Minimal non-odd-transversal hypergraphs and minimal non-odd-bipartite hypergraphs

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Abstract

Among all uniform hypergraphs with even uniformity, the odd-transversal or odd-bipartite hypergraphs are closer to bipartite simple graphs than bipartite hypergraphs from the viewpoint of both structure and spectrum. A hypergraph is called odd-transversal if it contains a subset of the vertex set such that each edge intersects the subset in an odd number of vertices, and it is called minimal non-odd-transversal if it is not odd-transversal but deleting any edge results in an odd-transversal hypergraph. In this paper we give an equivalent characterization of the minimal non-odd-transversal hypergraphs by means of the degrees and the rank of its incidence matrix over $\mathbb{Z}_2$. If a minimal non-odd-transversal hypergraph is uniform, then it has even uniformity, and hence is minimal non-odd-bipartite. We characterize 2-regular uniform minimal non-odd-bipartite hypergraphs, and give some examples of $d$-regular uniform hypergraphs which are minimal non-odd-bipartite. Finally we give upper bounds for the least H-eigenvalue of the adjacency tensor of minimal non-odd-bipartite hypergraphs.

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1 Introduction

Let $G = (V, E)$ be a hypergraph, where $V =: V(G)$ is the vertex set, and $E =: E(G)$ is the edge set whose elements $e \subseteq V$. If for each edge $e$ of $G$, $|e| = k$, then $G$ is called a $k$-uniform hypergraph. The degree $d(v)$ of a vertex $v$ of $G$ is defined to be the number of edges of $G$ containing $v$. If $d(v) = d$ for all vertices $v$ of $G$, then $G$ is called $d$-regular.

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A hypergraph $G$ is called $2$-colorable if there exists a 2-coloring of the vertices of $V(G)$ such that $G$ contains no monochromatic edges; and it is called minimal non-$2$-colorable if it is not 2-colorable but deleting any edge from $E(G)$ results in a 2-colorable hypergraph. Seymour [27] proved that if $G$ is minimal non-$2$-colorable and $V(G) = \bigcup\{e : e \in E(G)\}$, then $|E(G)| \geq |V(G)|$. Aharoni and Linial [1] presented an infinite version of Seymour’s result. Alon and Bregman [3] proved that if $k \geq 8$ then every $k$-regular $k$-uniform hypergraph is 2-colorable. Henning and Yeo [15] showed that Alon-Bergman result is true for $k \geq 4$.

A subset $U$ of $V(G)$ is called a transversal (also called vertex cover [4] or hitting set [17]) of $G$ if each edge of $G$ has a nonempty intersection with $U$. The transversal number of $G$ is the minimum size of transversals in $G$, which was well studied by Alon [2], Chvátal and McDiarmid [7], Henning and Yeo [16]. A hypergraph $G$ is called bipartite if for some nonempty proper subset $U \subseteq V(G)$, $U$ and its complement $U^c$ are both transversal; or equivalently the vertex set $V(G)$ has a bipartition into two parts such that every edge of $E(G)$ intersects both parts. Surely, $G$ is bipartite if and only if $G$ is 2-colorable.

A subset $U$ of $V(G)$ is called an odd transversal of $G$ if each edge of $G$ has an odd number of vertices [9, 26]. A hypergraph $G$ is called odd-transversal if it has an odd transversal; otherwise, $G$ is called non-odd-transversal Nikiforov [22] firstly uses odd transversal to investigate the spectral symmetry of tensors and hypergraphs. The notion of odd-bipartite hypergraphs was introduced by Hu and Qi [18] to study the zero eigenvalue of the signless Laplacian tensor.

Definition 1.1 ([18]). Let $G$ be a $k$-uniform hypergraph $G$, where $k$ is even. If there exists a bipartition $\{U, U^c\}$ of $V(G)$ such that each edge of $G$ intersects $U$ (and also $U^c$) in an odd number of vertices, then $G$ is called odd-bipartite, and $\{U, U^c\}$ is an odd-bipartition of $G$; otherwise, $G$ is called non-odd-bipartite.

So, odd-bipartite hypergraphs are surely odd-transversal hypergraphs and bipartite hypergraphs. For the uniform hypergraphs with even uniformity, the notion of odd-bipartite hypergraphs is equivalent to that of odd-transversal hypergraphs.

From the viewpoint of spectrum, a simple graph is bipartite if and only if its adjacency matrix has a symmetric spectrum. However, the adjacency tensor of a bipartite uniform hypergraph does not possess such property. We note that the hypergraphs under consideration are uniform when discussing their spectra. Shao et al. [28] proved that the adjacency tensor of a $k$-uniform hypergraph $G$ has a symmetric H-spectrum (the set of eigenvalues associated with real eigenvectors) if and only if $k$ is even and $G$ is odd-bipartite. So, the odd-bipartite hypergraphs are closer to bipartite simple graphs than the bipartite hypergraphs based on the following two reasons. First they both have a structural property, namely, there exists a bipartition of the vertex set such that every edge intersects the each part of the bipartition in an odd number of vertices. Second they both have a symmetric H-spectrum.

There are some examples of odd-bipartite hypergraphs, e.g. power of simple graphs and cored hypergraphs [19], hm-hypergraphs [18], $m$-partite $m$-uniform hypergraphs [8]. Nikiforov [23] gives two classes of non-odd-transversal hypergraphs. Fan et al. [20] construct non-odd-bipartite generalized power hypergraphs from non-bipartite simple graphs.
It is known that a connected bipartite simple graph has a unique bipartition up to isomorphism. However, an odd-bipartite hypergraph can have more than one odd-bipartition. Fan et al. [12] given a explicit formula for the number of odd-bipartition of a hypergraph by the rank of its incidence matrix over $\mathbb{Z}_2$. So, it seems hard to give examples of non-odd-bipartite hypergraphs.

