios. If we have no cross-ratios and thus every vertex in our tropical curves is 3-valent, then the movable component boils down to a string as introduced in [GM08], which can be thought of as a single chain.

**Classification 34 (Types of movable vertices).** Let \( \Gamma \) be a tropical curve as in Definition 30. If there is a vertex \( v \) in the movable component of \( \Gamma \) that is adjacent to a fixed component and all of its adjacent edges and ends which are non-contracted are parallel, then the movable component of \( \Gamma \) has exactly one vertex, namely \( v \). Otherwise \( \Gamma \) would not give rise to a 1-dimensional family only.

Hence the following classification is complete if we assume that the movable component of \( \Gamma \) has more than 1 vertex (if it has exactly 1 vertex, then we can directly jump to the proof of Proposition 29): We distinguish 4 types of vertices in the movable component.
Type (I) vertices are adjacent to a fixed component and not all adjacent edges and non-contracted ends are parallel.

Type (II) vertices are not 3-valent and adjacent to a contracted end which satisfies a multi-line condition.

Type (IIIa) vertices are 3-valent, adjacent to two bounded edges and adjacent to a contracted end which satisfies a multi-line condition.

Type (IIIb) vertices are 3-valent, adjacent to one bounded edge, a contracted end which satisfies a multi-line condition and an end in standard direction.

Throughout this section we use the assumption that the movable component of $\Gamma$ has more than 1 vertex whenever we refer to this classification of vertices.

**Construction 35.** In the following we often forget the vertices of type (IIIa) and type (IIIb) in $\Gamma$ by gluing the non-contracted edges adjacent to a vertex of type (IIIa) (resp. type (IIIb)) together and obtain a tropical curve denoted by $\tilde{\Gamma}$. We fix this notation of $\tilde{\Gamma}$ throughout this section.

If $\tilde{\Gamma}$ allows no 1-dimensional movement, then the only vertices in the movable component of $\tilde{\Gamma}$ are of type (IIIA) or (IIIB). Hence there is no type (I) vertex in the movable component of $\Gamma$. Thus $\Gamma$ has no fixed component. In particular $p_{[n]} = \emptyset$, but this case is treated separately in Lemma 51, Lemma 52 and Proposition 53. Therefore we can assume that $\tilde{\Gamma}$ allows an unbounded 1-dimensional movement.

![Figure 5: The cone $\sigma_{v_2}(b_1,e)$ in which the direction of movement of $v_2$ lies. The slope of the edge connecting $v_1, v_2$ is fixed during the movement. Hence the translation $b_2 + v_2$ of the direction of movement $b_2$ of $v_2$ is contained in the open half-plane $H$ whose boundary is $(e) + v_2$ and whose interior contains $b_1 + v_1$.](image)

**Lemma 36 (Angle Lemma).** Let $\tilde{\Gamma}$ be a tropical curve in $\mathbb{R}^2$ as in Construction 35 that allows an unbounded 1-dimensional movement. Let $v_1, v_2$ be adjacent vertices in the movable component of $\tilde{\Gamma}$, let $b_1 \neq 0$ be the direction of movement of $v_1$ and let $v(e, v_1) \neq b_1$ be the direction vector at $v_1$ of the edge $e$ that connects $v_1$ and $v_2$. Then the direction of movement $b_2$ of $v_2$ lies in the half-open cone

$$\sigma_{v_2}(b_1, e) := \{ x \in \mathbb{R}^2 \mid x = v_2 + \lambda_1 v(e, v_1) + \lambda_2 b_1, \lambda_1 \in \mathbb{R}_{\geq 0}, \lambda_2 \in \mathbb{R}_{>0} \}$$
centered at \(v_2\) that is spanned by \(b_1\) and \(v(e,v_1)\), where half-open means that the boundary of \(\sigma_{v_2}(b_1,e)\) that is generated by \(b_1\) is part of the cone and the boundary that is generated by \(v(e,v_1)\) is not part of the cone, while \(v_2\) itself is also not part of the cone.

Proof. This is true since the length of the edge \(e'\) that connects \(v_1\) and \(v_2\) cannot shrink when moving \(v_1\) and \(v_2\), otherwise the movement would be bounded. Therefore the (affine) lines \((b_1)+v_1\) and \((b_2)+v_2\) must either be parallel or their point of intersection does not lie in \(H\).

Let us now introduce a partial order which gives rise to chains. Our goal is to show that no chain has more than one element, i.e. we are at step (S3) now:

**Definition 37** (Partial order). We use the notation from Construction 35. Let \(\tilde{\Gamma}\) be a tropical curve in \(\mathbb{R}^2\) that allows an unbounded 1-dimensional movement and let \(H\) be an open half-plane. If we translate \(H\) to a vertex \(v\in\tilde{\Gamma}\), i.e. \(v\) is contained in the boundary of \(H\), then we denote the translated half-plane by \(H_v\). Let \(M\) be the set of all vertices of the movable component of \(\tilde{\Gamma}\), i.e. \(M\) consists of all type (I) and type (II) vertices of the movable component of \(\tilde{\Gamma}\). The half-plane \(H\) induces a partial order \(\Omega(H)\) on \(M\) as follows: For \(v_1,v_2 \in M\) define

\[
v_1 \geq v_2 : \iff \begin{cases} v_1 = v_2, & \text{or} \\ v_2 \text{ is adjacent to } v_1 \text{ and } v_2 \in H_{v_1}. \end{cases}
\]

Here, we only use open half-planes \(H\) such that \(b_1+v_1 \in H_{v_1}\). Therefore if \(v_1 \geq \cdots \geq v_n\) is a maximal chain and \(b_i\) is the direction of movement of \(v_i\) for \(i = 1,\ldots,n\), then \(b_i+v_i \in H_{v_i}\) for \(i = 1,\ldots,n\) by inductively applying Lemma 36.

**Notation 38.** Given a chain \(v_1 \geq \cdots \geq v_n\) in the movable component of \(\tilde{\Gamma}\), we denote the direction of movement of \(v_i\) by \(b_i\) for \(i = 1,\ldots,n\) throughout this section. If such a chain is maximal, then an edge connecting \(v_i\) and \(v_{i+1}\) is usually denoted by \(e_i\) for \(i = 1,\ldots,n-1\). By abuse of notation, we often write \(e_i\) instead of the direction vector \(v(e_i,v_i)\) at \(v_i\) from Definition 5.

**Lemma 39** (Maximal chains). We use Notation 38. Let \(\tilde{\Gamma}\) be a tropical curve in \(\mathbb{R}^2\) as in Construction 35, that allows an unbounded 1-dimensional movement. Let \(v_1 \geq \cdots \geq v_n\) be a maximal chain with \(n > 1\) and \(b_1+v_1 \in H_{v_1}\) in \(\tilde{\Gamma}\) with respect to \(\Omega(H)\) as in Definition 37. Then there is no vertex \(v_{n+1} \in \tilde{\Gamma}\) adjacent to \(v_n\) such that \(v_{n+1} \in H_{v_n}\).

Proof. We use Notation 38. By definition, \(v_n, b_{n-1}+v_{n-1} \in H_{v_{n-1}}\) and there is an edge \(e_{n-1}\) connecting \(v_{n-1}\) to \(v_n\). If \(\langle b_{n-1} \rangle = \langle e_{n-1} \rangle\), then \(b_{n-1}\) and \(b_n\) are parallel. Thus we have a 2-dimensional movement which yields a contradiction since we just allow a 1-dimensional movement. In total, the requirements of Lemma 36 are fulfilled such that \(b_n+v_n \in H_{v_n}\) follows. Since there is an edge \(e_n\) that connects \(v_n\) to \(v_{n+1}\) and \(v_{n+1} \in H_{v_n}\), Definition 37 yields \(v_n \geq v_{n+1}\) with respect to \(\Omega(H)\). This contradicts our maximality assumption. \(\square\)
Figure 6: This is an example of the partial order $\Omega(H)$ for $H \subset \mathbb{R}^2$ which is an open half-plane as shown on the left (the boundary of the half-plane is darkened). On the right there is a sketch of a tropical curve in $\mathbb{R}^2$ such that $v_1 \geq v_3 \geq v_4$ and $v_2 \geq v_3 \geq v_4$ with respect to the order $\Omega(H)$.

**Definition 40** (Special half-planes). Let $e \in \mathbb{R}^2$ be a vector of one of the standard directions $(-1,0), (0,-1), (1,1)$. An open half-plane is called *special half-plane* if the affine subspace $(e) + v \subset \mathbb{R}^2$ for some $v \in \mathbb{R}^2$ that is generated by $e$ is the boundary of $H$. There are six special half-planes up to translation, see Figure 7.

Figure 7: All six special half-planes up to translation. The boundary of each is darkened.

**Definition 41.** An open half-plane $H$ is called 1-ray (resp. 2-ray) half-plane if it contains exactly one (resp. two) rays of standard direction. Notice that special half-planes are 1-ray half-planes.

**Lemma 42.** Let $\tilde{\Gamma}$ be a tropical curve in $\mathbb{R}^2$ as in Construction 35 that allows an unbounded 1-dimensional movement. Let $v_1$ be a vertex of the movable component of $\tilde{\Gamma}$. Let $H$ be a 1-ray half-plane that contains a ray of standard direction $D$. If $v_1 \geq \cdots \geq v_n$ is a maximal chain starting at $v_1$ with respect to $\Omega(H)$ such that $n > 1$ and $b_1 + v_1 \in H_{v_1}$, then there is an end $e$ of $\tilde{\Gamma}$ adjacent to $v_n$ which is parallel to $D$. 
Proof. We use Notation 38. Notice that $v_{n-1} \geq v_n$. Hence $e_{n-1} + v_n \notin \overline{H}_{v_n}$, where $\overline{H}_{v_n}$ denotes the closure of $H_{v_n}$. Thus by balancing, there is an edge $e \in \Gamma$ adjacent to $v_n$ such that $e \in H_{v_n}$. If $e$ connects $v_n$ to a fixed component, then $b_n + v_n \notin \overline{H}_{v_n}$ because the movement of $v_n$ should be unbounded, i.e. $b_n$ moves $v_n$ away from that fixed component while $\langle e \rangle + v_n = \langle b_n \rangle + v_n$, which contradicts that $b_n + v_n \in H_{v_n}$ by Lemma 36. Hence $e$ is an end of $\Gamma$ by Lemma 39. Since $H_{v_n}$ is a 1-ray half-plane containing exactly 1 ray of standard direction $D$, the direction of $e$ is $D$. \hfill \Box

Lemma 43 (About maximal chains, weak version). Let $\tilde{\Gamma}$ be a tropical curve in $\mathbb{R}^2$ as in Construction 35 that allows an unbounded 1-dimensional movement. Let $v_1$ be a vertex of the movable component of $\tilde{\Gamma}$. If there is a 1-ray half-plane $H$ and $v_1 \geq \cdots \geq v_n$ is a maximal chain starting at $v_1$ with respect to $\Omega(H)$ such that $n > 1$ and $b_1 + v_1 \in H_{v_1}$, then $v_n$ is a 3-valent type (I) vertex.

