Minimal non-odd-transversal hypergraphs and minimal non-odd-bipartite hypergraphs

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Abstract

Among all uniform hypergraphs with even uniformity, the odd-transversal or odd-bipartite hypergraphs are closer to bipartite simple graphs than bipartite hypergraphs from the viewpoint of both structure and spectrum. A hypergraph is called odd-transversal if it contains a subset of the vertex set such that each edge intersects the subset in an odd number of vertices, and it is called minimal non-odd-transversal if it is not odd-transversal but deleting any edge results in an odd-transversal hypergraph. In this paper we give an equivalent characterization of the minimal non-odd-transversal hypergraphs by means of the degrees and the rank of its incidence matrix over $\mathbb{Z}_2$. If a minimal non-odd-transversal hypergraph is uniform, then it has even uniformity, and hence is minimal non-odd-bipartite. We characterize 2-regular uniform minimal non-odd-bipartite hypergraphs, and give some examples of d-regular uniform hypergraphs which are minimal non-odd-bipartite. Finally we give upper bounds for the least H-eigenvalue of the adjacency tensor of minimal non-odd-bipartite hypergraphs.

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1 Introduction

Let $G = (V, E)$ be a hypergraph, where $V =: V(G)$ is the vertex set, and $E =: E(G)$ is the edge set whose elements $e \subseteq V$. If for each edge e of G, $|e| = k$, then G is called a *k-uniform* hypergraph. The *degree* $d(v)$ of a vertex v of G is defined to be the number of edges of G containing v. If $d(v) = d$ for all vertices v of G, then G is called *d-regular.*

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A hypergraph G is called 2-*colorable* if there exists a 2-coloring of the vertices of $V(G)$ such that G contains no monochromatic edges; and it is called *minimal non*-2-*colorable* if it is not 2-colorable but deleting any edge from $E(G)$ results in a 2-colorable hypergraph. Seymour [27] proved that if G is minimal non-2-colorable and $V(G) = \cup\{e : e \in E(G)\}$, then $|E(G)| \geqslant |V(G)|$. Aharoni and Linial [1] presented an infinite version of Seymour's result. Alon and Bregman [3] proved that if $k \geqslant 8$ then every k-regular k-uniform hypergraph is 2-colorable. Henning and Yeo [15] showed that Alon-Bergman result is true for $k \geqslant 4$.

A subset U of $V(G)$ is called a *transversal* (also called *vertex cover* [4] or *hitting set* [17]) of G if each edge of G has a nonempty intersection with U. The *transversal number* of G is the minimum size of transversals in G, which was well studied by Alon [2], Chvátal and McDiarmid [7], Henning and Yeo [16]. A hypergraph G is called *bipartite* if for some nonempty proper subset $U \subseteq V(G)$, U and its complement U^c are both transversal; or equivalently the vertex set $V(G)$ has a bipartition into two parts such that every edge of $E(G)$ intersects both parts. Surely, G is bipartite if and only if G is 2-colorable.

A subset U of $V(G)$ is called an *odd transversal* of G if each edge of G intersects U in an odd number of vertices [9, 26]. A hypergraph G is called *odd-transversal* if it has an odd transversal; otherwise, G is called *non-odd-transversal* Nikiforov [22] firstly uses odd transversal to investigate the spectral symmetry of tensors and hypergraphs. The notion of odd-bipartite hypergraphs was introduced by Hu and Qi [18] to study the zero eigenvalue of the signless Laplacian tensor.

Definition 1.1 ([18])**.** Let G be a k-uniform hypergraph G, where k is even. If there exists a bipartition $\{U, U^c\}$ of $V(G)$ such that each edge of G intersects U (and also U^c) in an odd number of vertices, then G is called *odd-bipartite*, and $\{U, U^c\}$ is an *odd-bipartition* of G; otherwise, G is called *non-odd-bipartite*.

So, odd-bipartite hypergraphs are surely odd-transversal hypergraphs and bipartite hypergraphs. For the uniform hypergraphs with even uniformity, the notion of odd-bipartite hypergraphs is *equivalent to* that of odd-transversal hypergraphs.

From the viewpoint of spectrum, a simple graph is bipartite if and only if its adjacency matrix has a symmetric spectrum. However, the adjacency tensor of a bipartite uniform hypergraph does not possess such property. We note that the hypergraphs under consideration are uniform when discussing their spectra. Shao et al. [28] proved that the adjacency tensor of a k-uniform hypergraph G has a symmetric H-spectrum (the set of eigenvalues associated with real eigenvectors) if and only if k is even and G is odd-bipartite. So, the odd-bipartite hypergraphs are closer to bipartite simple graphs than the bipartite hypergraphs based on the following two reasons. First they both have a structural property, namely, there exists a bipartition of the vertex set such that every edge intersects the each part of the bipartition in an odd number of vertices. Second they both have a symmetric H-spectrum.

There are some examples of odd-bipartite hypergraphs, e.g. power of simple graphs and cored hypergraphs [19], hm-hypergraphs [18], m-partite m-uniform hypergraphs [8]. Nikiforov [23] gives two classes of non-odd-transversal hypergraphs. Fan et al. [20] construct non-odd-bipartite generalized power hypergraphs from non-bipartite simple graphs.

It is known that a connected bipartite simple graph has a unique bipartition up to isomorphism. However, an odd-bipartite hypergraph can have more than one odd-bipartition. Fan et al. [12] given a explicit formula for the number of odd-bipartition of a hypergraph by the rank of its incidence matrix over $\mathbb{Z}_2$. So, it seems hard to give examples of non-odd-bipartite hypergraphs.

To our knowledge, there is no characterization of non-odd-transversal or non-odd-bipartite hypergraphs. We observe that non-odd-transversal (or non-odd-bipartite) hypergraphs have a hereditary property, that is, if G contains a non-odd-transversal (or non-odd-bipartite) sub-hypergraph, then G is non-odd-transversal (or non-odd-bipartite). We call G *minimal non-odd-transversal* (or *minimal non-odd-bipartite*), if G is non-odd-transversal (or non-odd-bipartite) but deleting any edge from G results in an odd-transversal (or odd-bipartite) hypergraph, or equivalently, any nonempty proper edge-induced sub-hypergraph of G is odd-transversal (or odd-bipartite). In this paper we give an equivalent characterization of the minimal non-odd-transversal hypergraphs by means of the degrees and the rank of its incidence matrix over $\mathbb{Z}_2$. If a minimal non-odd-transversal hypergraph is uniform, then it has even uniformity, and hence is minimal non-odd-bipartite. We characterize 2-regular uniform minimal non-odd-bipartite hypergraphs, and give some examples of d-regular uniform hypergraphs which are minimal non-odd-bipartite. Finally we give upper bounds for the least H-eigenvalue of the adjacency tensor of minimal non-odd-bipartite hypergraphs.

2 Basic notions

Unless specified somewhere, all hypergraphs in this paper contain *no multiple edges or isolated vertices*, where vertex is called *isolated* if it is not contained in any edge of the hypergraph. Let $G = (V, E)$ be a hypergraph. The hypergraph G is called *square* if $|V| = |E|$. A *walk* of length t in G is a sequence of alternate vertices and edges: $v_0 e_1 v_1 e_2 \ldots e_t v_t$, where $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, t-1$; and G is said to be *connected* if every two vertices are connected by a walk.

The *vertex-induced sub-hypergraph* of G by the a subset $U \subseteq V(G)$, denoted by $G|_U$, is a hypergraph with vertex set U and edge set $\{e \cap U : e \in E(G), e \cap U \neq \varnothing\}$. For a connected hypergraph G, a vertex v is called a *cut vertex* of G if $G|_{V(G)\setminus\{v\}}$ is disconnected. The *edge-induced sub-hypergraph* of G by a subset $F \subseteq E(G)$, denoted by $G|_F$, is a hypergraph with vertex set $\cup\{e : e \in F\}$ and edge set F.