To our knowledge, there is no characterization of non-odd-transversal or non-odd-bipartite hypergraphs. We observe that non-odd-transversal (or non-odd-bipartite) hypergraphs have a hereditary property, that is, if $G$ contains a non-odd-transversal (or non-odd-bipartite) sub-hypergraph, then $G$ is non-odd-transversal (or non-odd-bipartite). We call $G$ minimal non-odd-transversal (or minimal non-odd-bipartite), if $G$ is non-odd-transversal (or non-odd-bipartite) but deleting any edge from $G$ results in an odd-transversal (or odd-bipartite) hypergraph, or equivalently, any nonempty proper edge-induced sub-hypergraph of $G$ is odd-transversal (or odd-bipartite). In this paper we give an equivalent characterization of the minimal non-odd-transversal hypergraphs by means of the degrees and the rank of its incidence matrix over $\mathbb{Z}_2$. If a minimal non-odd-transversal hypergraph is uniform, then it has even uniformity, and hence is minimal non-odd-bipartite. We characterize 2-regular uniform minimal non-odd-bipartite hypergraphs, and give some examples of $d$-regular uniform hypergraphs which are minimal non-odd-bipartite. Finally we give upper bounds for the least $H$-eigenvalue of the adjacency tensor of minimal non-odd-bipartite hypergraphs.

2 Basic notions

Unless specified somewhere, all hypergraphs in this paper contain no multiple edges or isolated vertices, where vertex is called isolated if it is not contained in any edge of the hypergraph. Let $G = (V, E)$ be a hypergraph. The hypergraph $G$ is called square if $|V| = |E|$. A walk of length $t$ in $G$ is a sequence of alternate vertices and edges: $v_0e_1v_1e_2\ldots e_tv_t$, where $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, t - 1$; and $G$ is said to be connected if every two vertices are connected by a walk.

The vertex-induced sub-hypergraph of $G$ by the a subset $U \subseteq V(G)$, denoted by $G|_U$, is a hypergraph with vertex set $U$ and edge set $\{e \cap U : e \in E(G), e \cap U \neq \emptyset\}$. For a connected hypergraph $G$, a vertex $v$ is called a cut vertex of $G$ if $G|_{V(G) \setminus \{v\}}$ is disconnected. The edge-induced sub-hypergraph of $G$ by a subset $F \subseteq E(G)$, denoted by $G|_F$, is a hypergraph with vertex set $U \cup \{e : e \in F\}$ and edge set $F$.

Let $G$ be a hypergraph and let $e$ be an edge of $G$. Denote by $G - e$ the hypergraph obtained from $G$ by deleting the edge $e$ from $E(G)$. For a connected hypergraph $G$, an edge $e$ is called a cut edge of $G$ if $G - e$ is disconnected.

A matching $M$ of $G$ is a set of pairwise disjoint edges of $G$. In particular, if $G$ is bipartite simple graph with a bipartition $\{V_1, V_2\}$, a vertex subset $U_1 \subseteq V_1$ is matched to $U_2 \subseteq V_2$ in $M$, if there exists a bijection $f : U_1 \rightarrow U_2$ such that $\{\{v, f(v)\} : v \in U_1\} \subseteq M$. A subset $U$ of $V_1$ (or $V_2$) is matched by $M$ if every vertex of $U$ is incident with an edge of $M$. $M$ is called a perfect matching if $V_1$ and $V_2$ are both matched by $M$. The following result is known as Hall’s Theorem.
Lemma 2.1. (Hall 1935) Let $G$ be a bipartite simple graph with a bipartition $\{V_1, V_2\}$. Then $G$ contains a matching $M$ such that $V_1$ is matched by $M$ if and only if $|N(S)| \geq |S|$ for every $S \subseteq V_1$, where $N(S)$ denotes the set of all vertices of $V_2$ adjacent to some vertex of $S$.

Let $G$ be a hypergraph. The incidence bipartite graph $\Gamma_G$ of $G$ is a bipartite simple graph with two parts $V(G)$ and $E(G)$ such that $\{v, e\} \in E(\Gamma_G)$ if and only if $v \in e$.

The edge-vertex incidence matrix of $G$, denoted by $B_G = (b_{e,v})$, is a matrix of size $|E(G)| \times |V(G)|$, whose entries $b_{e,v} = 1$ if $v \in e$, and $b_{e,v} = 0$ otherwise.

The dual of $G$, denoted by $G^*$, is the hypergraph whose vertex set is $E(G)$ and edge set is $\{\{e \in E(G) : v \in e\} : v \in V(G)\}$. If no two vertices of $G$ are contained in precisely the same edges of $G$, then $G^*$ contains no multiple edges. In this situation, $(G^*)^*$ is isomorphic to $G$, $\Gamma_G$ is isomorphic to $\Gamma_{G^*}$, and $B_G = B_{G^*}^\top$, where the latter denotes the transpose of $B_G$.

Let $G$ be a simple graph, and let $k$ be an even integer greater than 2. Denote by $G^{k, \frac{n}{2}}$ the hypergraph obtained from $G$ whose vertex set is $\bigcup_{e \in V(G)} v$ and edge set $\{u \cup v : \{u, v\} \in E(G)\}$, where $v$ denotes an $\frac{n}{2}$-set corresponding to $v$, and all those sets are pairwise disjoint; intuitively $G^{k, \frac{n}{2}}$ is obtained from $G$ by blowing up each vertex into an $\frac{n}{2}$-set and preserving the adjacency relation [20]. It is proved that $G^{k, \frac{n}{2}}$ is non-odd-bipartite if and only if $G$ is non-bipartite [20].

Next we will introduce some knowledge of eigenvalues of a tensor. For integers $k \geq 2$ and $n \geq 2$, a tensor (also called hypermatrix) $T = (t_{i_1i_2\ldots i_k})$ of order $k$ and dimension $n$ refers to a multidimensional array $t_{i_1i_2\ldots i_k} \in \mathbb{C}$ for all $i_j \in [n] := \{1, 2, \ldots, n\}$ and $j \in [k]$. The tensor $T$ is called symmetric if its entries are invariant under any permutation of their indices.

