

The Q_2 -free process in the hypercube

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Abstract

The generation of a random triangle-saturated graph via the triangle-free process has been studied extensively. In this short note our aim is to introduce an analogous process in the hypercube. Specifically, we consider the Q_2 -free process in Q_d and the random subgraph of Q_d it generates. Our main result is that with high probability the graph resulting from this process has at least $cd^{2/3}2^d$ edges. We also discuss a heuristic argument based on the differential equations method which suggests a stronger conjecture, and discuss the issues with making this rigorous. We conclude with some open questions related to this process.

Mathematics Subject Classifications: 05C35, 05C80

1 Introduction

Let F be a (typically small) graph. A graph G on vertex set V is F -saturated if it contains no copy of F as a subgraph but the addition of any new edge in $V^{(2)} \setminus E(G)$ creates a copy of F . The F -free process is a well-known way of generating a random F -saturated graph. We fix a finite set V and form a nested sequence G_0, G_1, \dots, G_M of F -free graphs with common vertex set V . For each i we have $|E(G_i)| = i$ and G_{i+1} is obtained from G_i by randomly adding a new edge chosen uniformly at random from all of the possible edges which do not create a copy of F . The process stops when no new edge can be added; in other words G_M is an F -saturated graph with M edges. We can now ask: what can be said about the properties of the graph G_M , and in particular the random variable M ?

Work in this direction was initiated in 1992 by Ruciński and Wormald [24], who studied the case $F = K_{1,3}$, the star with three leaves, investigating the structure of G_M .

A major breakthrough in the area was the 2009 paper of Bohman [5] on the case $F = K_3$, the so-called *triangle-free process*. Using the differential equations method for random graph processes introduced by Ruciński and Wormald in [24] (see for instance [29] for a survey of the subject), Bohman determined the order of M with high probability.

Theorem 1 (Bohman [5]). *Let G_M be the graph generated by the triangle-free process with $|V| = n$. Then with high probability,*

$$c_1(\log n)^{\frac{1}{2}}n^{\frac{3}{2}} \leq M \leq c_2(\log n)^{\frac{1}{2}}n^{\frac{3}{2}},$$

for some constants c_1 and c_2 .

This result was later refined by Pontiveros, Griffiths and Morris [23] and independently by Bohman and Keevash [6]. Both sets of authors used a substantial extension of the differential equations method to determine M asymptotically, with high probability. They also used their analysis of the triangle-free process to improve the known bounds on the Ramsey number $R(3, t)$.

There is now a large body of work on the F -free process for other graphs. See for instance [7, 21, 25] and the references therein.

Now let H be a (typically large) *host graph*. A graph G is (H, F) -saturated if it is an F -free subgraph of H but the addition of any new edge in $E(H) \setminus E(G)$ creates a copy of F . With this formulation the usual notion of F -saturation corresponds to (K_n, F) -saturation. Our particular interest is the case where the host graph H is the hypercube Q_d . Where Q_d is defined to have vertex set $\{0, 1\}^d$, with two vertices joined by an edge if they differ in exactly one coordinate.

Analogously to the usual triangle-free process, we define the (Q_d, Q_2) -free process as the random nested sequence of subgraphs of Q_d generated by repeatedly adding a new edge chosen uniformly at random from all those edges of Q_d which do not create a copy of Q_2 . We consider Q_2 as our forbidden subgraph since this is the most natural analogue of the triangle-free process (forbidding K_3 in K_n). However, the definition naturally extends to other forbidden subgraphs of Q_d . For comparison, the unconstrained random process with the hypercube as host graph is considered in [1, 8, 10].