Let G be a hypergraph and let e be an edge of G. Denote by $G - e$ the hypergraph obtained from G by deleting the edge e from $E(G)$. For a connected hypergraph G, an edge e is called a *cut edge* of G if $G - e$ is disconnected.

A *matching* M of G is a set of pairwise disjoint edges of G. In particular, if G is bipartite simple graph with a bipartition $\{V_1, V_2\}$, a vertex subset $U_1 \subseteq V_1$ is *matched to* $U_2 \subseteq V_2$ in M, if there exists a bijection $f : U_1 \to U_2$ such that $\{\{v, f(v)\} : v \in U_1\} \subseteq M$. A subset U of V_1 (or V_2) is *matched by* M if every vertex of U is incident with an edge of M. M is called a *perfect matching* if V_1 and V_2 are both matched by M. The following result is known as Hall's Theorem.

Lemma 2.1. (Hall 1935) Let G be a bipartite simple graph with a bipartition $\{V_1, V_2\}$. Then G contains a matching M such that V_1 is matched by M if and only if $|N(S)| \geqslant |S|$ for every $S \subseteq V_1$, where $N(S)$ denotes the set of all vertices of V_2 adjacent to some vertex of S.

Let G be a hypergraph. The *incidence bipartite graph* Γ_G of G is a bipartite simple graph with two parts $V(G)$ and $E(G)$ such that $\{v, e\} \in E(\Gamma_G)$ if and only if $v \in e$.

The edge-vertex *incidence matrix* of G, denoted by $B_G = (b_{e,v})$, is a matrix of size $|E(G)| \times |V(G)|$, whose entries $b_{e,v} = 1$ if $v \in e$, and $b_{e,v} = 0$ otherwise.

The *dual* of G, denoted by G^*, is the hypergraph whose vertex set is $E(G)$ and edge set is $\{\{e \in E(G) : v \in e\} : v \in V(G)\}$. If no two vertices of G are contained in precisely the same edges of G, then G^* contains no multiple edges. In this situation, $(G^*)^*$ is isomorphic to G, Γ_G is isomorphic to Γ_{G^*}, and $B_G = B_{G^*}^\top$, where the latter denotes the transpose of B_{G^*}.

Let G be a simple graph, and let k be an even integer greater than 2. Denote by $G^{k,\frac{k}{2}}$ the hypergraph obtained from G whose vertex set is $\cup_{v \in V(G)} \mathbf{v}$ and edge set $\{\mathbf{u} \cup \mathbf{v} : \{u, v\} \in E(G)\}$, where $\mathbf{v}$ denotes an $\frac{k}{2}$-set corresponding to v, and all those sets are pairwise disjoint; intuitively $G^{k,\frac{k}{2}}$ is obtained from G by blowing up each vertex into an $\frac{k}{2}$-set and preserving the adjacency relation [20]. It is proved that $G^{k,\frac{k}{2}}$ is non-odd-bipartite if and only if G is non-bipartite [20].

Next we will introduce some knowledge of eigenvalues of a tensor. For integers $k \geqslant 2$ and $n \geqslant 2$, a *tensor* (also called *hypermatrix*) $\mathcal{T} = (t_{i_1 i_2 \ldots i_k})$ of order k and dimension n refers to a multidimensional array $t_{i_1 i_2 \ldots i_k} \in \mathbb{C}$ for all $i_j \in [n] := \{1, 2, \ldots, n\}$ and $j \in [k]$. The tensor $\mathcal{T}$ is called *symmetric* if its entries are invariant under any permutation of their indices.

Given a vector $x \in \mathbb{C}^n$, $\mathcal{T}x^k \in \mathbb{C}$ and $\mathcal{T}x^{k-1} \in \mathbb{C}^n$, which are defined as follows:

$$\mathcal{T}x^k = \sum_{i_1, i_2, \ldots, i_k \in [n]} t_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} t_{i i_2 i_3 \ldots i_k} x_{i_2} x_{i_3} \cdots x_{i_k}, \text{ for } i \in [n].$$

Let $\mathcal{I}$ be the *identity tensor* of order k and dimension n, that is, $i_{i_1 i_2 \ldots i_k} = 1$ if and only if $i_1 = i_2 = \cdots = i_k \in [n]$ and zero otherwise.

Definition 2.2 ([21, 25])**.** Let $\mathcal{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})x^{k-1} = 0$, or equivalently $\mathcal{T}x^{k-1} = \lambda x^{[k-1]}$, has a solution $x \in \mathbb{C}^n \backslash \{0\}$, then λ is called an eigenvalue of $\mathcal{T}$ and x is an eigenvector of $\mathcal{T}$ associated with λ, where $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \in \mathbb{C}^n$.

The *characteristic polynomial* $\varphi_{\mathcal{T}}(\lambda)$ of $\mathcal{T}$ is defined as the resultant of the polynomials $(\lambda\mathcal{I} - \mathcal{T})x^{k-1}$; see [25, 6, 14]. It is known that λ is an eigenvalue of $\mathcal{T}$ if and only if it is a root of $\varphi_{\mathcal{T}}(\lambda)$. The *spectrum* of $\mathcal{T}$ is the multi-set of the roots of $\varphi_{\mathcal{T}}(\lambda)$. The *spectral radius* of $\mathcal{T}$ is defined as the maximum modulus of the eigenvalues of $\mathcal{T}$, denoted by $\rho(\mathcal{T})$.

Suppose that $\mathcal{T}$ is real. If x is a real eigenvector of $\mathcal{T}$, surely the corresponding eigenvalue λ is real. In this case, x is called an *H-eigenvector* and λ is called an *H-eigenvalue.* The *H-spectrum* of $\mathcal{T}$ is the set of all H-eigenvalues of $\mathcal{T}$, denote by $\mathrm{HSpec}(\mathcal{T})$. Denote by $\lambda_{\max}(\mathcal{T})$ and $\lambda_{\min}(\mathcal{T})$ the largest H-eigenvalue and the least H-eigenvalue of $\mathcal{T}$, respectively.

For a real symmetric tensor, Zhou et al. [32] and Qi [25] give the following results.

Lemma 2.3. Let $\mathcal{T}$ be a real symmetric tensor of order k and dimension n.

(1) ([32, Theorem 3.6]) If $\mathcal{T}$ is also nonnegative, then

$$\lambda_{\max}(\mathcal{T}) = \min\{\mathcal{T}x^k : x \in \mathbb{R}^n, x \geqslant 0, \|x\|_k = 1\},$$

where $\|x\|_k = \left(\sum_{i=1}^n |x_i^k|\right)^{\frac{1}{k}}$. Furthermore, x is an optimal solution of the above optimization if and only if it is an eigenvector of $\mathcal{T}$ associated with $\lambda_{\max}(\mathcal{T})$.

(2) ([25, Theorem 5]) If k is also even, then

$$\lambda_{\min}(\mathcal{T}) = \min\{\mathcal{T}x^k : x \in \mathbb{R}^n, \|x\|_k = 1\},$$

and x is an optimal solution of the above optimization if and only if it is an eigenvector of $\mathcal{T}$ associated with $\lambda_{\min}(\mathcal{T})$.

Let G be a k-uniform hypergraph on n vertices $v_1, v_2, \ldots, v_n$. The *adjacency tensor* of G [8] is defined as $\mathcal{A}(G) = (a_{i_1 i_2 \ldots i_k})$, a k-th order n-dimensional tensor, where

$$a_{i_1 i_2 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

The *spectral radius, the least H-eigenvalue* of G refer to those of its adjacency tensor $\mathcal{A}(G)$, denoted by $\rho(G), \lambda_{\min}(G)$, respectively. The H-spectrum of $\mathcal{A}(G)$ is denoted by $\mathrm{HSpec}(G)$.