Given a vector $x \in \mathbb{C}^n$, $Tx^k \in \mathbb{C}$ and $Tx^{k-1} \in \mathbb{C}^n$, which are defined as follows:

$$
Tx^k = \sum_{i_1i_2\ldots i_k \in [n]} t_{i_1i_2\ldots i_k}x_{i_1}x_{i_2}\ldots x_{i_k},
$$

$$(Tx^{k-1})_i = \sum_{i_2\ldots i_k \in [n]} t_{i_2i_3\ldots i_k}x_{i_2}x_{i_3}\ldots x_{i_k}, \text{ for } i \in [n].$$

Let $I$ be the identity tensor of order $k$ and dimension $n$, that is, $i_1i_2\ldots i_k = 1$ if and only if $i_1 = i_2 = \cdots = i_k \in [n]$ and zero otherwise.

Definition 2.2 ([21, 25]). Let $T$ be a $k$-th order $n$-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda I - T)x^{k-1} = 0$, or equivalently $T_x^{k-1} = \lambda x^{[k-1]}$, has a solution $x \in \mathbb{C}^n \setminus \{0\}$, then $\lambda$ is called an eigenvalue of $T$ and $x$ is an eigenvector of $T$ associated with $\lambda$, where $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \in \mathbb{C}^n$.

The characteristic polynomial $\varphi_T(\lambda)$ of $T$ is defined as the resultant of the polynomials $(\lambda I - T)x^{k-1}$; see [25, 6, 14]. It is known that $\lambda$ is an eigenvalue of $T$ if and only if it is a root of $\varphi_T(\lambda)$. The spectrum of $T$ is the multi-set of the roots of $\varphi_T(\lambda)$. The spectral radius of $T$ is defined as the maximum modulus of the eigenvalues of $T$, denoted by $\rho(T)$.
Suppose that $T$ is real. If $x$ is a real eigenvector of $T$, surely the corresponding eigenvalue $\lambda$ is real. In this case, $x$ is called an $H$-eigenvector and $\lambda$ is called an $H$-eigenvalue. The $H$-spectrum of $T$ is the set of all $H$-eigenvalues of $T$, denote by $H\text{Spec}(T)$. Denote by $\lambda_{\text{max}}(T)$ and $\lambda_{\text{min}}(T)$ the largest $H$-eigenvalue and the least $H$-eigenvalue of $T$, respectively.

For a real symmetric tensor, Zhou et al. [32] and Qi [25] give the following results.

**Lemma 2.3.** Let $T$ be a real symmetric tensor of order $k$ and dimension $n$.

1. ([32, Theorem 3.6]) If $T$ is also nonnegative, then
   
   $$\lambda_{\text{max}}(T) = \min \{ T x^k : x \in \mathbb{R}^n, x \geq 0, \|x\|_k = 1 \},$$

   where $\|x\|_k = \left( \sum_{i=1}^{n} |x_i^k|^\frac{k}{2} \right)^\frac{2}{k}$. Furthermore, $x$ is an optimal solution of the above optimization if and only if it is an eigenvector of $T$ associated with $\lambda_{\text{max}}(T)$.

2. ([25, Theorem 5]) If $k$ is also even, then
   
   $$\lambda_{\text{min}}(T) = \min \{ T x^k : x \in \mathbb{R}^n, \|x\|_k = 1 \},$$

   and $x$ is an optimal solution of the above optimization if and only if it is an eigenvector of $T$ associated with $\lambda_{\text{min}}(T)$.

Let $G$ be a $k$-uniform hypergraph on $n$ vertices $v_1, v_2, \ldots, v_n$. The adjacency tensor of $G$ [8] is defined as $A(G) = (a_{i_1 i_2 \ldots i_k})$, a $k$-th order $n$-dimensional tensor, where

$$a_{i_1 i_2 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

The spectral radius, the least $H$-eigenvalue of $G$ refer to those of its adjacency tensor $A(G)$, denoted by $\rho(G), \lambda_{\text{min}}(G)$, respectively. The H-spectrum of $A(G)$ is denoted by $H\text{Spec}(G)$.

The spectral hypergraph theory has been an active topic in algebraic graph theory recently; see e.g. [8, 10, 11, 23, 24]. By the Perron-Frobenius theorem for nonnegative tensors [5, 13, 29, 30, 31], $\rho(G)$ is exactly the largest $H$-eigenvalue of $A(G)$. If $G$ is connected, there exists a unique positive eigenvector up to scales associated with $\rho(G)$, called the Perron vector of $G$. Noting that the adjacency tensor $A(G)$ is nonnegative and symmetric, so $\rho(G)$ holds (1) of Lemma 2.3, and $\lambda_{\text{min}}(G)$ holds (2) of Lemma 2.3 if $k$ is even. By Perron-Frobenius theorem, $\lambda_{\text{min}}(G) \geq -\rho(G)$. By the following lemma, if $G$ is connected and non-odd-bipartite, then $\lambda_{\text{min}}(G) > -\rho(G)$.

**Lemma 2.4** ([23, 22, 28, 12]). Let $G$ be a $k$-uniform connected hypergraph. Then the following results are equivalent.

1. $k$ is even and $G$ is odd-bipartite.
2. $\lambda_{\text{min}}(G) = -\rho(G)$.
3. $H\text{Spec}(G) = -H\text{Spec}(G)$.

Finally, we introduce some notations used throughout the paper. Denote by $C_n$ a cycle.
of length \( n \) as a simple graph. Denote by \( \mathbf{1} \) an all-one vector whose size can be implicated by the context, \( \text{rank} A \) the rank of a matrix \( A \) over \( \mathbb{Z}_2 \), and \( F_q \) a field of order \( q \).

## 3 Characterization of minimal non-odd-transversal hypergraphs

In this section we will give some equivalent conditions in terms of degrees and rank of the incidence matrix over \( \mathbb{Z}_2 \) for a hypergraph to be minimal non-odd-transversal.