We will describe this process more formally using an equivalent definition based on random permutations which turns out to be easier to work with. The approach was first used (in the K_n host graph setting) by Erdős, Suen and Winkler [11] and has also appeared in a more refined form, for instance in [19, 26]. We first choose a uniformly random permutation of $E(Q_d)$ giving a labelling of these edges as $e_1, e_2, \dots, e_{|E(Q_d)|}$. From this, form a nested sequence of subgraphs of Q_d by looking at the edges in turn, and adding the next edge which does not create a copy of Q_2 . More precisely, suppose that we have constructed graphs G_0, G_1, \dots, G_i and have looked at edges $e_1, \dots, e_{t(i)}$. We look at the edges $e_{t(i)+1}, e_{t(i)+2}, \dots$ in turn, stopping when we get to some e_j which can be added to G_i without creating a copy of Q_2 . We add e_j to G_i to form G_{i+1} and let $t(i+1) = j$. The result is a nested sequence of subgraphs G_0, G_1, \dots, G_M of Q_d with $E(G_i) = i$ and G_M being (Q_d, Q_2) -saturated.

As in the graph case, our main question is what can be said about the random variable M ? Our main result, proved in Section 2, is that with high probability, the subgraph of Q_d generated by the Q_2 -free process in Q_d has at least $cd^{2/3}2^d$ edges, for some constant c . We also establish a local version of this result: with high probability, almost all vertices have degree at least $cd^{2/3}$.

In Section 3 we consider the differential equations heuristic in the hypercube context. This leads to the conjecture that the order of magnitude of the number of edges in the graph generated is $(\log d)^{1/3} d^{2/3} 2^d$. This conjecture is a consequence of the guiding heuristic for other constrained processes that the process should resemble an unconstrained graph process except in statistics involving the constraint. Indeed, let O_i be the number of ‘open pairs’ in G_i (ie edges of Q_d that can be added that can be added to G_i without producing a copy of Q_2). Clearly, for all i , we have that $i + O_i$ is an upper bound for the size of the final graph. If the heuristic is correct for our process then O_i should be close to $(1 - p^3)^{d-1} d 2^{d-1}$. Under this assumption, when $i = O((\log d)^{1/3} d^{2/3} 2^d)$ the upper bound $i + O_i$ is also $O((\log d)^{1/3} d^{2/3} 2^d)$ and so this will be the size of the final graph. We also see that the proof strategy used to make the differential equations method rigorous followed in the ordinary graph triangle-free process is unlikely to work in the hypercube context without some significant new ideas. Thus the problem of analysing this process may be a natural testing ground for extending the differential equations method further or developing new techniques.

Related to this heuristic, we now describing how our process relates to an important general framework for constrained graph process introduced by Bennett and Bohman [4]. This is based on independent sets in an auxiliary hypergraph. For the usual F -free process this auxiliary hypergraph H has *vertex set* $V(H) = E(K_n)$, with a set $\{e_1, \dots, e_r\} \subset V(H)$ forming an edge of H precisely when e_1, \dots, e_r form a copy of F when viewed as edges of K_n . Now, maximal independent sets in H correspond to F -saturated graphs. The F -free process corresponds to the natural random greedy algorithm for finding a maximal independent set in H . Namely, we grow our independent set by adding a new random vertex of H at each time step subject to the condition that the set remains independent (ie contains no edge of H). Bennett and Bohman’s result gives a lower bound of $\Omega\left(N\left(\frac{\log N}{D}\right)^{\frac{1}{r-1}}\right)$ on the independent set generated by this random greedy algorithm in an r -uniform, D -regular hypergraph with N vertices, subject to certain degree and co-degree conditions. One of these conditions is that H is reasonably dense in the sense that $D > N^\epsilon$. This bound is consistent with the heuristic mentioned above that the independent set generated after i steps should resemble a subset generated by independently picking each element of V with probability $p = i/N$ except in that it contains no edges of H .

Our (Q_d, Q_2) -free process can also be expressed in this framework. The auxiliary hypergraph has $N = d 2^{d-1}$, $r = 4$ (each Q_2 consists of 4 edges) and $D = d - 1$ (each edge of Q_d lies in $d - 1$ copies of Q_2). Notice that D is only $O(\log N)$ and so this hypergraph is too sparse to apply the Bennett and Bohman result. The (Q_d, Q_2) -free process then provides a natural example of a constrained graph process to which the Bennett and Bohman result does not apply.