The spectral hypergraph theory has been an active topic in algebraic graph theory recently; see e.g. [8, 10, 11, 23, 24]. By the Perron-Frobenius theorem for nonnegative tensors [5, 13, 29, 30, 31], $\rho(G)$ is exactly the largest H-eigenvalue of $\mathcal{A}(G)$. If G is connected, there exists a unique positive eigenvector up to scales associated with $\rho(G)$, called the *Perron vector* of G. Noting that the adjacency tensor $\mathcal{A}(G)$ is nonnegative and symmetric, so $\rho(G)$ holds (1) of Lemma 2.3, and $\lambda_{\min}(G)$ holds (2) of Lemma 2.3 if k is even. By Perron-Frobenius theorem, $\lambda_{\min}(G) \geqslant -\rho(G)$. By the following lemma, if G is connected and non-odd-bipartite, then $\lambda_{\min}(G) > -\rho(G)$.

Lemma 2.4 ([23, 22, 28, 12])**.** Let G be a k-uniform connected hypergraph. Then the following results are equivalent.

(1) k is even and G is odd-bipartite.

(2) $\lambda_{\min}(G) = -\rho(G)$.

(3) $\mathrm{HSpec}(G) = -\mathrm{HSpec}(G)$.

Finally, we introduce some notations used throughout the paper. Denote by C_n a cycle of length n as a simple graph. Denote by $\mathbb{1}$ an all-one vector whose size can be implicated by the context, $\text{rank}A$ the rank of a matrix A over $\mathbb{Z}_2$, and $\mathbb{F}_q$ a field of order q.

3 Characterization of minimal non-odd-transversal hypergraphs

In this section we will give some equivalent conditions in terms of degrees and rank of the incidence matrix over $\mathbb{Z}_2$ for a hypergraph to be minimal non-odd-transversal.

Lemma 3.1. If G is a minimal non-odd-transversal hypergraph, then G is connected and contains no cut vertices.

Proof. If G contains more than one connected component, then at least one of them is non-odd-transversal, a contradiction to the definition. So G itself is connected. Suppose G contains a cut vertex. Then G is obtained from two connected nontrivial sub-hypergraphs G_1, G_2 sharing exactly one vertex (the cut vertex). So, at least one of G_1, G_2 is non-odd-transversal, also a contradiction. $\square$

Lemma 3.2. Let G be a connected hypergraph, and B_G be the edge-vertex incidence matrix of G. Then G is odd-transversal if and only if the equation

$$B_G x = \mathbb{1} \text{ over } \mathbb{Z}_2 \tag{3.1}$$

has a solution, or equivalently

$$\text{rank}B_G = \text{rank}(B_G, \mathbb{1}) \text{ over } \mathbb{Z}_2. \tag{3.2}$$

Proof. If G is odd-transversal, then there is an odd-transversal U of G. Define a vector $x \in \mathbb{Z}_2^{V(G)}$ such that $x_v = 1$ if $v \in U$, and $x_v = 0$ otherwise. By the definition, it is easy to verify that x is a solution of the equation (3.1). On the other hand, if x is a solution of the equation (3.1), define $U = \{v : x_v = 1\}$. Then $U \neq \varnothing$, and for each edge e of G, $|e \cap U|$ is odd, implying that G is odd-transversal. $\square$

For each edge $e \in E(G)$, define an *indicator vector* $\chi_e \in \mathbb{Z}_2^{V(G)}$ such that $\chi_e(v) = 1$ if $v \in e$ and $\chi_e(v) = 0$ otherwise. Then B_G consists of those χ_e as row vectors for all $e \in E(G)$.

Lemma 3.3. Let G be a connected hypergraph with m edges. If m is odd, and each vertex has an even degree, or equivalently $\sum_{e\in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, then G is non-odd-transversal.

Proof. Let $e_1, \ldots, e_m$ be the edges of G. Write $(B_G, \mathbb{1})$ as the following form:

$$(B_G, \mathbb{1}) = \begin{pmatrix} \chi_{e_1} & 1 \\ \chi_{e_2} & 1 \\ \vdots & \vdots \\ \chi_{e_m} & 1 \end{pmatrix}. \tag{3.3}$$

Adding the first row to all other rows over $\mathbb{Z}_2$, we will have

$$\begin{pmatrix} \chi_{e_1} & 1 \\ \chi_{e_2}+\chi_{e_1} & 0 \\ \vdots & \vdots \\ \chi_{e_m}+\chi_{e_1} & 0 \end{pmatrix} =: \begin{pmatrix} \chi_{e_1} & 1 \\ C & O \end{pmatrix}. \tag{3.4}$$

So, $\mathrm{rank}(B_G, \mathbb{1}) = 1 + \mathrm{rank} C$. As m is odd and $\sum_{e\in E(G)} \chi_e = 0$,

$$\chi_{e_1} = \sum_{i=2}^{m} (\chi_{e_i} + \chi_{e_1}),$$

implying that $\mathrm{rank} B_G = \mathrm{rank} C$. By Lemma 3.2, G is non-odd-transversal. □

Theorem 3.4. Let G be a connected hypergraph with m edges. The following are equivalent.

(1) G is minimal non-odd-transversal.

(2) m is odd, $\sum_{e\in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, and $\sum_{e\in F} \chi_e \neq 0$ over $\mathbb{Z}_2$ for any nonempty proper subset F of $E(G)$.

(3) m is odd, $\sum_{e\in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, and $\mathrm{rank} B_G = m-1$ over $\mathbb{Z}_2$.

(4) m is odd, each vertex of G has an even degree, and any nonempty proper edge-induced sub-hypergraph of G contains vertices of odd degrees.

Proof. (1) $\Rightarrow$ (2). Suppose that G is minimal non-odd-transversal. Let $e_1, \ldots, e_m$ be the edges of G. By Eq. (3.3) and Eq. (3.4), as $\mathrm{rank} B_G \neq \mathrm{rank}(B_G, \mathbb{1})$ over $\mathbb{Z}_2$ by Lemma 3.2, χ_{e_1} is a linear combination of $\chi_{e_i} + \chi_{e_1}$ for $i = 2, \ldots, m$. So there exist $a_i \in \mathbb{Z}_2$ for $i = 2, \ldots, m$ such that

$$\chi_{e_1} = \sum_{i=2}^{m} a_i(\chi_{e_i} + \chi_{e_1}) = \left(\sum_{i=2}^{m} a_i\right) \chi_{e_1} + \sum_{i=2}^{m} a_i \chi_{e_i}. \tag{3.5}$$

We assert that $a_i = 1$ for $i = 2, \ldots, m$. Otherwise, there exists a j, $2 \leqslant j \leqslant m$, such that $a_j = 0$. Then χ_{e_1} is also a linear combination of $\chi_{e_i} + \chi_{e_1}$ for $i = 2, \ldots, m$ and $i \neq j$. So, $\mathrm{rank} B_{G-e_j} \neq \mathrm{rank}(B_{G-e_j}, \mathbb{1})$, implying that $G - e_j$ is non-odd-transversal by Lemma 3.2, a contradiction to the definition.

If m is even, then $\sum_{i=2}^{m} \chi_{e_i} = 0$ by Eq. (3.5), implying the vertices of $V(G)$ all have even degrees in $G - e_1$. So the vertices of e_1 all have odd degrees in G. By the arbitrariness of e_1, each vertex has an odd degree in G. However, there exists a vertex $v \notin e_1$ so that $d_G(v) = d_{G-e_1}(v)$, which is an even number, a contradiction.

So, m is odd, and $\sum_{e\in E(G)} \chi_e = 0$ by Eq. (3.5). Assume to the contrary there exists a nonempty proper subset F of $E(G)$, $\sum_{e\in F} \chi_e = 0$ over $\mathbb{Z}_2$. If $|F|$ is odd, then by Lemma

3.3, the sub-hypergraph $G|_F$ induced by F is non-odd-transversal, a contradiction to the definition. Otherwise, $|F|$ is even, then $|E(G)\backslash F|$ is odd as m is odd, and the sub-hypergraph $G|_{E(G)\backslash F}$ is non-odd-transversal, also a contradiction.