Proof. We use Notation 38. By Lemma 42 there is an end $e$ of $\tilde{\Gamma}$ adjacent to $v_n$. Moreover, since $H$ is a 1-ray half-plane containing exactly 1 ray of standard direction $D$, the direction of $e$ is $D$. Assume that the valency of $v_n$ is greater than 3, i.e. there is a cross-ratio in $\lambda_{v_n}$. Since all cross-ratios have only labels of contracted ends as entries (see Definition 17), we can apply Corollary 27. Therefore there is a vertex $v \in \Gamma$ connected to $v_n$ via $e$ such that $v$ is of type (IIIa) or type (IIIb) such that $v$ satisfies a multi-line condition. Since the movement of $v$ is unbounded, its direction of movement, denoted by $b_v$ is parallel to $e$ (cf. Remark 31). Therefore the movable component of $\Gamma$ allows a 2-dimensional movement, which is a contradiction.

In total, $v_n$ can only be a 3-valent type (I) vertex since we ruled out the other cases. \hfill \Box

Corollary 44. If we make the same assumptions as in Lemma 43 and additionally require that $H$ is a special half-plane (see Definition 40), then there exists no chain $v_1 \geq \cdots \geq v_n$ with respect to $\Omega(H)$ such that $n > 1$ and $b_1 + v_1 \in H_{v_1}$.

Proof. We use Notation 38. It is sufficient to show the statement for maximal chains $v_1 \geq \cdots \geq v_n$ starting at $v_1$. So we assume that our chain is maximal. The vertex $v_n$ is 3-valent of type (I) by Lemma 43. Let $D$ denote the ray of standard direction that is contained in $H$. By Lemma 42, there is an end $e$ adjacent to $v_n$ of standard direction $D$. Denote the edge that connects $v_n$ to a fixed component by $f$, and because $\langle f \rangle + v_n = \langle b_n \rangle + v_n$, we know that $f + v_n \notin \overline{H}_{v_n}$. Since all ends are of weight 1, the end $e$ is also of weight 1. Using balancing and the definition of special half-planes, we conclude that the edge $e_{n-1}$ that connects $v_{n-1}$ to $v_n$ lies in the boundary of $H_{v_n}$, which contradicts $v_{n-1} \geq v_n$. \hfill \Box

Observation 45. Let $v_1, v_2$ be two vertices of the movable component of $\tilde{\Gamma}$. Let $e$ be an edge that connects $v_1$ and $v_2$ and let $b_1$ be the direction of movement of $v_1$. Corollary 44 shows that there cannot be an open half-plane $H$ such that $b_1 + v_1, e + v_1 \in H_{v_1}$, and such that $H_{v_1}$ is a special half-plane. Note that $\langle b_1 \rangle \neq \langle e \rangle$, otherwise our movable component would move in a 2-dimensional way. Therefore, for each pair of directions of $b_1$ and $e$, there are open half-planes that contain $b_1$ and $e$. But each of these open half-planes is not a special half-plane. This observation gives rise to the following classification.

**Classification 46** (Dependence of $b_1$ and $e$). Let $\tilde{\Gamma}$ be as in Construction 35. In particular, we assume that $\tilde{\Gamma}$ has more than one vertex. Use the notation of Observation 45, i.e. let $v_1 \in \tilde{\Gamma}$ be a vertex with direction of movement $b_1$. If $b_1 + v_1$ is in one of the dashed red cones in Figure 8, then $e + v_1$ has to lie in the opposite cone. Otherwise there would be a special half-plane $H$ such that $b_1 + v_1, e + v_1 \in H_{v_1}$, which contradicts Observation 45. We distinguish the 3 cases depicted in Figure 8: If $b_1 + v_1$ and $e + v_1$ lie in the red cones depicted on the left, then $v_1$ is said to be of type $F_1$. The other two cases can be seen in Figure 8.

![Figure 8: A vertex $v_1$ with its cones in which $b_1 + v_1$ and $e + v_1$ can lie. From left to right: A vertex $v_1$ of type $F_1$, $F_2$ and $F_3$.](image)

The other way round, given a vertex $v_1 \in \tilde{\Gamma}$ and its type $F_i$, we can estimate the positions of $b_1 + v_1$ and $e + v_1$. See Figure 8 for the following: If $v_1$ is of type $F_i$, then $b_1 + v_1$ and $e + v_1$ need to lie in the red cones depicted in Figure 8 in such a way that $b_1 + v_1$ and $e + v_1$ lie in opposite cones.

**Remark 47.** If there is some maximal chain $v_1 \geq \cdots \geq v_n$ in $\tilde{\Gamma}$ with respect to $\Omega(H)$ such that $b_1 + v_1 \in H_{v_1}$ and $v_1$ is of type $F_i$, then $v_j$ is also of type $F_i$ for $j = 2, \ldots, n$.

**Proof.** We use Notation 38. By induction, is is sufficient to show the statement for $v_1 \geq v_2$. Let $e_1$ be the edge adjacent to $v_1, v_2$. Let $F_i$ be the type of $v_1$ such that $\sigma_{e_1} + v_1$ and $\sigma_{b_1} + v_1$ are its two opposing cones, where $e_1 + v_1 \in \sigma_{e_1} + v_1$ and $b_1 + v_1 \in \sigma_{b_1} + v_1$. Hence $-e_1 + v_2 \in \sigma_{b_1} + v_2$. By Observation 45, we obtain $b_2 + v_2 \in \sigma_{e_1} + v_2$. \hfill $\Box$

**Lemma 48.** We use Notation 38. Let $\tilde{\Gamma}$ be a tropical curve in $\mathbb{R}^2$ as in Construction 35 that allows an unbounded 1-dimensional movement. Let $v_1$ be a vertex of the movable component of $\tilde{\Gamma}$. Let $H$ be an open half-plane. Let $v_1 \geq \cdots \geq v_n$ be a maximal chain with respect to $\Omega(H)$ such that $n > 1$ and $b_1 + v_1 \in H_{v_1}$. If $b_n$ is of non-standard direction, then $v_n$ is adjacent to two ends of $\tilde{\Gamma}$ of different standard directions. If $b_n$ is of standard direction, then $v_n$ is adjacent to one end of $\tilde{\Gamma}$ of standard direction parallel to $b_n$.

**Proof.** Assume that $v_n$ is of type $F_i$ for an $i = 1, 2, 3$ and that $b_n$ is of non-standard direction. Thus, by Classification 46, $b_n + v_n$ lies in the interior of one of the dashed red cones of Figure 8 and all bounded edges adjacent to $v_n$ lie in the opposite cone. Therefore, by the balancing condition, $v_n$ needs to be adjacent to at least two ends of different standard directions.
Next, assume that $b_n$ is of standard direction. Hence $b_n + v_n$ appears in the boundary of two of the red cones $\sigma_1, \sigma_2$ of Classification 46. Therefore all edges which are no ends adjacent to $v_n \in \bar{\Gamma}$ are in the union $\sigma_1' \cup \sigma_2'$ of the opposite cones $\sigma_j'$ of $\sigma_j$ for $j = 1, 2$. Therefore balancing guarantees that there is an end adjacent to $v_n \in \bar{\Gamma}$ which is parallel to $b_n$. \hfill \Box

The following Lemma generalizes Lemma 43 from 1-ray half-planes to arbitrary half-planes.

**Lemma 49** (About maximal chains, strong version). Let $\bar{\Gamma}$ be a tropical curve in $\mathbb{R}^2$ as in Construction 35 that allows an unbounded 1-dimensional movement. Let $v_1$ be a vertex of the movable component of $\bar{\Gamma}$. If there is an open half-plane $H$ such that $v_1 \geq \cdots \geq v_n$ is a maximal chain starting at $v_1$ with respect to $\Omega(H)$ such that $n > 1$ and $b_1 + v_1 \in H_{v_1}$, then $v_n$ is a 3-valent type (I) vertex.

**Proof.** We use Notation 38, assume that val($v_n$) > 3, that $v_n$ is of type $F_i$ for an $i = 1, 2, 3$ and that $b_n$ is of non-standard direction. By Lemma 48, $v_n$ needs to be adjacent to at least two ends $E_1, E_2$ of different standard directions. By Corollary 27, we can reach a type (IIIb) vertex via each of the edges $E_1, E_2$ in $\Gamma$. The direction of movement of such a type (IIIb) vertex cannot be parallel to the end of standard direction it is connected to, otherwise we would have a 2-dimensional movement. Recall that type (IIIb) vertices can only move in standard direction since their contracted ends satisfy multi-line conditions. See Figure 9 for the following: If $i = 1$, i.e. $v_n$ is of type $F_1$, we consider the cone in which $b_n + v_n$ lies and go through all different directions of movements of the type (IIIb) vertices. In each case we obtain a contradiction to your unbounded movement.

We still get a contradiction if $b_n + v_n$ would lie in the other red cone of Figure 9. More generally, the same arguments and conclusion of the case $i = 1$ are true for $i = 2, 3$ and lead to contradictions as well.

![Figure 9: A vertex $v_n$ of type $F_1$ connected to two type (IIIb) vertices which move along the directions of the arrows.](image)

Next, we assume that $b_n$ is of standard direction. By Lemma 48, there is an end $E_1$ adjacent to $v_n \in \bar{\Gamma}$ which is parallel to $b_n$. Since we assumed that val($v_n$) > 3, there must, again, be a type (IIIb) vertex adjacent to $v_n$ via $E_1$. Notice that this vertex can only move unboundedly in the direction of $b_n$, which is a contradiction because our movement is only 1-dimensional.

In total, $v_n$ can only be a type (I) vertex that is 3-valent. \hfill \Box
Corollary 50. Let $v_1, b_1$ and $H$ be an open half-plane as in Lemma 49. Then there is no chain $v_1 \geq \cdots \geq v_n$ with $n > 1$ and $b_1 + v_1 \in H_{v_1}$ in the movable component of $\tilde{\Gamma}$.

Proof. We use Notation 38 and assume that there is a maximal chain $v_1 \geq \cdots \geq v_n$ starting at $v_1$. Hence $v_n$ must be a 3-valent type (I) vertex by Lemma 49. By Lemma 48, there is an end $E$ of $\tilde{\Gamma}$ adjacent to $v_n$. Moreover, denote the direction vector at $v_n$ of the edge that connects $v_n$ to a fixed component by $f$. Therefore the direction of movement of $v_n$, denoted by $b_n$, is given by $-f$ since $v_n$ moves unboundedly, i.e. it moves away from the fixed component it is adjacent to. We distinguish all cases of Classification 46 for $v_n$. So let the type of the vertex $v_n$ be $F_i$ for an $i = 1, 2, 3$ (see Figure 8). Since $b_n = -f$, the edges $e_{n-1}$ and $f$ adjacent to $v_n$ lie in the same cone. Then there exists no end $E$ such that $v_n$ is balanced (for each possible end $E$ we find a half-plane $P$ such that $E + v_n, f + v_n, -e_{n-1} + v_n \in P_{v_n}$) which is a contradiction.