We conclude, in Section 4, by raising some related open problems.

Finally, we note that, complementing these probabilistic questions, saturated graphs have been studied from an extremal perspective. Indeed, the well-studied *Turán number* of F , denoted by $ex(n, F)$, can be defined as the maximum number of edges in an F -

saturated graph on n vertices. As a counterpart to this, the *saturation number* of F , denoted by $\text{sat}(n, F)$ is the minimum number of edges in an F -saturated graph. See the surveys [13] and [12, 22] and many references therein for more on Turán and saturation numbers. Both Turán and saturation numbers have been studied for the host graph Q_d (see for instance [2, 3, 9, 16] for the former and [15, 17, 20] for the later). However, to our knowledge, this associated random process has not.

2 Main Result

Theorem 2. *Let M be the number of edges in the subgraph G_M of Q_d generated by the (Q_d, Q_2) -free process. With high probability, $M > cd^{2/3}2^d$, for some constant, c .*

As we shall see, the constant c can be taken to be arbitrarily close to $1/e$.

The proof uses the random permutation formulation of the process. We identify a local condition on the permutation which guarantees that a particular edge appears in the final graph G_M . Calculating the probability that this condition is satisfied gives a lower bound on the expected number of edges. The fact that the condition is a local one means that dependence between edges is limited and the second moment method gives a lower bound on M which holds with high probability.

Proof of Theorem 2. Generate a random permutation of the edges of Q_d by assigning to each edge e , a random variable T_e , where T_e is uniformly distributed in the interval $[0, 1]$. We say that e precedes f in our order if $T_e < T_f$.

Let G_M be the saturated graph yielded by following the (Q_d, Q_2) -free process on this permutation.

We say that an edge $e \in E(Q_d)$ is *good* if, for every Q_2 containing e , the last of its four edges in our ordering is not e . It is easy to see that if e is good, then e is an edge of G_M .

Let A_e denote the indicator random variable taking the value 1 if e is good and 0 otherwise, and let $A = \sum_{e \in E(Q_d)} A_e$ be the total number of good edges. Considering how the permutation is generated from the variables T_e , we obtain:

$$\begin{aligned} \mathbb{P}(A_e = 1) &= \int_0^1 (1 - x^3)^{d-1} dx \\ &\geq \int_0^{d^{-\frac{1}{3}}} (1 - x^3)^{d-1} dx \\ &\geq d^{-\frac{1}{3}} \left(1 - \frac{1}{d}\right)^{d-1} \quad (\text{as the integrand is decreasing in } x) \\ \mathbb{P}(A_e = 1) &\geq (1/e + o(1)) d^{-\frac{1}{3}}, \end{aligned}$$

for large enough d .

Since Q_d has $d2^{d-1}$ edges, linearity of expectation gives that:

$$\mathbb{E}(A) \geq (1/e + o(1)) d^{2/3} 2^{d-1}.$$

The event A_e depends only on the variables T_f where f is one of the $3(d-1)$ edges contained in a Q_2 through e . It follows that A_e is independent of all but at most $9d^2$ other A_f .

$$\begin{aligned}\mathrm{Var}(A) &= d2^{d-1}\mathrm{Var}(A_e) + \sum_{e \neq f} \mathrm{Cov}(A_e, A_f) \\ &\leq \mathbb{E}(A) + 9d^3 2^d \\ &= o(\mathbb{E}(A)^2).\end{aligned}$$

Thus by Chebychev's inequality, $A \geq cd^{2/3}2^d$ with high probability, for some c (which can be taken to be arbitrarily close to $1/e$). This concludes the proof, since $M \geq A$. \square

A slightly more careful calculation gives that $d^{2/3}2^{d-1}$ is the correct order of magnitude of $\mathbb{E}(A)$ so $A = \Theta(d^{2/3}2^{d-1})$ with high probability. However, because the property of being good is sufficient but not necessary for an edge to be in G_M , this observation gives no upper bound for M .