(2) $\Rightarrow$ (3). As $\sum_{e\in E(G)} \chi_e = 0$, rank$B_G \leqslant m-1$ over $\mathbb{Z}_2$. If rank$B_G \leqslant m-2$ over $\mathbb{Z}_2$, then $\chi_{e_1}, \ldots, \chi_{e_{m-1}}$ are linear dependent. So there exist $a_1, \ldots, a_{m-1} \in \mathbb{Z}_2$, not all being zero, such that $\sum_{i=1}^{m-1} a_i\chi_{e_i} = 0$. Taking $F = \{e_i : a_i = 1, 1 \leqslant i \leqslant m-1\}$, then $\sum_{e\in F} \chi_e = \sum_{i=1}^{m-1} a_i\chi_{e_i} = 0$, a contradiction to (2). So rank$B_G = m-1$ over $\mathbb{Z}_2$.

(3) $\Rightarrow$ (1). By Lemma 3.3, G is non-odd-transversal. Let e be an arbitrary edge of G. Adding all rows χ_f for $f \neq e$ to the row χ_e will yield a zero row as $\sum_{e\in E(G)} \chi_e = 0$. So rank$B_{G-e}$ = rank$B_G = m-1$ over $\mathbb{Z}_2$, implying that B_{G-e} has full rank over $\mathbb{Z}_2$ with respect to rows. Hence, rankB_{G-e} = rank$(B_{G-e}, \mathbb{1})$ over $\mathbb{Z}_2$, and $G-e$ is odd-transversal by Lemma 3.2. So G is minimal non-odd-transversal.

Of course (2) is equivalent to (4). $\square$

Remark 3.5. From the proof of (3) $\Rightarrow$ (1) in Theorem 3.4, if G is a minimal non-odd-transversal hypergraph with m edges, then any $m-1$ rows of B_G are linear independent over $\mathbb{Z}_2$.

Example 3.6. The following are minimal non-odd-transversal hypergraphs by verifying the degrees and the rank of incidence matrix over $\mathbb{Z}_2$ according to Theorem 3.4, where the last two hypergraphs are square.

(1) $\{1, 2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{1, 5\}$.

(2) $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{1, 4, 5\}$, $\{3, 4\}$.

(3) $\{1, 2, 3\}$, $\{1, 3, 4, 5\}$, $\{1, 2, 4, 6\}$, $\{1, 5, 6, 7\}$, $\{2, 4, 7\}$, $\{2, 5, 6, 7\}$, $\{4, 5, 6, 7\}$.

From Example 3.6, we know a minimal non-odd-transversal hypergraph can contain both even-sized edges and odd-sized edges. In the following we will discuss minimal non-odd-transversal hypergraphs only with even-sized edges.

Corollary 3.7. Let G be a minimal non-odd-transversal hypergraph with n vertices and m edges. Then the following results hold.

(1) $n \geqslant m-1$, and any t edges contain at least t vertices for $1 \leqslant t \leqslant m-1$.

(2) The incidence bipartite graph Γ_G has a matching M such that $E(G)$ is matched by M, namely, there exists an injection $f : E(G) \to V(G)$ such that $f(e) \in e$ and $\{e, f(e)\} \in M$ for each $e \in E(G)$.

(3) If further G contains only even-sized edges, then $n \geqslant m$, and any t edges intersect at least $t+1$ vertices for $1 \leqslant t \leqslant m-1$.

Proof. Consider the incidence matrix B_G of G. By Theorem 3.4(3), rank$B_G = m-1 \leqslant n$. Let $e_1, \ldots, e_t$ be t edges of G, where $1 \leqslant t \leqslant m-1$. Let $U = \cup_{i=1}^t e_i$, and let $B_G[e_1, \ldots, e_t|U]$ be the sub-matrix of B_G with rows indexed $e_1, \ldots, e_t$ and columns indexed by the vertices of U. By Remark 3.5, rank$B_G[e_1, \ldots, e_t|U] = t \leqslant |U|$, yielding the result (1).

The result (2) follows from Hall's Theorem; see Lemma 2.1.

If G contains only even-sized edges, then each row sum of B_G is zero over $\mathbb{Z}_2$, which implies $\mathrm{rank} B_G \leqslant n-1$, and $\mathrm{rank} B_G[e_1,\ldots,e_t|U] \leqslant |U|-1$. So we get the result (3). $\square$

Corollary 3.8. Let G be a square minimal non-odd-transversal hypergraph only with even-sized edges. Then the following results hold.

(1) The incidence bipartite graph Γ_G has a perfect matching, namely, there exists a bijection $f: E(G) \to V(G)$ such that $f(e) \in e$ for each $e \in E(G)$.

(2) For each nonempty proper subset U of $V(G)$, $G|_U$ contains at least $|U|+1$ edges, and also contains odd-sized edges.

Proof. Surely (1) comes from (2) of Corollary 3.7 as G is square. Now let U be a nonempty proper subset of $V(G)$. Let F be the set of edges that intersect U so that $G|_U$ has edges $e \cap U$ for all $e \in F$. If $F = E(G)$, then $|F| = |V(G)| \geqslant |U|+1$ as G is square. Otherwise, we consider the submatrix $B_G[F^c|U^c]$, which has rank $|F^c|$ from the its rows by Remark 3.5. So, $|F^c| = |V(G)| - |F| \leqslant |V(G)| - |U| - 1$ as each row sum of $B_G[F^c|U^c]$ is zero over $\mathbb{Z}_2$, implying that $|F| \geqslant |U|+1$.

Assume to the contrary that each edge of $G|_U$ has even size. Then $B_G[E(G)|U]$, and $B_G[E(G)|U^c]$ as well, has zero row sums. So

$$\begin{aligned} \mathrm{rank} B_G &\leqslant \mathrm{rank} B_G[E(G)|U] + \mathrm{rank} B_G[E(G)|U^c] \\ &\leqslant |U| - 1 + |U^c| - 1 = |V(G)| - 2 = |E(G)| - 2, \end{aligned}$$

a contradiction to Theorem 3.4(3). $\square$

Corollary 3.9. Let G be a square hypergraph only with even-sized edges and even-degree vertices. Then G is minimal non-odd-transversal if and only if its dual G^* is minimal non-odd-transversal.

Proof. Suppose G is minimal non-odd-transversal with n vertices (edges). By Corollary 3.8(2), no two vertices of G lie in precisely the same edges of G. So G^* is also square, and $B_{G^*} = B_G^\top$. As each edge of G is even sized, each vertex of G^* has even degree. So G^* is minimal non-odd-transversal by Theorem 3.4. As G is isomorphic to $(G^*)^*$, G is minimal non-odd-transversal if G^* is. $\square$

4 Minimal non-odd-bipartite regular hypergraphs

In this section we mainly discuss minimal non-odd-transversal k-uniform hypergraphs. By the following lemma, k is necessarily even. So the minimal non-odd-transversal uniform hypergraphs are *exactly* the minimal non-odd-bipartite hypergraphs.

Lemma 4.1. Let G be a minimal non-odd-transversal k-uniform hypergraph. Then k is even. If G is further d-regular, then d is even and $d \leqslant k$.

Proof. Suppose that G has n vertices and m edges. By Theorem 3.4(4), each vertex of G has an even degree so that the sum of degrees of the vertices of G is even, which is equal to mk. As m is odd by Theorem 3.4, k is necessarily even, which implies that $n \geqslant m$ by Corollary 3.7(3). If G is d-regular, d is even by Theorem 3.4(4). Surely $nd = mk$, so $d \leqslant k$ as $n \geqslant m$. $\square$

So, in the following discussion we only deal with non-odd-bipartite regular hypergraphs with even uniformity and even degree.