The following proof builds on ideas of Proposition 5.1 in [GM08]. It is step (S4):

Proof of Proposition 29. Consider the 1-dimensional cycle

$$Y = \prod_{k \in \mathbb{Z}} \text{ev}^*_k(L_k) \cdot \prod_{i \in \mathbb{Z}} \text{ev}^*_i(p_i) \cdot \prod_{j=1}^{l-1} \text{ft}^*_\lambda_j(0) \cdot \mathcal{M}_{0, m}(\mathbb{R}^2, d)$$

from Definition 30. We need to show that

$$\{\text{ft}^*_\lambda(C) \mid C \in Y \text{ has no contracted bounded edge}\}$$

is bounded in $\mathcal{M}_{0,4}$. If it is unbounded, then there is a curve $C$ coming from a stable map in $Y$ without a contracted bounded edge which allows an unbounded movement. Hence the movable component of $C$ has exactly one vertex $v$ by Corollary 50 which is not of type (IIIa) or (IIIb) as in Classification 34. Notice that $C$ has at least one fixed component as well since we assume that there is at least one point condition that $C$ satisfies.

We distinguish different cases for $v$.

1. Assume that $\text{val}(v) = 3$ and that $v$ is adjacent to two edges $E_1, E_2$ which are parallel to two ends of different direction. The edges $E_1, E_2$ lead to other vertices in the movable component moving $v$ varies $\text{ft}^*_\lambda(C)$ and Corollary 27 applies. There are 3 cases (choose 2 different directions for $E_1, E_2$ from the 3 standard directions) we need to distinguish. Moving $v$ unboundedly, we obtain an end adjacent to $v$. More precisely, Figure 10 shows one of the 3 case where the directions are $(1, 1)$ and $(0, -1)$ (the other two cases are analogous). Hence moving $v$ further in its direction of movement eventually produces a combinatorial type that does not allow $\text{ft}^*_\lambda(C)$ to become larger as $v$ is moved.

2. Assume that $\text{val}(v) = 3$ and that all edges adjacent to $v$ are parallel. Since all ends of $C$ are of weight 1, the two edges $E_1, E_2$ adjacent to $v$, which lead to other vertices in the movable component, are on the same side of $v$. Therefore moving $v$ as before (analogous to Figure 10 but with $v_1, v_2$ lying on parallel ends) does not make the coordinate $\text{ft}^*_\lambda(C)$ larger.
Figure 10: The movable vertex $v$ and its movement away from the fixed component.

(3) Assume that $\text{val}(v) > 3$, then there are edges $E_1, E_2$ adjacent to $v$ (by Corollary 27) which connect $v$ to vertices $v_1, v_2$ of the movable component that satisfy multi-line conditions $L_{v_1}, L_{v_2}$. The same movement as in the case of $\text{val}(v) = 3$ yields a combinatorial type where there is an end adjacent to $v$ which contradicts Corollary 27 since $\text{val}(v) > 3$, see again Figure 10.

In total, choosing a large value for $|\lambda'|$ implies that only curves with a contracted bounded edge can contribute to $N_d \left( p_{2l}, L_{\kappa}, \lambda_{[l-1]}, \lambda' \right)$. Moreover, there is exactly one contracted bounded edge. Otherwise a stable map $C$ contributing to $N_d \left( p_{2l}, L_{\kappa}, \lambda_{[l-1]}, \lambda' \right)$ would give rise to a 1-dimensional family of stable maps contributing to $N_d \left( p_{2l}, L_{\kappa}, \lambda_{[l-1]}, \lambda' \right)$ which is a contradiction.

Notice that in Proposition 29 we assumed that $n \geq 1$, i.e. that there is at least one point condition. However, even without point conditions we can still assume that there is a contracted bounded edge, see Proposition 53.

**Lemma 51.** Let $C$ be a tropical stable map that contributes to $N_d \left( L_{\kappa}, \lambda_{[1]} \right)$. Then there is a vertex $v$ of $C$ which is adjacent to two contracted ends $e_1, e_2$ such that $e_1$ satisfies a multi-line condition $L_a$ and $e_2$ satisfies a multi-line condition $L_b$, respectively.

**Proof.** Assume that each vertex of $C$ is at most adjacent to one contracted end that satisfies a multi-line condition. Hence each vertex of the tropical curve associated to $C$
allows a 1-dimensional movement since its movement is only restricted by at most one multi-line condition (we have no point conditions). Thus \( C \) give rise to a 1-dimensional family which is a contradiction.

\[ \text{Lemma 52. Let } v \text{ be the vertex adjacent to } e_1, e_2 \text{ from Lemma 51. Then } \text{val}(v) > 3 \text{ and there is a degenerated cross-ratio } \lambda \in \lambda_{[l]} \text{ such that } \lambda = \{e_1, e_2, \beta_3, \beta_4\}. \]

\[ \text{Proof. We use the notation from Lemma 51. If } \text{val}(v) = 3, \text{ then, by Lemma 51, there is a contracted bounded edge adjacent to } v. \text{ Hence } C \text{ cannot be fixed by the set of given conditions which is a contradiction. Thus } \text{val}(v) > 3. \]

By Corollary 27 there is a cross-ratio \( \lambda \) as desired or there are cross-ratios \( \lambda_1 = \{e_1, \ldots \} \) and \( \lambda_2 = \{e_2, \ldots \} \) such that \( e_2 \notin \lambda_1 \) and \( e_1 \notin \lambda_2 \). Assume that there is no cross-ratio \( \lambda \) as desired. Then \( v \) can be resolved by adding a contracted bounded edge \( e \) to \( C \) that is adjacent to \( v \) and a new 3-valent vertex \( v' \) which is adjacent to \( e_1, e_2 \). Notice that this resolution of \( v \) is compatible with \( \lambda_1, \lambda_2 \) but gives rise to a 1-dimensional family of tropical stable maps satisfying \( L_{e_1}, \lambda_{[l]} \) which is a contradiction.

\[ \text{Proposition 53. We use notation from Lemma 51 and Lemma 52 and assume without loss of generality that } e_1, e_2 \text{ are entries of the cross-ratio } \lambda_l. \text{ Let } \lambda' \text{ be a non-degenerated cross-ratio that degenerates to } \lambda_l, \text{ where } e_1, e_2 \text{ are grouped together. Then every tropical stable map } C' \text{ that contributes to } N_d(L_{e_1}, \lambda_{[l-1]}, \lambda'_l) \text{ arises from a tropical stable map } C \text{ that contributes to } N_d(L_{e_2}, \lambda_{[l]}) \text{ by adding a contracted bounded edge } e \text{ to } C \text{ that is adjacent to } v \text{ and a new vertex } v' \text{ which is in turn adjacent to } e_1, e_2. \]

\[ \text{Proof. Let } C \text{ be a tropical stable map that contributes to } N_d(L_{e_1}, \lambda_{[l]}) \text{ and let } v \text{ the vertex from Lemma 51 at which } \lambda_l \text{ is satisfied. Assume that the edge } e' \text{ we add by resolving } v \text{ according to } \lambda'_l \text{ is not contracted and denote the tropical stable map obtained this way by } C''. \text{ Denote the vertex adjacent to } e' \text{ and } e_1, e_2 \text{ by } \tilde{v}. \text{ Consider } C'' \text{ as a point in the cycle that arises from dropping the cross-ratio condition } \lambda'_l \text{ (cf. Definition 30). Then } C'' \text{ is in the boundary of a 2-dimensional cell of the same cycle that arises from } C'' \text{ by adding a contracted bounded edge } e \text{ to } C'' \text{ that separates } \tilde{v} \text{ from } e_1, e_2. \text{ Hence there is a 2-dimensional cell inside a 1-dimensional cycle, which is a contradiction.} \]

Each tropical stable map \( C \) contributing to the number \( N_d(L_{e_1}, \lambda_{[l]}) \) yields a contribution to \( N_d(L_{e_2}, \lambda_{[l-1]}, \lambda'_l) \) if the vertex \( v \) at which \( \lambda_l \) is satisfied is resolved according to \( \lambda'_l \) and each resolution of \( v \) according to \( \lambda'_l \) produces a contracted bounded edge \( e \). Hence Remark 19 and the description of mult(\( C \)) via resolutions of vertices (see also [Gol20]) guarantees that there cannot be more stable maps \( C' \) contributing to \( N_d(L_{e_2}, \lambda_{[l-1]}, \lambda'_l) \) than the ones obtained from adding a contracted bounded edge \( e \) to tropical stable maps \( C \).

\[ \text{Behavior of cut contracted bounded edges} \]

After we identified a contracted bounded edge \( e \) in Propositions 29, 53, we can cut this edge which yields a split of the original tropical stable map into two new ones. The aim of this subsection is to prove Corollary 59, in which the behavior of the two new ends that arise from cutting \( e \) is described.
Construction 54 (Cutting the contracted bounded edge). Let $C$ be a tropical stable map that contributes to $N_d(p_2, L_w, \lambda_{[l-1]})$, where $\lambda'_l$ is a non-degenerated tropical cross-ratio such that $|\lambda'_l|$ is large. Assume that $C$ has a contracted bounded edge $e$.

If we cut $e$, we obtain two tropical stable maps $C_1$ and $C_2$ with contracted ends $e_1$ and $e_2$ that come from $e$ (notice that $e$ is of weight zero and does therefore not effect balancing). By abuse of notation, the label of $e_i$ is also $e_i$ for $i = 1, 2$. We usually denote the vertices adjacent to the ends $e_1, e_2$ by $v_1, v_2$. Notice that $C_i$ is of degree $d_i$ for $i = 1, 2$ such that $d_1 + d_2 = d$ since $C$ is balanced and of degree $d$.

If a contracted bounded edge $e$ is cut, the cross-ratios can be adapted the following way: If $\lambda_j$ is a degenerated cross-ratio that is satisfied at some vertex $v \in C_i$ for $i = 1, 2$, then, by the path criterion (Remark 20), either all entries of $\lambda_j$ are labels of contracted ends of $C_i$ or 3 entries of $\lambda_j$ are labels of contracted ends of $C_i$ and one entry $\beta$ is a label of a contracted end of $C_i$ for $t \neq i$. In the first case, we do not change $\lambda_j$ and in the latter case, we replace the entry $\beta$ of $\lambda_j$ by $e_i$. We denote a degenerated cross-ratio that we adapted to $e_i$ by $\lambda_j^{-e_i}$.

Each $C_i$ of degree $d_i$ for $i = 1, 2$ satisfies point conditions $p_{\underline{n}}$, multi-line conditions $L_{\underline{\kappa}}$, and cross-ratio conditions $\lambda_{\underline{i}}^{-e_i}$ such that $n_{\underline{i}} \cup n_{\underline{2}} = n$, $\kappa_{\underline{i}} \cup \kappa_{\underline{2}} = \kappa$ and $l_{\underline{i}} \cup l_{\underline{2}} = [l-1]$, where we adapted all cross-ratios to the cut edge $e$. We say that $C$ splits into the two tropical stable maps $C_1$ and $C_2$ and the splitting type of $C$ is $(d_1, n_{\underline{i}}, \kappa_{\underline{i}}, l_{\underline{i}}, f_{\underline{i}} \mid d_2, n_{\underline{2}}, \kappa_{\underline{2}}, l_{\underline{2}}, f_{\underline{2}})$, where $f_{\underline{i}} \cup f_{\underline{2}} = f$ is a partition of the ends of $C$ that satisfy no point or multi-line condition as in Definition 15.