Notice that the way in which we bounded the integral, means that only edges with $T_e \leq d^{-1/3}$ are considered. This means that with high probability, not only do we finish the process with at least $cd^{2/3}2^d$ edges, but at least this many edges must be added from among the first $d^{2/3}2^{d-1}$ edges considered.

The same approach can be used to give some information on the degrees in G_M . The degree of v in G_M is bounded by the number of good edges among the edges of Q_d incident to v . Unfortunately, there are no independent pairs among the events that each of these edges is good. However, the dependence is very limited and so a local analogue to Theorem 2 can be established.

Theorem 3. *Let v be a fixed vertex of Q_d and let G_M be the subgraph of Q_d generated by the (Q_d, Q_2) -free process. With high probability, the degree of v in G_M is at least $cd^{2/3}$, for some constant, c .*

Proof. Let e_1, \dots, e_d be the edges of Q_d incident to v . We define good edges as in the proof of Theorem 2. Let A_i be the indicator variable of the event ' e_i is good', and $D_v = \sum_{i=1}^d A_i$. We have that

$$\mathrm{Var}(D_v) = d\mathrm{Var}(A_e) + \sum_{i \neq j} \mathrm{Cov}(A_i, A_j) = dp(1-p) + d(d-1)(r-p^2)$$

where

$$\begin{aligned}p &= \mathbb{P}(A_1 = 1) = cd^{-1/3} \\ r &= \mathbb{P}(A_1 = 1, A_2 = 1)\end{aligned}$$

It will suffice to show that $r - p^2 = o(d^{-2/3})$. From this we deduce that $\mathrm{Var}(D_v) = o(d^{4/3}) = o(\mathbb{E}(D_v)^2)$, and the result follows as in the proof of Theorem 2.

Now, using the same method of generating a random permutation as in the proof of Theorem 2, we have:

$$p^2 = \int_0^1 \int_0^1 f(x, y, d) dx dy$$

$$r = \int_0^1 \int_0^1 g(x, y, d) dx dy$$

where

$$f(x, y, d) = (1 - x^3)^{d-1} (1 - y^3)^{d-1}$$

$$g(x, y, d) = (1 - x^3 - y^3 + x^2 y^2 \min\{x, y\})^{d-2} (1 - (\max\{x, y\})^2).$$

It is easy to check that if one of x, y is greater than $d^{-1/4}$ then both $f(x, y, d)$ and $g(x, y, d)$ are at most $\exp(d^{-1/4})$ and so the contribution to $r - p^2$ from this range of x, y is certainly $o(d^{-2/3})$.

On the other hand, if $x < y < d^{-1/4}$ then, writing

$$g(x, y, d) - f(x, y, d) = (1 - x^3)^{d-1} (1 - y^3)^{d-1} (h(x, y) - 1)$$

where

$$\begin{aligned} h(x, y) &= \left(\frac{1 - x^3 - y^3 + x^3 y^2}{(1 - x^3)(1 - y^3)} \right)^{d-2} \left(\frac{1 - y^2}{(1 - x^3)(1 - y^3)} \right) \\ &= \left(1 + \frac{x^3 y^2 - x^3 y^3}{(1 - x^3)(1 - y^3)} \right)^{d-2} \left(1 + \frac{x^3 + y^3 - x^3 y^3 - y^2}{(1 - x^3)(1 - y^3)} \right) \\ &< (1 + (d-2)2d^{-5/4}) (1 + 2d^{-3/4}) \\ &< (1 + 3d^{-1/4}), \end{aligned}$$

we conclude that

$$\int_0^{d^{-1/4}} \int_0^{d^{-1/4}} g(x, y, d) - f(x, y, d) dx dy \leq 3(d^{-1/4})^3.$$

That is, the contribution to $r - p^2$ from this range of x, y is at most $3d^{-3/4} = o(d^{-2/3})$ as required. \square

We do not know whether or not, with high probability *all* vertices have degree at least $cd^{2/3}$.