4.1 2-regular minimal non-odd-bipartite hypergraphs

It is known that the only minimal non-bipartite simple graph is an odd cycle C_{2l+1}, which is 2-regular. As a simple generalization, the generalized power hypergraph $C_{2l+1}^{k,\frac{k}{2}}$ is a 2-regular minimal non-odd-bipartite k-uniform hypergraph. However, the above hypergraph is not the only 2-regular minimal non-odd-bipartite k-uniform hypergraph. For example, the following 4-uniform hypergraph on 10 vertices with 5 edges below is minimal non-odd-bipartite:

$$\{1,2,3,4\}, \{3,4,5,6\}, \{5,6,7,8\}, \{1,7,9,10\}, \{2,8,9,10\}.$$

Lemma 4.2. Let G be a 2-regular uniform hypergraph with an odd number of edges. If G is connected, then G is minimal non-odd-bipartite.

Proof. Let H be a nonempty proper edge-induced sub-hypergraph of G. As G is connected, H contains a vertex v, which is also contained in some edge not in H. So v has degree 1 in H. The result follows by Theorem 3.4(4). $\square$

Next we give a construction of 2-regular k-uniform hypergraphs, where k is an even integer greater than 2.

Construction 4.3. Let k be an even integer greater than 2, and let n, m be positive integers such that $n = \frac{km}{2}$. Let $K_{n,n}$ be a complete bipartite simple graph with two parts V and E, where $V = [n]$ and $E = \cup_{t=1}^{m}\{e_t^1, \ldots, e_t^{\frac{k}{2}}\}$. Let $\hat{K}_{n,n}$ be obtained from $K_{n,n}$ by deleting the edges between the vertices of $V_t := \{\frac{k}{2}(t-1)+1, \ldots, \frac{k}{2}t\}$ and the vertices of $E_t := \{e_t^1, \ldots, e_t^{\frac{k}{2}}\}$ for each $t \in [m]$.

Take a perfect matching $\hat{M}$ of $\hat{K}_{n,n}$, where E_t are matched to $\tilde{V}_t := \{i_{t1}, \ldots, i_{t,\frac{k}{2}}\}$ respectively for $t \in [m]$, such that if $\tilde{V}_t = V_s$ for some $s \neq t$, then $\tilde{V}_s \neq V_t$.

Define a hypergraph $G_{\hat{M}}$ with vertex set $[n]$, whose edges are

$$e_t = V_t \cup \tilde{V}_t, \text{ for } t \in [m]. \tag{4.1}$$

Lemma 4.4. The hypergraph $G_{\hat{M}}$ defined in Construct 4.3 is a 2-regular k-uniform hypergraph on n vertices.

Proof. As there is no edge between V_t and E_t in $\hat{K}_{n,n}$, $V_t \cap \tilde{V}_t = \varnothing$ for each $t \in [m]$. So each edge e_t contains exactly k vertices. Note that $\{V_1, \ldots, V_t\}$ and $\{\tilde{V}_1, \ldots, \tilde{V}_t\}$ both form a t-partition of $[n]$. For each vertex v of G, $v \in V_s$ for a unique $s \in [m]$ and $v \in \tilde{V}_t$ for a unique $t \in [m]$, where $t \neq s$ as $V_s \cap \tilde{V}_s = \varnothing$. So v contained in exactly two edges e_s and e_t, implying v has degree 2. Finally we note that G contains no multiple edges; otherwise, if $e_s = e_t$ for $s \neq t$, then $V_s \cup \tilde{V}_s = V_t \cup \tilde{V}_t$, which implies that $V_s = \tilde{V}_t$ and $V_t = \tilde{V}_s$ as $V_s \cap V_t = \varnothing$ and $\tilde{V}_s \cap \tilde{V}_t = \varnothing$, a contradiction to the assumption. The result follows. $\square$

Corollary 4.5. Any 2-regular k-uniform hypergraph on n vertices can be constructed as in Construction 4.3, where k is even integer greater than 2.

Proof. Let G be a 2-regular k-uniform hypergraph with vertex set $V(G) = [n]$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Surely, $n = \frac{km}{2}$. Let $\bar{G} := \frac{k}{2} \cdot G$ be a k-uniform hypergraph with vertex set $V(G)$ and edge set $\frac{k}{2} \cdot E(G) := \{\frac{k}{2} \cdot e : e \in E(G)\}$, where $\frac{k}{2} \cdot e$ means the $\frac{k}{2}$ copies of e, written as $e^1, \ldots, e^{\frac{k}{2}}$. Then $\bar{G}$ is a k-regular k-uniform multi-hypergraph on n vertices. The incidence bipartite graph $\Gamma_{\bar{G}}$ of $\bar{G}$ is k-regular.

Let $K_{\bar{G}}$ be a complete bipartite graph with two parts $V(\bar{G})$ and $E(\bar{G})$. Let $\hat{K}_{\bar{G}}$ be obtained from $K_{\bar{G}}$ by deleting the edges between the vertices of $V_t := \{\frac{k}{2}(t-1)+1, \ldots, \frac{k}{2}t\}$ and the vertices of $E_t := \{e_t^1, \ldots, e_t^{\frac{k}{2}}\}$ for $t \in [m]$. Then $\hat{K}_{\bar{G}}$ is an $(n - \frac{k}{2})$-regular bipartite graph.

Considering the k-regular bipartite graph $\Gamma_{\bar{G}}$, it contains a perfect matching M. By a possible relabeling of the vertices, we may assume that for each $t \in [m]$, $V_t := \{\frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t\}$ is matched to $E_t := \{e_t^1, \ldots, e_t^{\frac{k}{2}}\}$ in M. By the construction of $\bar{G}$, returning to G, $V_t \subseteq e_t$ for $t \in [m]$.

Now deleting the edges between the vertices of V_t and the vertices of E_t from $\Gamma_{\bar{G}}$ for $t \in [m]$, we arrive at a $\frac{k}{2}$-regular bipartite graph denoted by $\hat{\Gamma}_{\bar{G}}$, which is a subgraph of $\hat{K}_{\bar{G}}$. Now $\hat{\Gamma}_{\bar{G}}$, and hence $\hat{K}_{\bar{G}}$ has a perfect matching $\hat{M}$, where, for each $t \in [m]$, E_t is matched to $\tilde{V}_t := \{i_{t1}, \ldots, i_{t,\frac{k}{2}}\}$ in $\hat{M}$. So, returning to G, $\tilde{V}_t \subseteq e_t$ for $t \in [m]$. As there is no edge between V_t and E_t in $\hat{\Gamma}_{\bar{G}}$, $V_t \cap \tilde{V}_t = \varnothing$ for each $t \in [m]$, which implies that $e_t = V_t \cup \tilde{V}_t$ for $t \in [m]$.

As G contains no multiple edges, if $\tilde{V}_t = V_s$ for some $s \neq t$, surely $\tilde{V}_s \neq V_t$; otherwise $e_t = e_s = V_t \cup V_s$, a contradiction.

From the above discussion, $K_{\bar{G}}$ and $\hat{K}_{\bar{G}}$ are respectively isomorphic to $K_{n,n}$ and $\hat{K}_{n,n}$ in Construction 4.3. A perfect matching $\hat{M}$ in $\hat{\Gamma}_{\bar{G}}$ or $\hat{K}_{\bar{G}}$ is exactly a perfect matching in $\hat{K}_{n,n}$. So G is isomorphic to the hypergraph $G_{\hat{M}}$ as in Construction 4.3. $\square$

Theorem 4.6. Let G be a 2-regular k-uniform hypergraph with n vertices and m edges, where m is odd and k is even. Then G is a minimal non-odd-bipartite hypergraph if and only if G can be constructed as in Construction 4.3 and G is connected.