Definition 55 (1/1 and 2/0 splits). Let $d$ be a degree, let $p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}$ be given conditions and let $f$ be labels of contracted ends that satisfy no conditions as in Definition 14. We refer to $(d_1, n_{\underline{i}}, \kappa_{\underline{i}}, l_{\underline{i}}, f_{\underline{i}} \mid d_2, n_{\underline{2}}, \kappa_{\underline{2}}, l_{\underline{2}}, f_{\underline{2}})$ as a split (of conditions) if $d_1 + d_2 = d$, $n_{\underline{i}} \cup n_{\underline{2}} = n$, $\kappa_{\underline{i}} \cup \kappa_{\underline{2}} = \kappa$, $l_{\underline{i}} \cup l_{\underline{2}} = [l-1]$, $f_{\underline{i}} \cup f_{\underline{2}} = f$ holds and each cross-ratio in $\lambda_{\underline{i}}$ has at least 3 of its entries in $n_{\underline{i}} \cup \kappa_{\underline{i}} \cup f_{\underline{i}}$. If we write $\lambda_{\underline{i}}^{-e_i}$, we mean that each entry of each cross-ratio in $\lambda_{\underline{i}}$ that is not in $n_{\underline{i}} \cup \kappa_{\underline{i}} \cup f_{\underline{i}}$ is replaced by the label $e_i$. Such a split is called a 1/1 split if

$$3d_i = \#n_{\underline{i}} + \#l_{\underline{i}} - \#f_{\underline{i}} + 1$$

holds for $i = 1, 2$. If

$$3d_i = \#n_{\underline{i}} + \#l_{\underline{i}} - \#f_{\underline{i}} \quad \text{and} \quad 3d_i = \#n_{\underline{i}} + \#l_{\underline{i}} - \#f_{\underline{i}} + 2$$

holds for $i = 1, 2$ with $t \neq i$ for some choice of $i, t \in \{1, 2\}$, then we refer to $(d_1, n_{\underline{i}}, \kappa_{\underline{i}}, l_{\underline{i}}, f_{\underline{i}} \mid d_2, n_{\underline{2}}, \kappa_{\underline{2}}, l_{\underline{2}}, f_{\underline{2}})$ as a 2/0 split.

Definition 56 (1/1 and 2/0 edges). Let $(d_1, n_{\underline{i}}, \kappa_{\underline{i}}, l_{\underline{i}}, f_{\underline{i}} \mid d_2, n_{\underline{2}}, \kappa_{\underline{2}}, l_{\underline{2}}, f_{\underline{2}})$ be a split of conditions as in Definition 55. Define for the (adapted) conditions $p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{\underline{i}}^{-e_i}$ and for $i = 1, 2$ the cycles

$$Y_i := ev_{e_i} \left( \prod_{k_{\underline{i}}} \ev^*_k (L_k) \cdot \prod_{l_{\underline{i}}} \ev^*_l \left( p_{l_{\underline{i}}} \right) \cdot \prod_{j \in \underline{j}} \ft^{*_{e_i}} \left( 0 \right) \cdot M_{0,m_i} \left( \mathbb{R}^2, d_i \right) \right) \subset \mathbb{R}^2,$$
where \( m_i := \#n_i + \#\kappa_i + \#f_i \). Notice that \((d_1, n_1, \kappa_1, l_1, f_1 | d_2, n_2, \kappa_2, l_2, f_2)\) is a 1/1 split if and only if both \( Y_i \) are 1-dimensional. It is a 2/0 split if and only if \( Y_i \) is 0-dimensional and \( Y_1 \) is 2-dimensional (see (3) in Definition 55).

Let \( C \) be a tropical stable map with a contracted bounded edge \( e \) such that \( C \) is of splitting type \((d_1, n_1, \kappa_1, l_1, f_1 | d_2, n_2, \kappa_2, l_2, f_2)\). Then \( m_i \) is the number of contracted ends of \( C_i \) and the cycle \( Y_i \) is the condition \( C_i \) imposes on \( C_i \) for \( t \neq i \) via \( e \). For example, if \( Y_1 \) is 0-dimensional, then the position of \( v_2 \) is completely determined by \( Y_1 \) since \( v_2 \) is connected to \( v_1 \) via \( e \) in \( C \) and \( C \) is fixed by the given conditions \( p_{2r}, L_{\kappa}, \lambda_{[l-1]}, \lambda_l \). Since all given conditions are in general position, the dimension of \( Y_2 \) is 2 in this case, i.e. \( v_2 \) cannot impose a condition via \( e \) to \( v_1 \). In general, we have two cases for \( C \):

1. One of the cycles \( Y_i \) is 0-dimensional and the other one is 2-dimensional. We then refer to \( e \) as a 2/0 edge.

2. Both of the cycles \( Y_i \) are 1-dimensional. We then refer to \( e \) as a 1/1 edge.

Which case occurs depends only on \( d_i, \#n_i, \#\kappa_i, \#l_i, \#f_i \) for \( i = 1, 2 \).

**Example 57.** An example for a 1/1 split is provided below, see Example 63. An example for a 2/0 split is the following: Let \( C \) be a degree 2 tropical stable map that satisfies point conditions \( p_{[2]} \), multi-line conditions \( L_{[4]} \), degenerated cross-ratios \( \lambda_1 = \{p_1, L_1, L_2, L_3\} \), \( \lambda_2 = \{p_1, p_2, L_1, L_2\} \) and a non-degenerated cross-ratio \( \lambda'_3 = (p_1 L_4 | p_2 L_4) \) whose length is large enough such that \( C \) has a contracted bounded edge \( e \). Construction 54 yields a split of \( C \) into \( C_1 \) and \( C_2 \), where the vertices adjacent to the split edge \( e \) are denoted by \( v_i \in C_i \) for \( i = 1, 2 \). Figure 11 shows \( C_1 \) and \( C_2 \), where we shifted \( C_2 \) in order to get a better picture (in fact \( v_1 \) and \( v_2 \) are the same point in \( \mathbb{R}^2 \)). Observe that the cycle \( Y_1 \) associated to \( C_1 \) is 0-dimensional while the cycle \( Y_2 \) associated to \( C_2 \) is 2-dimensional.

![Figure 11: The curve \( C_1 \) satisfying \( p_1, L_{[3]}, \lambda_{[2]} \) is shown on the left, the curve \( C_2 \) satisfying \( p_2, L_4 \) is shown on the right. Notice that the length of \( e \) in \( C \) is given by \( \lambda'_3 \), i.e. \( C \) is fixed by the given conditions.](image)

**Remark 58.** Fix a degree \( d \), point conditions \( p_{2r} \), multi-line conditions \( L_{\kappa} \) and cross-ratio conditions \( \lambda_{[l-1]} \). Let \((d_1, n_1, \kappa_1, l_1, f_1 | d_2, n_2, \kappa_2, l_2, f_2)\) denote a split of these conditions.
Consider degree $d_i$ tropical stable maps $C_i$ for $i = 1, 2$ with $\#n_i + \#k_i + \#f_i + 1$ contracted ends that satisfy the point conditions $p_{n_i}$, the multi-line conditions $L_{n_i}$, and the cross-ratio conditions $\lambda_i^{\epsilon_i}$. The cycles $Y_i$ for $i = \overline{1,2}$ tell us how to glue the end $e_1$ of $C_1$ to the end $e_2$ of $C_2$ to form a contracted bounded edge $e$ such that the new tropical stable map $C$ satisfies all given conditions.

If $Y_1$ is 0-dimensional and $p_{e_2}$ is a point in $Y_1$, then considering tropical stable maps $C_2$ that satisfy $p_{n_2}, L_{n_2}, \lambda_2^{\epsilon_2}$ and that satisfy $p_{e_2}$ with the end $e_2$ allows us to glue $C_1$ to $C_2$, where the contracted bounded edge is contracted to $p_{e_2} \in \mathbb{R}^2$.

If both $Y_i$ are 1-dimensional, then we can consider tropical stable maps $C_2$ that satisfy $p_{n_2}, L_{n_2}, \lambda_2^{\epsilon_2}$ and $Y_1$. Since $ev_{e_2}(C_2) \in Y_2$, i.e. $C_2$ satisfies $Y_2$ by definition, the position of the contracted end $e_2$ of $C_2$ in $\mathbb{R}^2$ is a point $p$ contributing to the 0-dimensional cycle $Y_1 \cdot Y_2$. On the other hand, there is a tropical stable map $C_1$ that satisfies $p_{n_2}, L_{n_2}, \lambda_1^{\epsilon_1}$ and $Y_2$ such that its end $e_1$ is contracted to $p$. Thus $e_1$ of $C_1$ and $e_2$ of $C_2$ can be glued to form a bounded edge $e$ that is contracted to $p$.

**Corollary 59** (of Proposition 29). *If $C$ is a tropical stable map as in Proposition 29 whose contracted bounded edge is a 1/1 edge, then the 1-dimensional cycles $Y_i$ from Definition 56 have ends of primitive directions $(1,1), (-1,0)$ and $(0,-1) \in \mathbb{R}^2$ only. In other words, the 1-dimensional conditions that a contracted bounded 1/1 edge passes from one vertex to the other has ends of standard directions.*

**Proof.** Proposition 53 implies that each contracted bounded edge that appears in the no-point-conditions case is a 2/0 edge. Hence we may assume that at least one point condition is given.

Let $\Gamma$ be a tropical curve associated to a tropical stable map in $Y_1$ whose movement is unbounded, i.e. that gives rise to an end of $Y_i$. Corollary 50 yields that the movable component of $\Gamma$ consists of exactly one vertex $v_i$ of type (I) or (II). Thus $v_i$ is of type (I) since we assumed that there is at least one point condition. If there is a cross-ratio $\lambda_1 \in \lambda_{[l_{-1}]}$ such that $\lambda_1^{\epsilon_1}$ is satisfied at $v_i$, i.e. $\lambda_1^{\epsilon_1} \in \lambda_{v_i}$, then Corollary 27 guarantees that $v_i$ is not adjacent to unbounded edges. This yields a contradiction when $v_i$ moves unboundedly as the proof of Proposition 29 shows. Hence $v_i$ is a 3-valent type (I) vertex which is adjacent to $e_i$ and an end $E$ of $\Gamma$. Therefore, $v_i$ moves parallel to $E$. \hfill $\square$

**Corollary 60.** *We use Notation from Construction 54, i.e. we denote the vertex adjacent to the end $e_i$ of $C_i$ by $v_i$. Under the same assumptions of Corollary 59, it follows that $v_i$ is 3-valent and adjacent to an end of $C_i$ for $i = 1, 2$.*

**Proof.** This follows immediately from the proof of Corollary 59. \hfill $\square$

### 3 Multiplicities of split curves

This section answers the question of how multiplicities behave under splitting a tropical stable map $C$ into $C_1, C_2$. Note that the multiplicity of $C$ does not have to be equal to $\text{mult}(C_1) \cdot \text{mult}(C_2)$. We have to deal with this problem later.
Definition 61 (Degenerated tropical lines). The following tropical intersections \( L_{10} := \max_{(x,y) \in \mathbb{R}^2}(x,0) \cdot \mathbb{R}^2, \ L_{01} := \max_{(x,y) \in \mathbb{R}^2}(y,0) \cdot \mathbb{R}^2 \) and \( L_{11} := \max_{(x,y) \in \mathbb{R}^2}(x,-y) \cdot \mathbb{R}^2 \) and any translations thereof are called *degenerated tropical lines*.