**Lemma 3.1.** If \( G \) is a minimal non-odd-transversal hypergraph, then \( G \) is connected and contains no cut vertices.

**Proof.** If \( G \) contains more than one connected component, then at least one of them is non-odd-transversal, a contradiction to the definition. So \( G \) itself is connected. Suppose \( G \) contains a cut vertex. Then \( G \) is obtained from two connected nontrivial sub-hypergraphs \( G_1, G_2 \) sharing exactly one vertex (the cut vertex). So, at least one of \( G_1, G_2 \) is non-odd-transversal, also a contradiction. \( \Box \)

**Lemma 3.2.** Let \( G \) be a connected hypergraph, and \( B_G \) be the edge-vertex incidence matrix of \( G \). Then \( G \) is odd-transversal if and only if the equation

\[
B_G x = \mathbf{1} \quad \text{over} \quad \mathbb{Z}_2
\]

has a solution, or equivalently

\[
\text{rank} B_G = \text{rank}(B_G, \mathbf{1}) \quad \text{over} \quad \mathbb{Z}_2.
\]

**Proof.** If \( G \) is odd-transversal, then there is an odd-transversal \( U \) of \( G \). Define a vector \( x \in \mathbb{Z}_2^{|V(G)|} \) such that \( x_v = 1 \) if \( v \in U \), and \( x_v = 0 \) otherwise. By the definition, it is easy to verify that \( x \) is a solution of the equation (3.1). On the other hand, if \( x \) is a solution of the equation (3.1), define \( U = \{ v : x_v = 1 \} \). Then \( U \neq \emptyset \), and for each edge \( e \) of \( G \), \( |e \cap U| \) is odd, implying that \( G \) is odd-transversal. \( \Box \)

For each edge \( e \in E(G) \), define an *indicator vector* \( \chi_e \in \mathbb{Z}_2^{V(G)} \) such that \( \chi_e(v) = 1 \) if \( v \in e \) and \( \chi_e(v) = 0 \) otherwise. Then \( B_G \) consists of those \( \chi_e \) as row vectors for all \( e \in E(G) \).

**Lemma 3.3.** Let \( G \) be a connected hypergraph with \( m \) edges. If \( m \) is odd, and each vertex has an even degree, or equivalently \( \sum_{e \in E(G)} \chi_e = 0 \) over \( \mathbb{Z}_2 \), then \( G \) is non-odd-transversal.

**Proof.** Let \( e_1, \ldots, e_m \) be the edges of \( G \). Write \( (B_G, \mathbf{1}) \) as the following form:

\[
(B_G, \mathbf{1}) = \begin{pmatrix}
\chi_{e_1} & 1 \\
\chi_{e_2} & 1 \\
\vdots & \vdots \\
\chi_{e_m} & 1
\end{pmatrix}
\]
Adding the first row to all other rows over $\mathbb{Z}_2$, we will have

$$\begin{pmatrix} \chi_{e_1} & 1 \\ \chi_{e_2} + \chi_{e_1} & 0 \\ \vdots & \vdots \\ \chi_{e_m} + \chi_{e_1} & 0 \end{pmatrix} = \begin{pmatrix} \chi_{e_1} & 1 \\ \chi_{e_2} & 0 \\ \vdots & \vdots \\ \chi_{e_m} & 0 \end{pmatrix} =: \begin{pmatrix} \chi_{e_1} \\ \chi_{e_2} \\ \vdots \\ \chi_{e_m} \end{pmatrix}.$$ (3.4)

So, $\text{rank}(B_G, 1) = 1 + \text{rank}C$. As $m$ is odd and $\sum_{e \in E(G)} \chi_e = 0$,

$$\chi_{e_1} = \sum_{i=2}^m (\chi_{e_i} + \chi_{e_1}),$$

implying that $\text{rank}B_G = \text{rank}C$. By Lemma 3.2, $G$ is non-odd-transversal. \hfill $\square$

**Theorem 3.4.** Let $G$ be a connected hypergraph with $m$ edges. The following are equivalent.

1. $G$ is minimal non-odd-transversal.
2. $m$ is odd, $\sum_{e \in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, and $\sum_{e \in F} \chi_e \neq 0$ over $\mathbb{Z}_2$ for any nonempty proper subset $F$ of $E(G)$.
3. $m$ is odd, $\sum_{e \in E(G)} \chi_e = 0$ over $\mathbb{Z}_2$, and $\text{rank}B_G = m - 1$ over $\mathbb{Z}_2$.
4. $m$ is odd, each vertex of $G$ has an even degree, and any nonempty proper edge-induced sub-hypergraph of $G$ contains vertices of odd degrees.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $G$ is minimal non-odd-transversal. Let $e_1, \ldots, e_m$ be the edges of $G$. By Eq. (3.3) and Eq. (3.4), as rank$B_G \neq \text{rank}(B_G, 1)$ over $\mathbb{Z}_2$ by Lemma 3.2, $\chi_{e_1}$ is a linear combination of $\chi_{e_i} + \chi_{e_1}$ for $i = 2, \ldots, m$. So there exist $a_i \in \mathbb{Z}_2$ for $i = 2, \ldots, m$ such that

$$\chi_{e_1} = \sum_{i=2}^m a_i (\chi_{e_i} + \chi_{e_1}) = \left( \sum_{i=2}^m a_i \right) \chi_{e_1} + \sum_{i=2}^m a_i \chi_{e_i}. \quad (3.5)$$

We assert that $a_i = 1$ for $i = 2, \ldots, m$. Otherwise, there exists a $j$, $2 \leq j \leq m$, such that $a_j = 0$. Then $\chi_{e_1}$ is also a linear combination of $\chi_{e_i} + \chi_{e_1}$ for $i = 2, \ldots, m$ and $i \neq j$. So, rank$B_{G-e_j} \neq \text{rank}(B_{G-e_j}, 1)$, implying that $G - e_j$ is non-odd-transversal by Lemma 3.2, a contradiction to the definition.