Proof. The sufficiency follows from Lemmas 4.4 and 4.2, and the necessity follows from Corollary 4.5 and Lemma 3.1. $\square$

Remark 4.7. The hypergraph constructed as in Construction 4.3 may not be connected. However, by Lemma 4.2 at least one component is minimal non-odd-bipartite as the total number of edges is odd. For example, the following 4-uniform hypergraph G on 18 vertices with edges

$$e_t = \{2t-1, 2t, 2t+5, 2t+6\}, t \in [9],$$

where the labels of the vertices are modulo 18. G has 3 connected components G_1, G_2, G_3 with edge sets listed below, each of which is isomorphic to $C_3^{4,2}$ (a minimal non-odd-bipartite hypergraph).

$$\begin{array}{llll} E(G_1): & \{1,2,7,8\}, & \{7,8,13,14\}, & \{13,14,1,2\}. \\ E(G_2): & \{3,4,9,10\}, & \{9,10,15,16\}, & \{15,16,3,4\}. \\ E(G_3): & \{5,6,11,12\}, & \{11,12,17,0\}, & \{17,0,5,6\}. \end{array}$$

In Fig. 4.1 we give an illustration of G constructed as in the way of Construction 4.3, where the dotted lines indicate a perfect matching in $K_{18,18}$, and the solid lines indicate a perfect matching in $\hat{K}_{18,18}$.

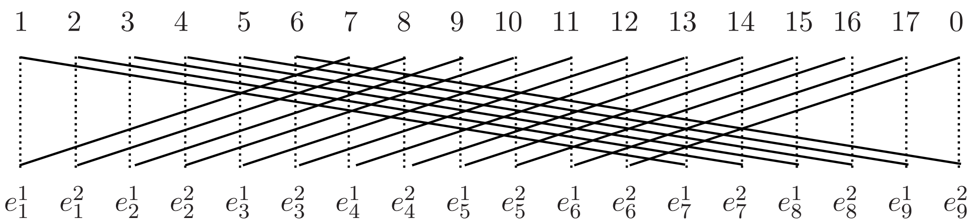

Figure 4.1: An illustration of Construction 4.3

4.2 Examples of d-regular minimal non-odd-bipartite hypergraphs

We first give an example of k-regular k-uniform minimal non-odd-bipartite hypergraph by using Cayley hypergraph. Let $G = (\mathbb{Z}_n; \{1, 2, \ldots, k-1\})$ be a *Cayley hypergraph*, where $V(G) = \mathbb{Z}_n$, and $E(G)$ consists of edges $\{i, i+1, \ldots, i+k-1\}$ for $i \in \mathbb{Z}_n$. Then G is connected, k-uniform and k-regular, with n vertices and n edges.

Theorem 4.8. Let k be an even integer greater than 2, and n be an odd integer greater than k. The $G = (\mathbb{Z}_n; \{1, 2, \ldots, k-1\})$ is minimal non-odd-bipartite if and only if $\gcd(k, n) = 1$.

Proof. By Theorem 3.4(3), it suffices to show that $\text{rank} B_G = n-1$ over $\mathbb{Z}_2$ if and only if $\gcd(k, n) = 1$. Consider the equation $B_G x = 0$ over $\mathbb{Z}_2$. For each $i \in \mathbb{Z}_n$, as $\{i, \ldots, i+k-1\}$ and $\{i+1, \ldots, k\}$ are edges of G, by the above equation we have

$$x_i + \cdots + x_{i+k-1} = 0, x_{i+1} + \cdots + x_{i+k} = 0.$$

So $x_i = x_{i+k}$ for each $i \in \mathbb{Z}_n$. Let $t := \gcd(k, n)$. Then there exist integers p, q such that $pk + qn = t$. Note that t is odd as n is odd, and if writing $k = st$, then s is even as k is even.

For each $i \in \mathbb{Z}_n$,

$$x_i = x_{i+k} = \cdots = x_{i+pk} = x_{i+t-qn} = x_{i+t}. \tag{4.2}$$

As s is even, for any $x_1, \ldots, x_t \in \mathbb{Z}_2$, and for any edge $\{i, \ldots, i+k-1\}$, by Eq. (4.2),

$$\begin{aligned} x_i + \cdots + x_{i+k-1} &= x_1 + \cdots + x_k \\ &= (x_1 + \cdots + x_t) + \cdots + (x_{(s-1)t+1} + \cdots + x_{st}) \\ &= s(x_1 + \cdots + x_t) = 0. \end{aligned}$$

So, the solution space of $B_G x = 0$ over $\mathbb{Z}_2$ has dimension t, which implies that $\text{rank} B_G = n - t$ over $\mathbb{Z}_2$. The result now follows. □

Let G be a k-uniform hypergraph with n vertices and m edges. Let $G^1, \ldots, G^t$ be t disjoint copies of G. For each vertex v (or each edge e) of G, it has t copies $v^1, \ldots, v^t$ (or $e^1, \ldots, e^t$) in $G^1, \ldots, G^t$ respectively. Let $t \circ G$ be a hypergraph whose vertex set is $\cup_{i=1}^t V(G^i)$, and edge set is $\{e^1 \cup \cdots \cup e^t : e \in E(G)\}$. Then $t \circ G$ is a tk-uniform hypergraph with tn vertices and m edges, and the degree of v^i in $t \circ G$ is same as the degree of v in G for each $v \in V(G)$ and $i \in [t]$. If further G is d-regular, then $t \circ G$ is also d-regular.

Lemma 4.9. Let G be a k-uniform hypergraph. Then G is minimal non-odd-bipartite if and only if $t \circ G$ is minimal non-odd-bipartite.

Proof. By a suitable labeling of the vertices of $t \circ G$, we have $B_{t \circ G} = (B_G, B_G, \ldots, B_G)$, where B_G occurs t times in the latter matrix. As $\text{rank} B_G = \text{rank} B_{t \circ G}$, the result follows by Theorem 3.4(3). □

Next we give an example of d-regular k-uniform minimal non-odd-bipartite hypergraph G with n vertices and m edges, where m is odd and d is even such that $\gcd(d, m) = 1$. Obviously $nd = km$, and $d | k$ as $\gcd(d, m) = 1$. Suppose $k = td$, where $t > 1$. By Theorem 4.8, the hypergraph $H = (\mathbb{Z}_m; \{1, 2, \ldots, d-1\})$ is minimal non-odd-bipartite, which is d-regular, d-uniform, with m edges. By Lemma 4.9, $t \circ H$ is minimal non-odd-bipartite with m edges, which is d-regular and $td(= k)$-uniform.

Corollary 4.10. Let $H = (\mathbb{Z}_m; \{1, 2, \ldots, d-1\})$, where m is odd and d is even such that $\gcd(d, m) = 1$. Then $t \circ H$ is minimal non-odd-bipartite with m edges, which is d-regular and td-uniform.

Note that in Corollary 4.10, if $d = 2$, then H is an odd cycle C_m, and $t \circ C_m = C_m^{2t,t}$ (a generalized power hypergraph), both of which are minimal non-odd-bipartite.

Thirdly we use a projective plane $(X, \mathcal{B})$ of order q to construct a regular minimal non-odd-bipartite hypergraph. Recall a *projective plane* of order q consists of a set X of $q^2 + q + 1$ elements called *points*, and a set $\mathcal{B}$ of $(q+1)$-subsets of X called *lines*, such that any two points lie on a unique line. It can be derived from the definition that any point lies on $q + 1$ lines, and two lines meet in a unique point, and there are $q^2 + q + 1$ lines. Now define a hypergraph based on $(X, \mathcal{B})$, denoted by $G = (X, \mathcal{B})$, whose vertices are the points of X and edges are the lines of $\mathcal{B}$. Then $G = (X, \mathcal{B})$ is a $(q+1)$-regular $(q+1)$-uniform hypergraph with $q^2 + q + 1$ vertices.

Theorem 4.11. Let $(X, \mathcal{B})$ be a projective plane of order q, and let $G = (X, \mathcal{B})$ be a hypergraph defined as in the above. If q is odd, then $G = (X, \mathcal{B})$ is minimal non-odd-bipartite.