3 Heuristic

For the triangle-free process, Bohman [5] introduces a heuristic that assumes certain random variables follow some trajectories closely. Using this assumption he deduces the

values of those trajectories in order to bound the number of edges in the resulting graph. This approach can be made rigorous using martingales.

We use the analogous heuristic for the (Q_d, Q_2) -free process to suggest a possible order for M . However, we also point out some differences between the (Q_d, Q_2) -free process and the triangle-free process that cause difficulties in making this argument rigorous.

Let G_0, \dots, G_M be the sequence of graphs generated by the (Q_d, Q_2) -free process. Let u and v be a pair of vertices that are adjacent in Q_d . We say that uv is open in G_i if there is no path of three G_i -edges that connect u to v . In other words uv is open if adding it to G_i does not form a copy of Q_2 . We write O_i for the number of open pairs in G_i . This definition of open pairs is analogous to a definition in [5].

We also define, for each Q_d -adjacent pair of vertices u and v , three other random variables. Let $W_i(uv)$ denote the number of paths of length 3 from u to v consisting of three open pairs in G_i , let $X_i(uv)$ be the number of paths of length 3 from u to v consisting of two open pairs and one G_i -edge and let $Y_i(uv)$ count the paths of length 3 from u to v consisting of one open pair and two G_i -edges.

For convenience, we also introduce a scaling $t = \frac{i}{d^{2/3}2^d}$. We assume there are continuous functions q, w, x and y such that for all i and all Q_d -adjacent u and v :

$$O_i \approx q(t)d2^d, \quad W_i(uv) \approx w(t)d, \quad X_i(uv) \approx x(t)d^{2/3}, \quad Y_i(uv) \approx y(t)d^{1/3}.$$

Note that adding a single edge uv to G_i to form G_{i+1} removes $Y_i(uv)$ open edges. Thus for small ϵ , we expect

$$q(t + \epsilon)d2^d \approx O_{i+\epsilon d^{2/3}2^d} \approx O_i - \epsilon d^{2/3}2^d \cdot y(t)d^{1/3} \approx (q(t) - \epsilon y(t))d2^d.$$

This suggests that $\frac{dq}{dt} = -y$. Similar arguments give:

$$\frac{dx}{dt} = \frac{3w}{q} - \frac{2xy}{q}, \quad \frac{dy}{dt} = \frac{2x}{q} - \frac{y^2}{q}, \quad \frac{dw}{dt} = \frac{-3yw}{q}.$$

Solving these equations with initial conditions $q(0) = 1/2, w(0) = 1, x(0) = y(0) = 0$ gives

$$q(t) = \frac{1}{2}e^{-8t^3}, \quad w(t) = e^{-24t^3}, \quad x(t) = 6te^{-16t^3}, \quad y(t) = 12t^2e^{-8t^3}.$$

For any i , the final number of edges in the process is bounded from above by $i + O(i)$ and from below by i . If indeed $O(i) \approx \frac{1}{2}e^{-8t^3}d2^d$, then when $t = \Theta(\log^{1/3} d)$, these bounds are both $\Theta\left((\log d)^{\frac{1}{3}}d^{\frac{2}{3}}2^d\right)$.

As we noted in the introduction, this solution fits with the heuristic that a constrained process should resemble an unconstrained process except in statistics involving the constraint.

Motivated by this heuristic, we propose the following,

Conjecture 4. Let G_M be the graph generated by the (Q_d, Q_2) -free process. With high probability,

$$c_1(\log d)^{1/3}d^{2/3}2^d \leq M \leq c_2(\log d)^{1/3}d^{2/3}2^d,$$

for some constants c_1 and c_2 .

For the triangle-free process, Bohman uses martingales to show that with high probability all the relevant random variables do indeed follow their trajectories closely. By contrast, in our process the situation is more complicated; the random variables we use to track the evolution of the graph do not all follow the trajectory indicated by the differential equations heuristic.