If $m$ is even, then $\sum_{i=2}^m \chi_{e_i} = 0$ by Eq. (3.5), implying the vertices of $V(G)$ all have even degrees in $G - e_1$. So the vertices of $e_1$ all have odd degrees in $G$. By the arbitrariness of $e_1$, each vertex has an odd degree in $G$. However, there exists a vertex $v \notin e_1$ so that $d_G(v) = d_{G-e_1}(v)$, which is an even number, a contradiction.

So, $m$ is odd, and $\sum_{e \in E(G)} \chi_e = 0$ by Eq. (3.5). Assume to the contrary there exists a nonempty proper subset $F$ of $E(G)$, $\sum_{e \in F} \chi_e = 0$ over $\mathbb{Z}_2$. If $|F|$ is odd, then by Lemma
The sub-hypergraph $G|_F$ induced by $F$ is non-odd-transversal, a contradiction to the definition. Otherwise, $|F|$ is even, then $|E(G)\setminus F|$ is odd as $m$ is odd, and the sub-hypergraph $G|_{E(G)\setminus F}$ is non-odd-transversal, also a contradiction.

$(2) \Rightarrow (3)$. As $\sum_{e \in E(G)} \chi_e = 0$, $\text{rank}_G \leq m - 1$ over $\mathbb{Z}_2$. If $\text{rank}_G \leq m - 2$ over $\mathbb{Z}_2$, then $\chi_{e_1}, \ldots, \chi_{e_{m-1}}$ are linear dependent. So there exist $a_1, \ldots, a_{m-1} \in \mathbb{Z}_2$, not all being zero, such that $\sum_{i=1}^{m-1} a_i \chi_{e_i} = 0$. Taking $F = \{ e_i : a_i = 1, 1 \leq i \leq m - 1 \}$, then $\sum_{e \in F} \chi_e = \sum_{i=1}^{m-1} a_i \chi_{e_i} = 0$, a contradiction to $(2)$. So $\text{rank}_G = m - 1$ over $\mathbb{Z}_2$.

$(3) \Rightarrow (1)$. By Lemma 3.3, $G$ is non-odd-transversal. Let $e$ be an arbitrary edge of $G$. Adding all rows $\chi_f$ for $f \neq e$ to the row $\chi_e$ will yield a zero row as $\sum_{e \in E(G)} \chi_e = 0$. So $\text{rank}_{G-e} = \text{rank}_G = m - 1$ over $\mathbb{Z}_2$, implying that $B_{G-e}$ has full rank over $\mathbb{Z}_2$ with respect to rows. Hence, $\text{rank}_{G-e} = \text{rank}(B_{G-e}, 1)$ over $\mathbb{Z}_2$, and $G - e$ is odd-transversal by Lemma 3.2. So $G$ is minimal non-odd-transversal.

Of course $(2)$ is equivalent to $(4)$.

**Remark 3.5.** From the proof of $(3) \Rightarrow (1)$ in Theorem 3.4, if $G$ is a minimal non-odd-transversal hypergraph with $m$ edges, then any $m - 1$ rows of $B_G$ are linear independent over $\mathbb{Z}_2$.

**Example 3.6.** The following are minimal non-odd-transversal hypergraphs by verifying the degrees and the rank of incidence matrix over $\mathbb{Z}_2$ according to Theorem 3.4, where the last two hypergraphs are square.

- $(1)\, \{1,2,3,4\}, \{2,3,4,5\}, \{1,5\}$.
- $(2)\, \{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{1,4,5\}, \{3,4\}$.
- $(3)\, \{1,2,3\}, \{1,3,4,5\}, \{1,2,4,6\}, \{1,5,6,7\}, \{2,4,7\}, \{2,5,6,7\}, \{4,5,6,7\}$.

From Example 3.6, we know a minimal non-odd-transversal hypergraph can contain both even-sized edges and odd-sized edges. In the following we will discuss minimal non-odd-transversal hypergraphs only with even-sized edges.

**Corollary 3.7.** Let $G$ be a minimal non-odd-transversal hypergraph with $n$ vertices and $m$ edges. Then the following results hold.

- $(1)$ $n \geq m - 1$, and any $t$ edges contain at least $t$ vertices for $1 \leq t \leq m - 1$.
- $(2)$ The incidence bipartite graph $\Gamma_G$ has a matching $M$ such that $E(G)$ is matched by $M$, namely, there exists an injection $f : E(G) \to V(G)$ such that $f(e) \in e$ and $\{e, f(e)\} \in M$ for each $e \in E(G)$.
- $(3)$ If further $G$ contains only even-sized edges, then $n \geq m$, and any $t$ edges intersect at least $t + 1$ vertices for $1 \leq t \leq m - 1$.

**Proof.** Consider the incidence matrix $B_G$ of $G$. By Theorem 3.4(3), $\text{rank}_G = m - 1 \leq n$. Let $e_1, \ldots, e_t$ be $t$ edges of $G$, where $1 \leq t \leq m - 1$. Let $U = \cup_{i=1}^t e_i$, and let $B_G[e_1, \ldots, e_t]|U$ be the sub-matrix of $B_G$ with rows indexed $e_1, \ldots, e_t$ and columns indexed by the vertices of $U$. By Remark 3.5, $\text{rank}_{B_G[e_1, \ldots, e_t]|U} = t \leq |U|$, yielding the result (1).
The result (2) follows from Hall’s Theorem; see Lemma 2.1. If $G$ contains only even-sized edges, then each row sum of $B_G$ is zero over $\mathbb{Z}_2$, which implies $\text{rank} B_G \leq n - 1$, and $\text{rank} B_{G[e_1, \ldots, e_t]} \leq |U| - 1$. So we get the result (3).