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

Figure 12: Degenerated tropical lines (from left to right) \( L_{10}, L_{01} \) and \( L_{11} \) in \( \mathbb{R}^2 \) with ends of weight one.

Notation 62 (Replacing 1/1 edge conditions). Let \( C \) be a tropical stable map that contributes to \( N_d(p_2, L_2, \lambda_{[t-1]}, \lambda_t) \) such that \( C \) has a contracted bounded edge \( e \) that is a 1/1 edge. Split \( e \) as in Construction 54 to obtain \( C_1, C_2 \) and let \( Y_t \) denote the 1-dimensional condition \( C_i \) satisfies for \( i \neq t \) as in Definition 56. Let \( v_i \) be the vertex of \( C_i \) that is adjacent to \( e_i \) (\( e_i \) is the contracted end of \( C_i \) that came from cutting \( e \) ) which satisfies \( Y_t \). Let \( st \in \{01, 10, 1-1\} \) and let \( L_{st} \) be a degenerated line as in Definition 61 such that its vertex is translated to \( v_i \). Let \( C_{i, st} \) denote the tropical curve that equals \( C_i \), but where we replaced the \( Y_t \) conditions with \( L_{st} \), i.e. \( C_{i, st} \) satisfies \( L_{st} \) instead of \( Y_t \).

Notice that only the multiplicities of \( C_i \) and \( C_{i, st} \) may differ. In particular, the multiplicity of \( C_{i, st} \) may be zero, whereas the multiplicity of \( C_i \) can be nonzero.

Example 63. Let \( C \) be a degree 3 tropical stable map that satisfies point conditions \( p_{[3]} \), multi-line conditions \( L_{[3]} \), degenerated cross-ratios \( \lambda_1 = \{p_1, p_2, p_5, L_1\} \), \( \lambda_2 = \{p_1, p_5, L_2, L_3\} \) and a non-degenerated cross-ratio \( \lambda'_2 = (p_1 p_2 L_2 L_3) \) whose length is large enough such that \( C \) has a contracted bounded edge \( e \). Construction 54 yields a split of \( C \) into \( C_1 \) and \( C_2 \), where the vertices adjacent to the split edge \( e \) are denoted by \( v_i \in C_i \) for \( i = 1, 2 \). Figure 13 shows \( C_1 \) and \( C_2 \), where we shifted \( C_2 \) in order to get a better picture (in fact \( v_1 \) and \( v_2 \) are the same point in \( \mathbb{R}^2 \) as in Example 57).

Notice that \( e \) is a 1/1 edge, so we use Notation 62 to replace conditions. For example, \( C_{2,10} \) equals \( C_2 \), where the end \( e_2 \) adjacent to \( v_2 \) satisfies the degenerated line condition \( L_{10} \). Figure 13 shows that \( C_{2,10} \) is not fixed by its conditions, i.e. \( \text{mult}(C_{2,10}) = 0 \). If we consider \( C_{2,01} \) instead, its multiplicity is 1 since it is the absolute value of the determinant the following matrix \( M(C_{2,01}) \) (see Definition 25)

\[
\begin{pmatrix}
\text{Base } p_5 & l_1 & l_2 & l_3 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

where \( p_5 \) is chosen as base point and the third row is associated to \( L_{01} \) satisfied by \( e_2 \).
Figure 13: The curve $C_1$ satisfying $p_{[4]}, L_1, \lambda_1$ is shown on the left, the curve $C_2$ satisfying $p_5, L_2, L_3, \lambda_2$ is shown on the right. Notice that the length of $e$ in $C$ is given by $\lambda_3'$, i.e. $C$ is fixed by the given conditions.

**Proposition 64.** Let $C$ be a tropical stable map contributing to $N_d(p_2, L_2, \lambda_{[l-1]}, \lambda'_l)$ such that $C$ has a contracted bounded edge $e$. The components arising from cutting $e$ as in Construction 54 are denoted by $C_1, C_2$.

(a) If $e$ is a 2/0 edge, then

$$\mult(C) = \mult(C_1) \cdot \mult(C_2).$$

(b) If $e$ is a 1/1 edge, then

$$\mult(C) = \frac{\det(M(C_{1,10})) \cdot \det(M(C_{2,01})) - \det(M(C_{1,01})) \cdot \det(M(C_{2,10}))}{\det(M(C_{1,01}))},$$

where $C_{i,st}$ is defined in Notation 62. In particular, if one of the determinants $\det(M(C_{1,01}))$ or $\det(M(C_{2,10}))$ vanishes, then

$$\mult(C) = \mult(C_{1,10}) \cdot \mult(C_{2,01}).$$

**Proof.** It is sufficient to prove (a), (b) for ev-multiplicities only since the cross-ratio multiplicities can be expressed locally at vertices (see Proposition 26). Thus contributions from vertices to cross-ratio multiplicities do not depend on cutting edges.

(a) Denote the vertices adjacent to $e$ by $v_1, v_2$ such that $v_1 \in C_1$ and $v_2 \in C_2$ and assume without loss of generality that $Y_1$ (notation from Definition 56) is 0-dimensional.

Consider the ev-matrix $M(C)$ of $C$ of Definition 25 with base point $v_1$, i.e.

$$M(C) = \begin{pmatrix} \text{Base } v_1 & \text{lengths in } C_1 & \text{lengths in } C_2 \\ \text{conditions in } C_1 & * & * \\ \text{conditions in } C_2 & * & 0 \end{pmatrix}$$
Let \( y_1 \) be the number of rows that belong to the conditions in \( C_1 \), let \( x_1 \) be the number of columns belonging to the base point and the lengths in \( C_1 \). Using notation from Definition 56, we obtain

\[
x_1 = 2 + 3d_1 - 3 + \#n_1 + \#\kappa_1 - \#l_1 + \#f_1 + 1,
\]

\[
y_1 = 2 \cdot \#n_1 + \#\kappa_1.
\]

On the other hand, \( C_1 \) is fixed by its set of conditions since \( Y_1 \) is 0-dimensional, i.e. we can apply (1) for \( m = \#n_1 + \#\kappa_1 + (\#f_1 + 1) \) to obtain \( x_1 = y_1 \). Thus the bold red lines in \( M(C) \) above divide \( M(C) \) into squares, hence

\[
|\det(M(C))| = \text{mult}(C_1) \cdot |\det(M)|,
\]

where \( M \) is the square matrix on the bottom right. We define the matrix

\[
M(C_{2,v_2}) := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
* & M
\end{pmatrix}
\]

where the first two columns are chosen in such a way that \( M(C_{2,v_2}) \) is the \( \text{ev} \)-matrix of \( C_2 \) with respect to the base point \( v_2 \). Notice that

\[
|\det(M)| = |\det(M(C_{2,v_2}))|
\]

and

\[
|\det(M(C_{2,v_2}))| = \text{mult}(C_2)
\]

hold, where \( C_2 \) satisfies the additional point condition imposed on \( e_2 \) by \( Y_1 \).

(b) We assume that the weights of each multi-line \( \omega(L_k) \) (see Definition 9) for \( k \in \kappa \) equals 1 since we can pull out the factor \( \omega(L_k) \) from each row of the \( \text{ev} \)-matrix, apply all the following arguments and multiply with \( \omega(L_k) \) later.

Denote the vertex of \( C_1 \) adjacent to the cut edge \( e \) by \( v_1 \) and the other vertex adjacent to \( e \) by \( v_2 \). The \( \text{ev} \)-matrix \( M(C) \) of \( C \) with respect to the base point \( v_1 \) is given by

\[
M(C) = \begin{pmatrix}
* & * & * & \cdots & 0 \\
* & 0 & \cdots & * \\
* & \cdots & * & 0
\end{pmatrix}
\]
The bold red lines divide $M(C)$ into square pieces at the upper left and the lower right. This follows from similar arguments used in the proof of part (a). Let $M$ be the matrix consisting of the lower right block of $M(C)$ whose entries (see above) are indicated by $*$ and its columns are associated to lengths in $C_2$. Let $A = (a_{ij})_{ij}$ be the submatrix of $M(C)$ given by the rows that belong to conditions of $C_1$ and by the base point’s columns and the columns that are associated to lengths in $C_1$, i.e. $A$ consists of all the $*$-entries above the bold red line in $M(C)$.

Consider the Laplace expansion of the rightmost column of $A$. Recursively, use Laplace expansion on every column that belongs to the lengths in $C_1$ starting with the rightmost column. Eventually, we end up with a sum in which each summand contains a factor $\det(M_{s,t})$ for a matrix $M_{s,t}$, which is one of the following three matrices, namely

$$M_{a_1,a_2} := \begin{pmatrix} a_{r_1} & a_{r_2} & 0 & \ldots & 0 \\ * & M \\ \end{pmatrix},$$

where $(a_{r_1}, a_{r_2}) = (1,0)$, $(a_{r_1}, a_{r_2}) = (0,1)$ or $(a_{r_1}, a_{r_2}) = (1,-1)$ are the remaining entries of $A$ in its $r$-th row after the recursive procedure. Notice that in each of the three cases the entries of the first two columns are of such a form that the matrix consisting of the lower right block of $M$ right. This follows from similar arguments used in the proof of part (a). Let $\sigma$ be the submatrix of $M$ consisting of the lower right block of $M$.

$$|\det(M(C))| = |F_{10} \cdot \det(M_{10}) + F_{01} \cdot \det(M_{01}) + F_{11} \cdot \det(M_{11})|,$$

where $F_{st} \in \mathbb{R}$ for $st = 10,01,1-1$ are factors occurring due to the recursive Laplace expansion. More precisely, let $b$ be the number of bounded edges in $C_1$, i.e. the number of Laplace expansions we applied. Then

$$F_{st} = \sum_{r(a_{r_1},a_{r_2})=(s,t)} \sum_{\sigma} \sgn(\sigma) \prod_{j=3}^{3+b} a_{\sigma(j)j},$$

where the second sum goes over all bijections $\sigma : \{3, \ldots, 3+b\} \rightarrow \{1, \ldots, r-1, r+1, \ldots, b+1\}$, i.e. it goes over all possibilities of choosing for each column Laplace expansion was used on an entry in a row of $A$ which is not the $r$-th row.