Proof. Let e be an edge of $G = (X, \mathcal{B})$ or a line of $(X, \mathcal{B})$. Then

$$B_{G-e}B_{G-e}^{\top} = qI + J,$$

where I is the identity matrix, and J is an all-ones matrix, both of size $q^2 + q$. So

$$\det(B_{G-e}B_{G-e}^{\top}) = \det(qI + J) = (q^2 + 2q)q^{q^2+q-1} \equiv 1 \mod 2,$$

implying that $\mathrm{rank} B_G = m - 1$ over $\mathbb{Z}_2$, where m is the number of edges of G. The result follows by Theorem 3.4(3). $\square$

It is known that if q is an odd prime power, then there always exists a projective plane of order q by using the vector space $\mathbb{F}_q^3$. By Lemma 4.9 and Theorem 4.11, we easily get the following result.

Corollary 4.12. Let q be an odd prime power. There exists a $(q + 1)$-regular $(q + 1)$-uniform minimal non-odd-bipartite hypergraph with $q^2 + q + 1$ edges. For any positive integer $t > 1$, there exists a $(q + 1)$-regular $t(q + 1)$-uniform minimal non-odd-bipartite hypergraph with $q^2 + q + 1$ edges.

Remark 4.13. From Corollary 4.8 and Corollary 4.12, the minimal non-odd-bipartite hypergraphs G have degree d and edge number m such that $\gcd(d, m) = 1$. (Note that $\gcd(q + 1, q^2 + q + 1) = 1$.) As $\gcd(d, m) = 1$, from the equality $nd = mk$, we have $d \mid k$, where n, k are the number of vertices and the uniformity of G respectively.

In fact, there exist d-regular minimal non-odd-bipartite hypergraphs with m edges such that $\gcd(d, m) > 1$. For example, let G be a 6-uniform 6-regular hypergraph with 9 edges below:

$$\begin{array}{lllll}\{1,2,3,4,5,6\}, & \{1,4,5,6,7,9\}, & \{1,3,5,6,7,8\}, & \{1,2,4,6,7,8\}, & \{1,3,5,7,8,9\}, \\ \{1,2,6,7,8,9\}, & \{2,3,4,5,7,9\}, & \{2,3,4,5,8,9\}, & \{2,3,4,6,8,9\}. & \end{array}$$

By Theorem 3.4, it is easy to verify that G is minimal non-odd-bipartite.

There also exist d-regular k-uniform minimal non-odd-bipartite hypergraphs such that $d \nmid k$. For example, let G be a 6-regular 8-uniform hypergraph with 9 edges below:

$$\begin{array}{lll}\{1,2,3,4,5,6,7,8\}, & \{1,3,4,5,6,7,9,11\}, & \{1,4,5,6,7,8,9,10\}, \\ \{1,5,7,8,9,10,11,12\}, & \{1,2,3,6,7,9,10,12\}, & \{1,2,4,5,8,10,11,12\}, \\ \{2,3,4,6,8,10,11,12\}, & \{2,3,4,5,8,9,11,12\}, & \{2,3,6,7,9,10,11,12\}.\end{array}$$

By Theorem 3.4, it is also easy to verify that G is minimal non-odd-bipartite.

Example 4.14. *The minimal non-odd-bipartite uniform hypergraph with fewest edges.* By Theorem 3.4, if G is a k-uniform minimal non-odd-bipartite hypergraph with n vertices and m edges, then m is odd. If $m = 1$, G is surely odd-bipartite. So, $m \geqslant 3$, and hence the maximum degree is at most 3 if $m = 3$. Assume that $m = 3$. By Theorem 3.4, each vertex has an even degree, implying that G is 2-regular. So $2n = 3k$, and $3|n$. Letting $n = 3l$, we have $k = 2l$. So $G = C_3^{2l,l}$, which is the unique example of minimal non-odd-bipartite hypergraph with 3 edges. It is consistent with the fact that C_3 is the unique minimal non-bipartite simple graph with 3 edges by taking $k = 2$.

Example 4.15. *The minimal non-odd-bipartite uniform hypergraph with fewest vertices.* If G is a k-uniform minimal non-odd-bipartite hypergraph with n vertices and m edges. Then $n \geqslant k+1$, as an edge is odd-bipartite. Assume that $n = k+1$. Then $m \leqslant \binom{k+1}{k} = k+1$, with equality if and only if G is a $(k+1)$-*simplex* denoted by $K_{k+1}^{[k]}$ ([8]), i.e. any k vertices of $K_{k+1}^{[k]}$ forms an edge. Let Δ be the maximum degree of G, which is even by Theorem 3.4. As m is odd by Theorem 3.4, we have

$$m - 1 \geqslant \Delta \geqslant \frac{km}{k+1} = m - \frac{m}{k+1} \geqslant m - 1,$$

which implies that $m = k+1$ and k is even. So, $K_{k+1}^{[k]}$ is the unique example of k-uniform minimal non-odd-bipartite hypergraph with $k+1$ vertices by Theorem 3.4. If taking $k = 2$, then C_3 is the the unique minimal non-bipartite simple graph with 3 vertices.

Example 4.16. *Example of non-regular minimal non-odd-bipartite hypergraph.* Let G be a 4-uniform hypergraph on vertices $1, 2, \ldots, 9$ with edges

$$\{1,2,3,4\}, \{1,2,4,5\}, \{1,3,6,7\}, \{1,5,8,9\}, \{6,7,8,9\}.$$

It is easy to verify that G is non-regular minimal non-odd-bipartite by Theorem 3.4.

Remark 4.17. *Example of minimal non-odd-bipartite hypergraph with cut edges.* Let G be a 4-uniform hypergraph G with vertices $1, 2, \ldots, 18$ and 9 edges:

$$\begin{array}{lllll} \{1,3,4,5\}, & \{2,3,4,6\}, & \{5,7,8,9\}, & \{6,7,8,9\}, & \{1,2,10,11\}, \\ \{10,12,13,14\}, & \{11,12,13,15\}, & \{14,16,17,18\}, & \{15,16,17,18\}, & \end{array}$$

where $\{1,2,10,11\}$ is a cut edge of G. By Theorem 3.4, G is minimal non-odd-bipartite.

5 Least H-eigenvalue of minimal non-odd-bipartite hypergraphs

Let G be a k-uniform minimal hypergraph. Let $x \in \mathbb{C}^{V(G)}$ whose entries are indexed by the vertices of G. For a subset U of $V(G)$, denote $x^U := \Pi_{v \in U} x_v$. Then we have

$$\mathcal{A}(G)x^k = \sum_{e \in E(G)} kx^e, \tag{5.1}$$

Theorem 5.1. Let G be a k-uniform minimal non-odd-bipartite hypergraph with n vertices and m edges, where k is even. Then

(1) $\lambda_{\min}(G) \leqslant -\rho(G) + \frac{2k}{n^{1/k}}$.

(2) $\lambda_{\min}(G) \leqslant -(1 - \frac{2}{m})\rho(G)$.