Associated with the (Q_d, Q_2) -free process, we have a sequence of graphs, $H(j)$, for $j = 0, \dots, n2^{d-1}$, where $H(j)$ is the graph formed by the first j edges in the randomly chosen permutation. This nested sequence of graphs is a natural analogue of the unconstrained Erdős-Rényi random graph process

We again let G_i denote the graphs of the (Q_d, Q_2) -free process for $i = 0, \dots, M$, but consider i as a function of j . That is, we write $i(j)$ for the number of edges added from among the first j edges looked at. For Q_d -adjacent vertices u and v , note that $Y_{i(j)}(uv) = 0$ whenever u and v are isolated in $H(j)$. Thus,

$$\begin{aligned} \mathbb{P}(Y_{i(j)}(uv) = 0) &\geq \frac{\binom{d2^{d-1}-2d}{j}}{\binom{d2^{d-1}}{j}} \\ &= \frac{(d2^{d-1}-j) \cdots (d2^{d-1}-2d+1-j)}{(d2^{d-1}) \cdots (d2^{d-1}-2d+1)} \\ &\geq \left(1 - \frac{j+d}{d2^{d-1}}\right)^{2d-1} \\ &\geq \exp\left(-\frac{j+d}{2^{d-2}}\right). \end{aligned}$$

It follows that there is some constant c such that while $j \leq cd2^{d-1}$, we have, in expectation, a large number of pairs uv with $Y_{i(j)}(uv) = 0$. It seems likely that i is approximately concave as a function of j (the number of edges added should grow faster early on in the process when fewer edges have been looked at). If true this would imply that for some uv , the random variable $Y_{i(j)}(uv)$ equals zero for a constant proportion of the process. Thus, unlike in the triangle-free process, we will not typically have every variable following its expected trajectory closely. It is still possible that this approach can be salvaged, for instance by showing that the collection of variables are approximated by their trajectories in some weaker sense, but this does not appear to be straightforward.

4 Further Questions

Given the apparent obstacles to adapting the techniques from the triangle-free process to the hypercube, the main open problem is to develop tools to understand the Q_2 -free process in Q_d . This could involve either refining the differential equations method or introducing a completely new approach.

The most immediate open problem is to give a good upper bound for M and in particular to answer the following question:

Question 5. Is the true order of magnitude of M given by $(\log d)^{1/3}d^{2/3}2^d$ as predicted by the differential equations heuristic?

The number of edges is just one of the properties of G_M that could be considered. It would be interesting to study other properties of this graph, for example the minimum and maximum degree of G_M . Note that Theorem 3 does not give any information about the minimum degree since it bounds almost all degrees rather than every degree.

Bohman proves that with high probability the triangle-free process produces a graph with no large independent set. This was used to give improved lower bounds on the Ramsey number $R(3, k)$. In the hypercube there is no analogous Ramsey result; indeed for any d there is a 2-colouring of $e(Q_d)$ with no monochromatic Q_2 . Nevertheless, one could ask about the existence of empty subcubes.

Question 6. What can be said about the number of copies of Q_k in Q_n which contain no edges of G_M ? For which k is the expected number of empty Q_k bounded away from 0?

More generally, what can be said about the appearance of fixed subgraphs in G_M ? This question has been addressed for the triangle-free process by Wolfowitz [28] (sparse subgraphs) and Gerke and Makai [14] (dense subgraphs). Extensions of their results to the H -free process were proved by Warnke [27].

Finally, we studied the Q_2 -free process as a natural special case of the F -free process. What can be said about the F -free process in the hypercube for other fixed graphs F ? Two particularly appealing instances for F are fixed dimension subcubes Q_k and the star $K_{1,t}$ (the bounded degree process).

Question 7. what can be said about the graph G_M generated by the F -free process in Q_d ? In particular when $F = Q_k$ or $F = K_{1,t}$.

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