Let $A_{10}, A_{01}, A_{11}$ be the square matrices obtained from $A$ by adding the new first row $(1,0,0,\ldots,0)$, $(0,1,0,\ldots,0)$ or $(1,-1,0,\ldots,0)$ to $A$. Again, notice that $A_{st}$ for $st = 10,01,1-1$ is the ev-matrix of $C_1$ (see Notation 62, Definition 25) with base point $v_1$. We claim that

$$\det(A_{10}) = F_{01} - F_{11}$$
holds. Let $N$ be the number of columns and rows of $A_{st}$. Denote the entries of the ev-matrix $M(C)$ by $m(C)_{ij}$. Define

$$S_{st} := \{ r \in [N - 1] \mid m(C)_{r1} = s, \ m(C)_{r2} = t \}$$

for $(s, t) = (1, 0), (0, 1), (1, -1)$ and notice that $\#S_{10} + \#S_{01} + \#S_{1-1} = N - 1$. Denote the entries of $A_{10}$ by $a_{ij}^{(10)}$ and apply Leibniz’ determinant formula to obtain

$$\det(A_{10}) = \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(j)j}^{(10)}$$

$$= \sum_{\sigma(2) \in S_0} \text{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} + \sum_{\sigma(2) \in S_{-1}} \text{sgn}(\sigma) \prod_{j=1}^{N} a_{\sigma(j)j}^{(10)} = F_{01} - F_{1-1},$$

where the second equality holds by definition of $S_{st}$ and the third equality holds by considering how contributions of $F_{01}$ and $F_{1-1}$ arise as choices of entries of $A$, see (5). The minus sign comes from the factor $a_{\sigma(2),2}^{(10)} = -1$ in each product in the last sum. Thus (6) holds.

We can show in a similar way that

$$\det(A_{01}) = -(F_{10} + F_{1-1}) = - F_{10} - F_{1-1}, \quad (7)$$

$$\det(A_{1-1}) = F_{10} + F_{1-1} + F_{01} - F_{1-1} = F_{10} + F_{01} \quad (8)$$

hold. Solving the system of linear equations (6), (7), (8) for $F_{10}, F_{01}, F_{1-1}$ yields

$$\begin{pmatrix} F_{10} \\ F_{01} \\ F_{1-1} \end{pmatrix} \varepsilon \begin{pmatrix} - \det(A_{01}) \\ \det(A_{10}) \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad (9)$$

where the 1-dimensional part appears because of the relation

$$- \det(M_{10}) + \det(M_{01}) + \det(M_{1-1}) = 0.$$}

Combining (4) with (9) proves part (b), where $A_{st} = C_{1,st}$ and $M_{st} = C_{2,st}$. In particular,

$$\text{mult}(C) = | \det(M(C_{1,10})) \cdot \det(M(C_{2,01})) - \det(M(C_{1,01})) \cdot \det(M(C_{2,10})) |$$

$$= | \det(M(C_{1,10})) \cdot \det(M(C_{2,01})) |$$

$$= | \det(M(C_{1,10})) | \cdot | \det(M(C_{2,01})) |$$

$$= \text{mult}(C_{1,10}) \cdot \text{mult}(C_{2,01})$$

holds if $\det(M(C_{1,01}))$ or $\det(M(C_{2,10}))$ vanishes. \qed
4 General Kontsevich’s formula

In this section, we prove a general tropical Kontsevich’s formula. For that, we must first deal with the behavior of the multiplicity of tropical stable maps under a split. More precisely, we would like to see that one summand in part (b) of Proposition 64 always vanishes.

**Definition 65.** Given a split \((d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)\) and a cross-ratio \(\lambda'_i = (\beta_1 \beta_2 | \beta_3 \beta_4)\) with entries in \(n_1 \cup \kappa_1 \cup f_1 \cup n_2 \cup \kappa_2 \cup f_2\) and \(\beta_1 = \min_{l=1}^4(\beta_i)\) (the labels of ends of abstract tropical curves are natural numbers), we say that \((d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)\) is a split respecting \(\lambda'_i\) if \(\beta_1, \beta_2 \in n_1 \cup \kappa_1 \cup f_1\) and \(\beta_3, \beta_4 \in n_2 \cup \kappa_2 \cup f_2\). Using the minimum here prevents a factor of \(\frac{1}{2}\) later, which would come from renaming \(C_1\) to \(C_2\) and vice versa.

**Lemma 66.** Let \((d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)\) be a 2/0 split of given conditions in general position as in Remark 58 and Definition 55 that respects \(\lambda'_i\) such that additionally \(3d_1 = |n_1| + |l_1| - |f_1|\) holds. Then

\[
\sum_{C: (d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)} \text{mult}(C) = N_{d_1}\left(p_{n_1}, L_{\kappa_1}, \lambda_{l_1}^{-e_1}\right) \cdot N_{d_2}\left(p_{n_2}, p_{e_2}, L_{\kappa_2}, \lambda_{l_2}^{-e_2}\right)
\]

holds, where the sum goes over all tropical stable maps \(C\) with a contracted bounded edge such that \(C\) contributes to \(N_{d_1}\left(p_{n_1}, L_{\kappa_1}, \lambda_{l_1}^{-e_1}\right)\), where \(\lambda'_i\) is the large non-degenerated cross-ratio \(C\) satisfies such that \(C\) has a contracted bounded edge, and \(C\) is of splitting type \((d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)\), and \(p_{e_2}\) is a point condition imposed on \(e_2\).

**Proof.** Each tropical stable map \(C\) on the left-hand side of (10) can be cut at its contracted bounded edge as in Construction 54 to obtain a tropical stable map \(C_1\) that contributes to the number \(N_{d_1}\left(p_{n_1}, L_{\kappa_1}, \lambda_{l_1}^{-e_1}\right)\) and a tropical stable map \(C_2\) that contributes to the number \(N_{d_2}\left(p_{n_2}, p_{e_2}, L_{\kappa_2}, \lambda_{l_2}^{-e_2}\right)\).

The other way around, each pair of tropical stable maps \(C_1, C_2\) such that \(C_1\) contributes to \(N_{d_1}\left(p_{n_1}, L_{\kappa_1}, \lambda_{l_1}^{-e_1}\right)\) and \(C_2\) contributes to \(N_{d_2}\left(p_{n_2}, p_{e_2}, L_{\kappa_2}, \lambda_{l_2}^{-e_2}\right)\) can be glued to a tropical stable map \(C\) using Remark 58.

Proposition 64 states that

\[
\text{mult}(C) = \text{mult}(C_1) \cdot \text{mult}(C_2)
\]

and thus proves the lemma.

**Lemma 67.** Let \((d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)\) be a 1/1 split of given conditions in general position as in Remark 58 and Definition 55 that respects \(\lambda'_i\). Then

\[
\sum_{C: (d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)} \text{mult}(C) = N_{d_1}\left(p_{n_1}, L_{\kappa_1}, \lambda_{l_1}^{-e_1}\right) \cdot N_{d_2}\left(p_{n_2}, L_{\kappa_2}, \lambda_{l_2}^{-e_2}\right)
\]
holds, where the sum goes over all tropical stable maps $C$ with a contracted bounded edge $e$ such that $C$ contributes to $N_d \left( p_{n_1}, L_e, \lambda_{[l-1]}^i, \lambda_l^i \right)$, where $\lambda_l^i$ is the large non-degenerated cross-ratio $C$ satisfies such that $C$ has a contracted bounded edge, and $C$ is of splitting type $(d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)$, and $L_{e_i}$ for $i = 1, 2$ is a tropical multi-line condition with ends of weight one that is imposed on $e_i$.

Proof. The ends of $Y_1$ and $Y_2$ (see Definition 56) are of standard directions, i.e. of direction $(1, 1), (-1, 0)$ and $(0, -1)$ by Corollary 59. The position of $Y_1$ and $Y_2$ in $\mathbb{R}^2$ depends only on the position of the given conditions. In particular, moving the given conditions (while keeping the property of being in general position) moves $Y_1$ and $Y_2$ as well.

Assume that the given conditions are positioned in such a way that $Y_1$ and $Y_2$ intersect only in their ends as shown in Figure 14. Choose the multi-line conditions $L_{e_1}$ and $L_{e_2}$ with weights one as in Figure 14 and consider a tropical stable map $C_1$ that contributes to $N_{d_1} \left( p_{n_1}, L_{e_1}, L_{e_1}, \lambda_{e_1}^{-1} \right)$ and a tropical stable map $C_2$ that contributes to $N_{d_2} \left( p_{n_2}, L_{e_2}, L_{e_2}, \lambda_{e_2}^{-1} \right)$. The contracted end of $C_i$ for $i = 1, 2$ that satisfies $L_{e_i}$ is $e_i$. Let $v_i$ denote the vertex adjacent to $e_i$ for $i = 1, 2$. Notice that $e_2$ is uniquely associated to a point $p$ in $Y_1 \cdot Y_2$, see Figure 14. By Corollary 59, each of the vertices $v_i$ is 3-valent and adjacent to an end of $C_i$ for $i = 1, 2$. Hence (by moving $v_1, v_2$ along those ends) each pair of tropical stable maps $(C_1, C_2)$ as above can be glued to a tropical stable map $C$ as in Remark 58 such that the ends $e_1, e_2$ are glued to form a bounded edge that is contracted to $p$. On the other hand each tropical stable map $C$ on the left hand side of (11) can be split into a pair $(C_1, C_2)$ of tropical stable maps as above using Construction 54.

Moreover,

$$\text{mult}(C) = \text{mult}(C_1) \cdot \text{mult}(C_2)$$

holds by Proposition 64 since $\det (M(C_{1,0}))$ and $\det (M(C_{2,1}))$ both vanish by our choice of positions of $Y_1$ and $Y_2$. Therefore (11) follows.

To finish the proof, we need to see that we can always assume that $Y_1$ and $Y_2$ intersect as shown in Figure 14, i.e. we want to show that the left hand side of (11) does not depend on the position of $Y_1$ and $Y_2$. Let $C$ be a tropical stable map contributing to $N_{d_1+d_2} \left( p_{n_1}, p_{n_2}, L_{e_1}, L_{e_2}, \lambda_{l_1}, \lambda_{l_2}, \lambda_l^i \right)$ as in Proposition 29. Notice that $n \geq 1$ since we have a $1 \parallel 1$ edge by Proposition 53. The cross-ratio's length $|\lambda_l^i|$ is so large such that there is a contracted bounded edge $e$ in $C$, and $C$ is of splitting type $(d_1, n_1, \kappa_1, l_1, f_1 \mid d_2, n_2, \kappa_2, l_2, f_2)$. Consider the cycle $Z_i$ that arises from forgetting the point conditions $p_{n_1}$ and the multi-line conditions $L_{e_i}$ for $i = 1, 2$ imposed on $C$. Hence $C$ gives rise to a top-dimensional cell of $Z_i$, where points in that cell correspond to $C$ together with some movement of the conditions $p_{n_i}, L_{e_i}$. The proof of Proposition 29 implies that if $|\lambda_l^i|$ is large enough, then the given conditions can be moved in a bounded area $B$ (say $B \subset \mathbb{R}^2$ is a rectangular box) and all tropical stable maps that satisfy this moved conditions still have a contracted bounded edge. Moreover, the splitting type of those tropical stable maps cannot change since that would require two contracted bounded edges which would contradict that our given conditions are in general position. Since $Z_1, Z_2$ are balanced, we
might choose different positions for our point and multi-line conditions for every splitting type without effecting the overall count. Let \( B_1, B_2 \subset B \) be disjoint small rectangular boxes such that \( B_1 \) lies in the lower right corner of \( B \) and \( B_2 \) lies in the upper left corner of \( B \). Move the conditions \( p_{n_1}, L_{\kappa_1}, \lambda_{l_1} \) into \( B_1 \) and the conditions \( p_{n_2}, L_{\kappa_2}, \lambda_{l_2} \) into \( B_2 \) while maintaining their property of being in general position. By choosing \( B_1 \) and \( B_2 \) small enough, we can bring \( Y_1 \) and \( Y_2 \) in the desired position from Figure 14.