Proof. As G is connected by Lemma 3.1, by Perron-Frobenius theorem, there exists a positive eigenvector x of $\mathcal{A}(G)$ associated with the spectral radius $\rho(G)$. We may assume $\|x\|_k = 1$. Then

$$\rho(G) = \mathcal{A}(G)x^k = \sum_{e \in E(G)} kx^e. \tag{5.2}$$

Observe that there exists a vertex u such that $x_u \leqslant \frac{1}{n^{1/k}}$. Let $\bar{e}$ be an edge of G containing u. Then

$$x^{\bar{e}} = x_u \prod_{v \in \bar{e}, v \neq u} x_v \leqslant \frac{1}{n^{1/k}}.$$

By the definition, $G - \bar{e}$ is odd-bipartite with an odd-bipartition $\{U, U^c\}$. Now define a vector y on the vertices of $G - \bar{e}$ such that $y_v = x_v$ if $v \in U$ and $y_v = -x_v$ otherwise. Note that $\bar{e}$ intersects U^c in an even number of vertices as G is non-odd-bipartite, which implies that $y^{\bar{e}} = x^{\bar{e}} > 0$. By Lemma 2.3(2) and Eq. (5.2),

$$\lambda_{\min}(G) \leqslant \mathcal{A}(G)y^k = -\mathcal{A}(G)x^k + 2kx^{\bar{e}} \leqslant -\rho(G) + \frac{2k}{n^{1/k}}.$$

For the second result, from Eq. (5.2), there exists one edge $\hat{e}$ such that $kx^{\hat{e}}$ is not greater than the average of the summands kx^e over all m edges e of G, that is,

$$kx^{\hat{e}} \leqslant \frac{\rho(G)}{m}.$$

Note that $G - \hat{e}$ is also odd-bipartite with an odd-bipartition say $\{W, W^c\}$. Now define a vector z on the vertices of $G - \hat{e}$ such that $z_v = x_v$ if $v \in W$ and $z_v = -x_v$ otherwise. By a similar discussion as the above, we have

$$\lambda_{\min}(G) \leqslant A(G)z^k = -\mathcal{A}(G)x^k + 2kx^{\hat{e}} \leqslant -(1 - \frac{2}{m})\rho(G).$$

□

Corollary 5.2. Let k be a positive even integer. For any $\epsilon > 0$, for any k-uniform minimal non-odd-bipartite hypergraph G with sufficiently larger number of vertices or edges,

(1) $-\rho(G) < \lambda_{\min}(G) < -\rho(G) + \epsilon$,

(2) $-1 < \lambda_{\min}(G)/\rho(G) < -1 + \epsilon$.

For a connected k-uniform hypergraph G, where k is even, if we denote

$$\alpha(G) := \rho(G) + \lambda_{\min}(G),\ \beta(G) := -\lambda_{\min}(G)/\rho(G),$$

then by Lemma 2.4, $\alpha(G) \geqslant 0$, with equality if G is odd-bipartite; and $0 < \beta(G) \leqslant 1$, with right equality if and only if G is odd-bipartite. So we can use $\alpha(G)$ and $\beta(G)$ to measure the non-odd-bipartiteness of an even uniform hypergraph.

Furthermore, by Theorem 5.1 and Corollary 5.2, if G is minimal non-odd-bipartite, then $\alpha(G) \to 0$ and $\beta(G) \to 1$ when the number of vertices or edges of G goes to infinity. So, the minimal non-odd-bipartite hypergraphs are very close to be odd-bipartite in this sense.

References

[1] R. Aharoni and N. Linial. Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas. *J. Combinatorial Theory, Ser. A*, 43:196–204, 1986.

[2] N. Alon. Transversal numbers of uniform hypergraphs. *Graphs Combin.*, 6:1–4, 1990.

[3] N. Alon and Z. Bregman. Every 8-uniform 8-regular hypergraph is 2-colorable. *Graphs Combin.*, 4:303–306, 1988.

[4] J. Cardinal, M. Karpinski, R. Schmied, and C. Viehmann. Approximating vertex cover in dense hypergraphs. *J. Discrete Algorithms*, 13:67–77, 2012.

[5] K. C. Chang, K. Pearson, and T. Zhang. Perron-Frobenius theorem for nonnegative tensors. *Commun. Math. Sci.*, 6:507–520, 2008.

[6] K. C. Chang, K. Pearson, and T. Zhang. On eigenvalue problems of real symmetric tensors. *J. Math. Anal. Appl.*, 350:416–422, 2009.

[7] V. Chvátal and C. McDiarmid. Small transversals in hypergraphs. *Combinatorica*, 12:19–26, 1992.

[8] J. Cooper and A. Dutle. Spectra of uniform hypergraphs. *Linear Algebra Appl.*, 436(9):3268–3292, 2012.

[9] R. Cowen, S. H. Hechler, J. W. Kennedy, and A. Steinberg. Odd neighborhood transversals on grid graphs. *Discrete Math.*, 307:2200–2208, 2007.

[10] Y.-Z. Fan, Y.-H. Bao, and T. Huang. Eigenvariety of nonnegative symmetric weakly irreducible tensors associated with spectral radius and its application to hypergraphs. *Linear Algebra Appl.*, 564:72–94, 2019.

[11] Y.-Z. Fan, T. Huang, Y.-H. Bao, C.-L. Zhuan-Sun, and Y.-P. Li. The spectral symmetry of weakly irreducible nonnegative tensors and connected hypergraphs. *Trans. Amer. Math. Soc.*, 372(3):2213–2233, 2019.

[12] Y.-Z. Fan, Y. Wang, Y.-H. Bao, J.-C. Wan, M. Li, and Z. Zhu. Eigenvectors of Laplacian or signless Laplacian of hypergraphs associated with zero eigenvalue. *Linear Algebra Appl.*, 579:244–261, 2019.

[13] S. Friedland, S. Gaubert, and L. Han. Perron-Frobenius theorem for nonnegative multilinear forms and extensions. *Linear Algebra Appl.*, 438:738–749, 2013.

[14] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, New York, 1977.

[15] M. A. Henning and A. Yeo. 2-colorings in k-regular k-uniform hypergraphs. *European J. Combin.*, 34:1192–1202, 2013.

[16] M. A. Henning and A. Yeo. Hypergraphs with large transversal number. *Discrete Math.*, 313:959–966, 2013.

[17] M. E. Ouali, H. Fohlin, and A. Srivastav. A randomised approximation algorithm for the hitting set problem. *Theoret. Comput. Sci.*, 555:23–34, 2014.

[18] S. Hu and L. Qi. The eigenvectors associated with the zero eigenvalues of the Laplacian and signless Laplacian tensors of a uniform hypergraph. *Discrete Appl. Math.*, 169:140–151, 2014.

[19] S. Hu, L. Qi, and J.-Y. Shao. Cored hypergraphs, power hypergraphs and their Laplacian H-eigenvalues. *Linear Algebra Appl.*, 439:2980–2998, 2013.

[20] M. Khan and Y.-Z. Fan. On the spectral radius of a class of non-odd-bipartite even uniform hypergraphs. *Linear Algebra Appl.*, 480:93–106, 2015.

[21] L.-H. Lim. Singular values and eigenvalues of tensors: a variational approach. In *Proceedings of the 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, pages 129–132, 2005.

[22] V. Nikiforov. Analytic methods for uniform hypergraphs. *Linear Algebra Appl.*, 457:455–535, 2014.

[23] V. Nikiforov. Hypergraphs and hypermatrices with symmetric spectrum. *Linear Algebra Appl.*, 519:1–18, 2017.

[24] K. Pearson and T. Zhang. On spectral hypergraph theory of the adjacency tensor. *Graphs Combin.*, 30(5):1233–1248, 2014.

[25] L. Qi. Eigenvalues of a real supersymmetric tensor. *J. Symbolic Comput.*, 40(6):1302–1324, 2005.

[26] D. Rautenbach and Z. Szigeti. Greedy colorings of words. *Discrete Appl. Math.*, 160:1872–1874, 2012.

[27] P. D. Seymour. On the two-coloring of hypergraphs. *Quart. J. Math. Oxford Ser.*, 25(3): 303–312, 1974.

[28] J.-Y. Shao, H.-Y. Shan, and B.-F. Wu. Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs. *Linear Multilinear Algebra*, 63:2359–2372, 2015.

[29] Y. Yang and Q. Yang. Further results for Perron-Frobenius theorem for nonnegative tensors. *SIAM J Matrix Anal. Appl.*, 31(5):2517–2530, 2010.

[30] Y. Yang and Q. Yang. Further results for Perron-Frobenius theorem for nonnegative tensors II. *SIAM J Matrix Anal. Appl.*, 32(4):1236–1250, 2011.

[31] Y. Yang and Q. Yang. On some properties of nonnegative weakly irreducible tensors. arXiv:1111.0713v2, 2011.

[32] G. Zhou, L. Qi, and S.-Y. Wu. On the largest eigenvalue of a symmetric nonnegative tensor. *Numer. Linear Algebra Appl.*, 20(6):913–928, 2013.