**Corollary 3.8.** Let $G$ be a square minimal non-odd-transversal hypergraph only with even-sized edges. Then the following results hold.

1. The incidence bipartite graph $\Gamma_G$ has a perfect matching, namely, there exists a bijection $f : E(G) \to V(G)$ such that $f(e) \in e$ for each $e \in E(G)$.

2. For each nonempty proper subset $U$ of $V(G)$, $G|_U$ contains at least $|U| + 1$ edges, and also contains odd-sized edges.

**Proof.** Surely (1) comes from (2) of Corollary 3.7 as $G$ is square. Now let $U$ be a nonempty proper subset of $V(G)$. Let $F$ be the set of edges that intersect $U$ so that $G|_U$ has edges $e \cap U$ for all $e \in F$. If $F = E(G)$, then $|F| = |V(G)| \geq |U| + 1$ as $G$ is square. Otherwise, we consider the submatrix $B_G|F^c|U^c$, which has rank $|F^c|$ from the its rows by Remark 3.5. So, $|F^c| = |V(G)| - |F| \leq |V(G)| - |U| - 1$ as each row sum of $B_G|F^c|U^c$ is zero over $\mathbb{Z}_2$, implying that $|F| \geq |U| + 1$.

Assume to the contrary that each edge of $G|_U$ has even size. Then $B_G|E(G)|U]$, and $B_G(E(G)|U^c|$ as well, has zero row sums. So

$$\text{rank} B_G \leq \text{rank} B_G|E(G)|U] + \text{rank} B_G|E(G)|U^c|$$

$$\leq |U| - 1 + |U^c| - 1 = |V(G)| - 2 = |E(G)| - 2,$$

a contradiction to Theorem 3.4(3).

**Corollary 3.9.** Let $G$ be a square hypergraph only with even-sized edges and even-degree vertices. Then $G$ is minimal non-odd-transversal if and only if its dual $G^*$ is minimal non-odd-transversal.

**Proof.** Suppose $G$ is minimal non-odd-transversal with $n$ vertices (edges). By Corollary 3.8(2), no two vertices of $G$ lie in precisely the same edges of $G$. So $G^*$ is also square, and $B_{G^*} = B_G^\top$. As each edge of $G$ is even sized, each vertex of $G^*$ has even degree. So $G^*$ is minimal non-odd-transversal by Theorem 3.4. As $G$ is isomorphic to $(G^*)^*$, $G$ is minimal non-odd-transversal if $G^*$ is.

### 4 Minimal non-odd-bipartite regular hypergraphs

In this section we mainly discuss minimal non-odd-transversal $k$-uniform hypergraphs. By the following lemma, $k$ is necessarily even. So the minimal non-odd-transversal uniform hypergraphs are exactly the minimal non-odd-bipartite hypergraphs.

**Lemma 4.1.** Let $G$ be a minimal non-odd-transversal $k$-uniform hypergraph. Then $k$ is even. If $G$ is further $d$-regular, then $d$ is even and $d \leq k$. 


Proof. Suppose that $G$ has $n$ vertices and $m$ edges. By Theorem 3.4(4), each vertex of $G$ has an even degree so that the sum of degrees of the vertices of $G$ is even, which is equal to $mk$. As $m$ is odd by Theorem 3.4, $k$ is necessarily even, which implies that $n \geq m$ by Corollary 3.7(3). If $G$ is $d$-regular, $d$ is even by Theorem 3.4. Surely $nd = mk$, so $d \leq k$ as $n \geq m$.

So, in the following discussion we only deal with non-odd-bipartite regular hypergraphs with even uniformity and even degree.

4.1 2-regular minimal non-odd-bipartite hypergraphs

It is known that the only minimal non-bipartite simple graph is an odd cycle $C_{2l+1}$, which is 2-regular. As a simple generalization, the generalized power hypergraph $C_{2l+1}^{k,\frac{k}{2}}$ is a 2-regular minimal non-odd-bipartite $k$-uniform hypergraph. However, the above hypergraph is not the only 2-regular minimal non-odd-bipartite $k$-uniform hypergraph. For example, the following 4-uniform hypergraph on 10 vertices with 5 edges below is minimal non-odd-bipartite:

$$\{1,2,3,4\}, \{3,4,5,6\}, \{5,6,7,8\}, \{1,7,9,10\}, \{2,8,9,10\}.$$

Lemma 4.2. Let $G$ be a 2-regular uniform hypergraph with an odd number of edges. If $G$ is connected, then $G$ is minimal non-odd-bipartite.

Proof. Let $H$ be a nonempty proper edge-induced sub-hypergraph of $G$. As $G$ is connected, $H$ contains a vertex $v$, which is also contained in some edge not in $H$. So $v$ has degree 1 in $H$. The result follows by Theorem 3.4(4).

Next we give a construction of 2-regular $k$-uniform hypergraphs, where $k$ is an even integer greater than 2.

Construction 4.3. Let $k$ be an even integer greater than 2, and let $n, m$ be positive integers such that $n = \frac{km}{2}$. Let $K_{n,n}$ be a complete bipartite simple graph with two parts $V$ and $E$, where $V = [n]$ and $E = \cup_{t=1}^{m} \{e_{t,1}, \ldots, e_{t,k}^{2}\}$. Let $\hat{K}_{n,n}$ be obtained from $K_{n,n}$ by deleting the edges between the vertices of $V_{t} := \{\frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t\}$ and the vertices of $E_{t} := \{e_{t,1}, \ldots, e_{t,k}^{2}\}$ for each $t \in [m]$.

Take a perfect matching $M$ of $\hat{K}_{n,n}$, where $E_{t}$ are matched to $\tilde{V}_{t} := \{i_{t1}, \ldots, i_{tk}\}$ respectively for $t \in [m]$, such that if $\tilde{V}_{t} = V_{s}$ for some $s \neq t$, then $\tilde{V}_{s} \neq V_{t}$.