**Theorem 68** (General tropical Kontsevich’s formula). We use notation from Notation 2, Definition 56, 65 and Remark 58. Fix a degree \( d \), point conditions \( p_{n_2} \), multi-line conditions \( L_e \) and degenerated cross-ratios \( \lambda_{l[i]} \) such that these conditions are in general position. Let \( \lambda'_l \) denote a cross-ratio that degenerates to \( \lambda_l \).

\[(a) \quad \text{If there is at least one point condition, i.e. } p_{n_2} \neq \emptyset, \text{ then the equation}\]

\[
N_d \left( p_{n_2}, L_{e_2}, \lambda_{l[i]} \right) = \sum_{(d_1, n_2, \kappa_1, l_1) \mid d_2, n_2, \kappa_2, l_2, f_2} \text{ is a 1/1 split respecting } \lambda'_l \ N_{d_1} \left( p_{n_1}, L_{\kappa_1}, L_{e_1}, \lambda_{l_1}^{-e_1} \right) \cdot N_{d_2} \left( p_{n_2}, L_{\kappa_2}, L_{e_2}, \lambda_{l_2}^{-e_2} \right)
\]

\[+ \sum_{(d_1, n_2, \kappa_1, l_1) \mid d_2, n_2, \kappa_2, l_2, f_2} \text{ is a 2/0 split respecting } \lambda'_l \text{ and } 3d_1 = |n_1| + |l_1| - |f_1| \]

\[N_{d_1} \left( p_{n_1}, L_{\kappa_1, l_1}, \lambda_{l_1}^{-e_1} \right) \cdot N_{d_2} \left( p_{n_2}, p_{e_1}, L_{\kappa_2, l_2}, \lambda_{l_2}^{-e_2} \right) \tag{12}
\]

\[+ \sum_{(d_1, n_2, \kappa_1, l_1) \mid d_2, n_2, \kappa_2, l_2, f_2} \text{ is a 2/0 split respecting } \lambda'_l \text{ and } 3d_2 = |n_2| + |l_2| - |f_2| \]

\[N_{d_1} \left( p_{n_1}, p_{e_1}, L_{\kappa_1, l_1}, \lambda_{l_1}^{-e_1} \right) \cdot N_{d_2} \left( p_{n_2}, L_{\kappa_2, l_2}, \lambda_{l_2}^{-e_2} \right)
\]

holds.
(b) If there are no point conditions, i.e. \( p_n = \emptyset \), then the equation

\[
N_d(L, \lambda_{\{1\}}) = \sum_{(l_1, l_2, l_3) \text{ is a 2/0 split respecting } \lambda'_1} N_0\left( L_a, L_b, \lambda_{\{1\}}^{-c} \right) \cdot N_d\left( p, L \setminus \{ L_a, L_b \}, \lambda_{\{1\}}^{-c} \right)
\]

(13)

holds, where the multi-line conditions \( L_a, L_b \) are the ones of Lemma 51.

Moreover, (12) and (13) give rise to a recursion with two types of initial values:

1. The numbers \( N_d\left( p_n \right) \) which tropical Kontsevich’s formula (Corollary 71) provides.

2. The numbers \( N_0\left( L_a, L_b, \lambda_{\{1\}}^{-c} \right) \) which satisfy

\[
N_0\left( L_a, L_b, \lambda_{\{1\}}^{-c} \right) = \omega(L_a) \cdot \omega(L_b) \cdot \text{mult}_{cr}(v'),
\]

(14)

where \( v' \) denotes the only vertex of the only tropical stable map contributing to \( N_0\left( L_a, L_b, \lambda_{1}^{-c} \right) \) and \( \text{mult}_{cr}(v') \) is its cross-ratio multiplicity, see Definition 22. Notice that in the special case of \( \lambda_{\{1\}}^{-c} = \emptyset \) we have

\[
N_0\left( L_a, L_b \right) = \omega(L_a) \cdot \omega(L_b).
\]

(15)

Using Tyomkin’s correspondence theorem 28 and Remark 19, Theorem 68 immediately yields the following corollary.

**Corollary 69 (General Kontsevich’s formula).** Let \( N_d^{\text{class}}(p_n, \mu_{\{1\}}) \) denote the number of plane rational degree \( d \) curves that satisfy point conditions and classical cross-ratios \( \mu_1, \ldots, \mu_l \) as in Theorem 28 such that all conditions are in general position. Then Theorem 68 provides a recursive formula to calculate these numbers with initial values as in Theorem 68.

**Example 70.** We want to give an example of how to compute numbers we are looking for using our general tropical Kontsevich’s formula. Say we want to compute the number \( N_1\left( p_{\{3\}}, L_4, L_5, \lambda_{\{2\}} \right) \). For degenerated tropical cross-ratios

\[
\lambda_1 := \{1, 2, 3, 4\} \quad \text{and} \quad \lambda_2 := \{1, 2, 3, 5\}.
\]

Notice that (1) is satisfied so your input data makes sense. Recall the conventions we used for labeling ends: in this example, we want to count rational tropical stable maps \( C \) of degree 2 in \( \mathbb{R}^2 \) that have 5 contracted ends. A contracted end labeled with \( i \) satisfies the point condition \( p_i \) for \( i = 1, 2, 3 \) and satisfies the multi-line condition \( L_i \) for \( i = 4, 5 \). There is no non-contracted end which satisfies no condition. To use Theorem 68, we need to fix a tropical cross-ratio \( \lambda'_2 \) that degenerates to \( \lambda_2 \). We choose

\[
\lambda'_2 := (1235).
\]
If \( C \) splits into \( C_1, C_2 \), then by Definition 65 ends 1, 2 are contracted ends of \( C_1 \), i.e. \( p_1, p_2 \) are satisfied in \( C_1 \), and 3, 5 are contracted ends of \( C_2 \), i.e. \( p_3, L_5 \) are satisfied in \( C_2 \). Therefore \( \lambda_1 \) is satisfied in \( C_1 \) such that 4 is a contracted end of \( C_1 \) that satisfies \( L_4 \). Going through the three cases of different types of splits using (2) and (3), we see that the only possible splits are the 2/0 splits

\[
(1, p_1, p_2, L_4, \lambda_1 \mid 1, p_3, L_5).
\]

Since they only differ in the distribution of the labels of non-contracted ends, there are \( (\binom{2}{1})^3 = 8 \) of them (for notation, see Construction 54). Hence part (a) of Theorem 68 yields

\[
N_2 \left(p_{[3]}, L_4, L_5, \lambda_{[2]} \right) = 8 \cdot N_1 \left(p_1, p_2, L_4, \lambda_1^{e_1} \right) \cdot N_1 \left(p_3, p_{e_2}, L_5 \right),
\]

where the rightmost factor can be written as

\[
N_1 \left(p_3, p_{e_2}, L_5 \right) = \omega(L_5) \cdot N_1 \left(p_3, p_{e_2} \right)_{=1}
\]

by tropical Bézout’s Theorem [AR10].

So it remains to calculate \( N_1 \left(p_1, p_2, L_4, \lambda_1^{e_1} \right) \). For that, we want to use Theorem 68 again. A rational tropical stable map \( C \) contributing to \( N_1 \left(p_1, p_2, L_4, \lambda_1^{e_1} \right) \) has 4 contracted ends. A contracted end labeled with \( i \) satisfies \( p_i \) for \( i = 1, 2 \) and \( L_i \) for \( i = 4 \). The remaining contracted end is labeled with \( e_1 \) and satisfies no point condition. To stick to our convention of labeling ends with natural numbers, we relabel \( e_1 \) by 6. Again, fix a tropical cross-ratio \( \lambda_1^{e_1} \) that degenerates to \( \lambda_1^{e_1} = \{1, 2, 6, 4\} \). We choose

\[
\lambda_1^{e_1} := (12|46).
\]

If \( C \) splits into \( C_1, C_2 \) then 1, 2 are contracted ends of \( C_1 \), i.e. \( p_1, p_2 \) are satisfied in \( C_1 \), and 4, 6 are contracted ends of \( C_2 \), i.e. \( L_4 \) is satisfied by \( C_2 \) and there is one contracted end, labeled 6, in \( C_2 \) that satisfies no condition. As before, we can go through all splits and notice that

\[
(\Delta_1^2, p_1, p_2 \mid \Delta_0^2, L_4, 6)
\]

is the only possible split. Hence part (a) of Theorem 68 yields

\[
N_1 \left(p_1, p_2, L_4, \lambda_1^{e_1} \right) = N_1 \left(p_1, p_2, L_{e_1} \right) \cdot N_0 \left(L_4, L_{e_2} \right),
\]

where

\[
N_1 \left(p_1, p_2, L_{e_1} \right) = \omega(L_{e_1}) \cdot N_1 \left(p_1, p_2 \right)_{=1}
\]

holds by tropical Bézout’s Theorem and by definition of \( L_{e_1} \). Moreover,

\[
N_0 \left(L_4, L_{e_2} \right) = \omega(L_4)
\]
by Theorem 68.
In total, we calculated
\[ N_2(p_{[3]}, L_4, L_5, \lambda_{[2]}) = 8 \cdot \omega(L_4) \cdot \omega(L_5) \]
for \( \lambda_1, \lambda_2 \) defined as above.

We now prove Theorem 68, discuss the initial values of the recursion Theorem 68 provides and then proceed with tropical Kontsevich’s formula which is a corollary of part (a) of Theorem 68.

Proof of part (a) of Theorem 68. Using Remark 19, we obtain
\[ N_d(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l]}) = N_d(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l) \]
for a cross-ratio \( \lambda'_l \) that degenerates to \( \lambda_l \). Since \( N_d(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l) \) does not depend on \( |\lambda'_l| \), choose it to be large as in Proposition 29. Hence each stable map contributing to \( N_d(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l) \) has a contracted bounded edge \( e \) which can be cut as using Construction 54 and thus gives rise to some splitting type \((d_1, n_1, \kappa_1, l_1, f_1 | d_2, n_2, \kappa_2, l_2, f_2)\) that respects \( \lambda'_l \). Therefore
\[ N_d(p_{\underline{n}}, L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l) = \sum_{(d_1, n_1, \kappa_1, l_1, f_1 | d_2, n_2, \kappa_2, l_2, f_2)} \sum_{C \text{ is a split respecting } \lambda'_l} \text{mult}(C), \quad (16) \]
where the second sum goes over all stable maps \( C \) that give rise to the split \((d_1, n_1, \kappa_1, l_1, f_1 | d_2, n_2, \kappa_2, l_2, f_2)\).
Reordering the first sum of (16) as in (12) and applying Lemmas 66, 67 proves part (a) of Theorem 68.

Proof of part (b) of Theorem 68. We use notation from Lemma 51, 52 and Proposition 53. We use Remark 19, i.e.
\[ N_d(L_{\underline{\kappa}}, \lambda_{[l]}) = N_d(L_{\underline{\kappa}}, \lambda_{[l-1]}, \lambda'_l), \quad (17) \]
and conclude with Proposition 53 that each stable map contributing to the right hand side of (17) has a contracted bounded edge \( e \) which is adjacent to a vertex \( v' \) which is in turn adjacent to \( e_1, e_2 \). Notice that cutting \( e \) yields a 2/0 split. Thus Lemma 66 gives us equation (14).

Proof of the initial values part of Theorem 68. Notice that equations (12), (13) of Theorem 68 allow us to successively reduce the number of point, multi-line or cross-ratio conditions. There are three cases:
(1) We run out of cross-ratio conditions. Then, if there are point conditions left, tropical Bézout’s Theorem [AR10] can be applied to reduce the initial value problem to the numbers \( N_d(p_n) \) which tropical Kontsevich’s formula (Corollary 71) provides. If there are no point conditions left, then

\[ N_d(L_n) = 0 \]

for all \( d \neq 0 \) applies. Otherwise \( d = 0, \#\mathbb{K} = \#\{a, b\} = 2 \) and \( \#f = 1 \) such that

\[ N_0(L_\kappa) = \omega(L_a) \cdot \omega(L_b) \]

holds.

(2) We run out of point conditions. Then (13) reduces the initial value problem to calculating \( N_0(L_a, L_b, \lambda^{-e}_{12}) \). This can be done via (14).

For equation (14), notice that each edge of a tropical stable map of degree 0 must be contracted. Thus there cannot be a bounded edge since all cross-ratios are degenerated. Hence there is exactly one vertex \( v' \) in such a stable map whose position is determined by the unique point of intersection of \( L_a \) and \( L_b \). Therefore there is exactly one stable map contributing to \( N_0(L_a, L_b, \lambda^{-e}_{12}) \) whose multiplicity is \( \omega(L_a) \cdot \omega(L_b) \cdot \text{mult}_{cr}(v') \) by Proposition 26.

(3) We run out of multi-line conditions. Then (12) can still be applied, so cases (1) and (2) apply. \( \square \)

We now prove tropical Kontsevich’s formula using our general tropical Kontsevich’s formula (Theorem 68). Hence Kontsevich’s formula is indeed a special case of Theorem 68. Notice that the proof of Kontsevich’s formula that we present here is similar to the one given by Gathmann and Markwig in [GM08]. The main difference is that our proof uses the more general language of 1/slash.1 and 2/slash.0 splits.

**Corollary 71** (Tropical Kontsevich’s formula, [GM08]). For \( \#\mathbb{K} = 3d - 1 > 0 \) point conditions in general position the equality

\[ \frac{N_d(p_n)}{(d!)^3} = \sum_{d_1 + d_2 = d} \left( d_1^2 d_2^2 \cdot \left( \frac{3d - 4}{3d_1 - 2} \right) - d_1^3 d_2 \cdot \left( \frac{3d - 4}{3d_2 - 1} \right) \right) \cdot \frac{N_{d_1}(p_{n_1})}{(d_1!)^3} \cdot \frac{N_{d_2}(p_{n_2})}{(d_2!)^3} \]

holds and provides a recursion to calculate \( N_d(p_n) \) from the initial value \( N_1(p_1, p_2) = 1 \).

**Remark 72.** The factors \( \frac{1}{(d!)^3}, \frac{1}{(d_1!)^3}, \frac{1}{(d_2!)^3} \) in Corollary 71 appear since ends of our rational tropical stable maps are labeled. For each rational tropical stable map to \( \mathbb{R}^2 \) of degree \( d \) there are \( (d!)^3 \) ways to label its non-contracted ends. Often tropical Kontsevich’s formula is stated for rational tropical stable maps with unlabeled ends such that the factors \( \frac{1}{(d!)^3}, \frac{1}{(d_1!)^3}, \frac{1}{(d_2!)^3} \) disappear.
Proof of Corollary 71. Let \( p_a \) be point conditions, let \( L_a, L_b \) be multi-line conditions with weights \( \omega(L_a) = \omega(L_b) = 1 \) and let \( \lambda = \{ L_a, L_b, p_c, p_d \} \) be a degenerated cross-ratio, where \( p_c, p_d \in p_a \) are points and the labels are chosen in such a way that \( a < b < c < d \).

Consider the cross-ratio \( \lambda' := (L_a p_c|L_b p_d) \) that degenerates to \( \lambda \). We claim that (12) reduces to

\[
N_d(p_a, L_a, L_b, \lambda) = \sum_{(d_1, n_1 | d_2, n_2) \text{ is a } 1/1 \text{ split respecting } \lambda'} N_{d_1}(p_{n_1}, L_a, L_{e_1}) \cdot N_{d_2}(p_{n_2}, L_b, L_{e_2})
\]

in our case. Since we only have two multi-line conditions and no contracted ends without point or multi-line conditions, each split we deal with can be written as \( (d_1, n_1 | d_2, n_2) \) since \( \lambda' \) determines the distribution of \( L_a \) and \( L_b \) in each possible split respecting \( \lambda' \). To show the claim, it remains to show that the last two sums of (12) vanish. For that it is, because of symmetry, sufficient to show that the second sum vanishes. Let \( N_{d_1}(p_{n_1}, L_a) \cdot N_{d_2}(p_{n_2}, p_{e_2}, L_b) \) be a factor of the second sum. Let \( C_1 \) be a tropical stable map contributing to \( N_{d_1}(p_{n_1}, L_a) \) and let \( C_2 \) be a tropical stable map contributing to \( N_{d_2}(p_{n_2}, p_{e_2}, L_b) \). Using Remark 58, \( C_1 \) and \( C_2 \) can be glued to form a tropical stable map \( C \) which has a contracted bounded edge \( e \). Since our split was a 2/0 split, the 3-valent vertex \( v_1 \) of \( C \) that is adjacent to \( e \) is fixed. Hence there is a contracted end satisfying a point condition that is adjacent to \( v_1 \). Thus there is another contracted end adjacent to \( v_1 \) which needs to satisfy either a point or a multi-line condition which is a contradiction since all conditions are in general position.

Now consider the cross-ratio \( \lambda' := (L_a L_b|p_c p_d) \) that degenerates to \( \lambda \). We claim that (12) reduces to

\[
N_d(p_a, L_a, L_b, \lambda) = \sum_{(d_1, n_1 | d_2, n_2) \text{ is a } 1/1 \text{ split respecting } \lambda'} N_{d_1}(p_{n_1}, L_a, L_{b}, L_{e_1}) \cdot N_{d_2}(p_{n_2}, L_{b}, L_{e_2}) + N_0(L_a, L_b) \cdot N_d(p_{n_2}, p_{e_2})
\]

in this case. As before, splits can be written as \( (d_1, n_1 | d_2, n_2) \). The last sum of (12) vanishes with the same arguments from before. It remains to see that the second sum of (12) equals the product \( N_0(L_a, L_b) \cdot N_d(p_{n_2}, p_{e_2}) \). If \( d_1 > 0 \) and we consider a product contributing to the last sum, then the same arguments from before show that this product vanishes. Hence the only remaining contribution from the second sum that is possible is \( N_0(L_a, L_b) \cdot N_d(p_{n_2}, p_{e_2}) \).

Notice that there are no cross-ratios on the right-hand sides of (18) and (19) such that tropical Bézout’s Theorem [AR10] yields

\[
\sum_{(d_1, n_1 | d_2, n_2) \text{ is a } 1/1 \text{ split respecting } \lambda'} d_1^2 N_{d_1}(p_{n_1}) \cdot d_2^2 N_{d_2}(p_{n_2}) = \sum_{(d_1, n_1 | d_2, n_2) \text{ is a } 1/1 \text{ split respecting } \lambda'} d_1^2 N_{d_1}(p_{n_1}) \cdot d_2 N_{d_2}(p_{n_2}) + N_0(L_a, L_b) \cdot N_d(p_{n_2}, p_{e_2})
\]
since $\omega(L_a) = \omega(L_b) = \omega(L_{c_1}) = \omega(L_{c_2}) = 1$. Using $N_0(L_a, L_b) = \omega(L_a)\omega(L_b) = 1$, we obtain

$$N_d(p_n, p_{e_2}) = \sum_{(d_1, n_1, d_2, n_2)} d_1^2 d_2^2 N_{d_1}(p_{n_1}) N_{d_2}(p_{n_2}) - \sum_{(d_1, n_1, d_2, n_2)} d_1^3 d_2 N_{d_1}(p_{n_1}) N_{d_2}(p_{n_2}).$$

is a 1/1 split respecting $\lambda'$

Since all conditions we started with are in general position

$$3d = \#n + 1 + 1$$

holds, i.e. each choice of $n_1, n_2$ in a split for fixed $d_1, d_2$ is a choice of distributing the remaining $3d - 4$ points. There are $\binom{3d-4}{3d_1-2}$ choices if $p_c \in n_1$ and $\binom{3d-4}{3d_1-1}$ choices if $p_c, p_d \in n_2$. Using $3d_i = \#n_i + 1$ provides the index for the sum we are looking for, namely

$$N_d(p_n, p_{e_2}) = \sum_{(\Delta_1 | \Delta_2) \in \frac{p_c}{d_1, d_2 > 0}} \left( d_1^3 d_2^2 \cdot \frac{3d - 4}{3d_1 - 2} - d_1^2 d_2 \cdot \frac{3d - 4}{3d_1 - 1} \right) N_{d_1}(p_{n_1}) N_{d_2}(p_{n_2}),$$

where $\Delta_i$ denotes the labels of non-contracted ends associated to $d_i$ for $i = 1, 2$. For each choice of $d_1, d_2 > 0$ there are

$$\binom{d}{d_1}^3 = \frac{d!}{d_1!(d-d_1)!} = \frac{d!}{d_1! \cdot d_2!}$$

summands $(\Delta_1 | \Delta_2)$ associated to it, which record of how labels of non-contracted ends are split. Hence

$$N_d(p_n, p_{e_2}) = \sum_{d_1 + d_2 = d} \left( d_1^3 d_2^2 \cdot \frac{3d - 4}{3d_1 - 2} - d_1^2 d_2 \cdot \frac{3d - 4}{3d_1 - 1} \right) \cdot \frac{d!}{d_1! \cdot d_2!} \cdot N_{d_1}(p_{n_1}) N_{d_2}(p_{n_2})$$

holds, which yields the desired formula. □

Further generalizations

The same methods Gathmann and Markwig used to prove tropical Kontsevich’s formula [GM08] also yield a recursive formula for counting rational tropical stable maps of bidegree $(d_1, d_2)$ (i.e. with ends of directions $(\pm 1, 0), (0, \pm 1)$) to $\mathbb{R}^2$ that satisfy point conditions, see [FM11]. Analogously, the methods developed in this paper yield a recursive formula for rational tropical stable maps to $\mathbb{R}^2$ of bidegree $(d_1, d_2)$ that satisfy point conditions, degenerated multi-line conditions and cross-ratio conditions.
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