Define a hypergraph $G_{\hat{M}}$ with vertex set $[n]$, whose edges are

$$e_{t} = V_{t} \cup \tilde{V}_{t}, \text{ for } t \in [m].$$

(4.1)

Lemma 4.4. The hypergraph $G_{\hat{M}}$ defined in Construct 4.3 is a 2-regular $k$-uniform hypergraph on $n$ vertices.
Proof. As there is no edge between $V_t$ and $E_t$ in $\hat{K}_{n,n}$, $V_t \cap \hat{V}_t = \emptyset$ for each $t \in [m]$. So each edge $e_t$ contains exactly $k$ vertices. Note that $\{V_1, \ldots, V_t\}$ and $\{\hat{V}_1, \ldots, \hat{V}_t\}$ both form a $t$-partition of $[n]$. For each vertex $v$ of $G$, $v \in V_t$ for a unique $s \in [m]$ and $v \in \hat{V}_t$ for a unique $t \in [m]$, where $t \neq s$ as $V_s \cap \hat{V}_s = \emptyset$. So $v$ contained in exactly two edges $e_s$ and $e_t$, implying $v$ has degree 2. Finally we note that $G$ contains no multiple edges; otherwise, if $e_s = e_t$ for $s \neq t$, then $V_s \cup \hat{V}_t = V_t \cup \hat{V}_t$, which implies that $V_s = \hat{V}_t$ and $V_t = V_s$ as $V_s \cap V_t = \emptyset$ and $\hat{V}_s \cap \hat{V}_t = \emptyset$, a contradiction to the assumption. The result follows.

Corollary 4.5. Any 2-regular $k$-uniform hypergraph on $n$ vertices can be constructed as in Construction 4.3, where $k$ is even integer greater than 2.

Proof. Let $G$ be a 2-regular $k$-uniform hypergraph with vertex set $V(G) = [n]$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Surely, $n = \frac{km}{2}$. Let $\hat{G} := \frac{k}{2} \cdot G$ be a $k$-uniform hypergraph with vertex set $V(G)$ and edge set $\frac{k}{2} \cdot E(G) := \{\frac{k}{2} \cdot e : e \in E(G)\}$, where $\frac{k}{2} \cdot e$ means the $\frac{k}{2}$ copies of $e$, written as $e^1, \ldots, e^{\frac{k}{2}}$. Then $\hat{G}$ is a $k$-regular $k$-uniform multi-hypergraph on $n$ vertices. The incidence bipartite graph $\Gamma_G$ of $G$ is $k$-regular.

Let $K_G$ be a complete bipartite graph with two parts $V(\hat{G})$ and $E(\hat{G})$. Let $\hat{K}_G$ be obtained from $K_G$ by deleting the edges between the vertices of $V_t := \{\frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t\}$ and the vertices of $E_t := \{e_t^1, \ldots, e_t^{\frac{k}{2}}\}$ for $t \in [m]$. Then $\hat{K}_G$ is an $(n-k^2)$-regular bipartite graph.

Considering the $k$-regular bipartite graph $\Gamma_{\hat{G}}$, it contains a perfect matching $M$. By a possible relabeling of the vertices, we may assume that for each $t \in [m]$, $V_t := \{\frac{k}{2}(t-1) + 1, \ldots, \frac{k}{2}t\}$ is matched to $E_t := \{e_t^1, \ldots, e_t^{\frac{k}{2}}\}$ in $\hat{M}$. By the construction of $\hat{G}$, returning to $G$, $V_t \subseteq e_t$ for $t \in [m]$.

Now deleting the edges between the vertices of $V_t$ and the vertices of $E_t$ from $\Gamma_{\hat{G}}$ for $t \in [m]$, we arrive at a $\frac{k}{2}$-regular bipartite graph denoted by $\Gamma_{\hat{G}}$, which is a subgraph of $K_{\hat{G}}$. Now $\Gamma_{\hat{G}}$, and hence $K_{\hat{G}}$ has a perfect matching $M$, where, for each $t \in [m]$, $E_t$ is matched to $V_t := \{i_1, \ldots, i_{\frac{k}{2}t}\}$ in $M$. So, returning to $G$, $V_t \subseteq e_t$ for $t \in [m]$. As there is no edge between $V_t$ and $E_t$ in $\Gamma_{\hat{G}}$, $V_t \cap \hat{V}_t = \emptyset$ for each $t \in [m]$, which implies that $e_t = V_t \cup \hat{V}_t$ for $t \in [m]$.

As $G$ contains no multiple edges, if $\hat{V}_t = V_s$ for some $s \neq t$, surely $\hat{V}_s \neq V_t$; otherwise $e_t = e_s = V_t \cup V_s$, a contradiction.

From the above discussion, $K_G$ and $\hat{K}_G$ are respectively isomorphic to $K_{n,n}$ and $\hat{K}_{n,n}$ in Construction 4.3. A perfect matching $M$ in $\hat{G}$ or $K_G$ is exactly a perfect matching in $\hat{K}_{n,n}$. So $G$ is isomorphic to the hypergraph $G_M$ as in Construction 4.3.

Theorem 4.6. Let $G$ be a 2-regular $k$-uniform hypergraph with $n$ vertices and $m$ edges, where $m$ is odd and $k$ is even. Then $G$ is a minimal non-odd-bipartite hypergraph if and only if $G$ can be constructed as in Construction 4.3 and $G$ is connected.

Proof. The sufficiency follows from Lemmas 4.4 and 4.2, and the necessity follows from Corollary 4.5 and Lemma 3.1.
Remark 4.7. The hypergraph constructed as in Construction 4.3 may not be connected. However, by Lemma 4.2 at least one component is minimal non-odd-bipartite as the total number of edges is odd. For example, the following 4-uniform hypergraph $G$ on 18 vertices with edges

$$e_t = \{2t - 1, 2t, 2t + 5, 2t + 6\}, t \in [9],$$

where the labels of the vertices are modulo 18. $G$ has 3 connected components $G_1, G_2, G_3$ with edge sets listed below, each of which is isomorphic to $C_3^{4,2}$ (a minimal non-odd-bipartite hypergraph).

$$E(G_1) : \{1, 2, 7, 8\}, \{7, 8, 13, 14\}, \{13, 14, 1, 2\}. $$

$$E(G_2) : \{3, 4, 9, 10\}, \{9, 10, 15, 16\}, \{15, 16, 3, 4\}. $$

$$E(G_3) : \{5, 6, 11, 12\}, \{11, 12, 17, 0\}, \{17, 0, 5, 6\}. $$

In Fig. 4.1 we give an illustration of $G$ constructed as in the way of Construction 4.3, where the dotted lines indicate a perfect matching in $K_{18,18}$, and the solid lines indicate a perfect matching in $\hat{K}_{18,18}$.

![Figure 4.1: An illustration of Construction 4.3](image)

4.2 Examples of $d$-regular minimal non-odd-bipartite hypergraphs

We first give an example of $k$-regular $k$-uniform minimal non-odd-bipartite hypergraph by using Cayley hypergraph. Let $G = (\mathbb{Z}_n; \{1, 2, \ldots, k - 1\})$ be a Cayley hypergraph, where $V(G) = \mathbb{Z}_n$, and $E(G)$ consists of edges $\{i, i + 1, \ldots, i + k - 1\}$ for $i \in \mathbb{Z}_n$. Then $G$ is connected, $k$-uniform and $k$-regular, with $n$ vertices and $n$ edges.

**Theorem 4.8.** Let $k$ be an even integer greater than 2, and $n$ be an odd integer greater than $k$. The $G = (\mathbb{Z}_n; \{1, 2, \ldots, k - 1\})$ is minimal non-odd-bipartite if and only if $\gcd(k, n) = 1$.

**Proof.** By Theorem 3.4(3), it suffices to show that rank $B_G = n - 1$ over $\mathbb{Z}_2$ if and only if $\gcd(k, n) = 1$. Consider the equation $B_G x = 0$ over $\mathbb{Z}_2$. For each $i \in \mathbb{Z}_n$, as $\{i, \ldots, i + k - 1\}$ and $\{i + 1, \ldots, k\}$ are edges of $G$, by the above equation we have

$$x_i + \cdots + x_{i+k-1} = 0, x_{i+1} + \cdots + x_{i+k} = 0.$$  

So $x_i = x_{i+k}$ for each $i \in \mathbb{Z}_n$. Let $t := \gcd(k, n)$. Then there exist integers $p, q$ such that $pk + qn = t$. Note that $t$ is odd as $n$ is odd, and if writing $k = st$, then $s$ is even as $k$ is even.
For each \( i \in \mathbb{Z}_n \),
\[
    x_i = x_{i+k} = \cdots = x_{i+pk} = x_{i+t} = x_{i+t-qn} = x_{i+t}.
\]
(4.2)

As \( s \) is even, for any \( x_1, \ldots, x_t \in \mathbb{Z}_2 \), and for any edge \( \{i, \ldots, i+k-1\} \), by Eq. (4.2),
\[
    x_i + \cdots + x_{i+k-1} = x_1 + \cdots + x_k \\
    = (x_1 + \cdots + x_t) + \cdots + (x_{(s-1)t+1} + \cdots + x_{st}) \\
    = s(x_1 + \cdots + x_t) = 0.
\]

So, the solution space of \( B_Gx = 0 \) over \( \mathbb{Z}_2 \) has dimension \( t \), which implies that \( \text{rank} B_G = n - t \) over \( \mathbb{Z}_2 \). The result now follows. \( \square \)

Let \( G \) be a \( k \)-uniform hypergraph with \( n \) vertices and \( m \) edges. Let \( G^1, \ldots, G^t \) be \( t \) disjoint copies of \( G \). For each vertex \( v \) (or each edge \( e \)) of \( G \), it has \( t \) copies \( v^1, \ldots, v^t \) (or \( e^1, \ldots, e^t \)) in \( G^1, \ldots, G^t \) respectively. Let \( t \circ G \) be a hypergraph whose vertex set is \( \bigcup_{i=1}^{t} V(G^i) \), and edge set is \( \{e^1 \cup \cdots \cup e^t : e \in E(G)\} \). Then \( t \circ G \) is a \( tk \)-uniform hypergraph with \( tn \) vertices and \( m \) edges, and the degree of \( v^i \) in \( t \circ G \) is same as the degree of \( v \) in \( G \) for each \( v \in V(G) \) and \( i \in [t] \). If further \( G \) is \( d \)-regular, then \( t \circ G \) is also \( d \)-regular.

**Lemma 4.9.** Let \( G \) be a \( k \)-uniform hypergraph. Then \( G \) is minimal non-odd-bipartite if and only if \( t \circ G \) is minimal non-odd-bipartite.

**Proof.** By a suitable labeling of the vertices of \( t \circ G \), we have \( B_{t \circ G} = (B_G, B_G, \ldots, B_G) \), where \( B_G \) occurs \( t \) times in the latter matrix. As \( \text{rank} B_G = \text{rank} B_{t \circ G} \), the result follows by Theorem 3.4(3). \( \square \)

Next we give an example of \( d \)-regular \( k \)-uniform minimal non-odd-bipartite hypergraph \( G \) with \( n \) vertices and \( m \) edges, where \( m \) is odd and \( d \) is even such that \( \gcd(d, m) = 1 \). Obviously \( nd = km \), and \( d \mid k \) as \( \gcd(d, m) = 1 \). Suppose \( k = td \), where \( t > 1 \). By Theorem 4.8, the hypergraph \( H = (\mathbb{Z}_m; \{1, 2, \ldots, d-1\}) \) is minimal non-odd-bipartite, which is \( d \)-regular, \( d \)-uniform, with \( m \) edges. By Lemma 4.9, \( t \circ H \) is minimal non-odd-bipartite with \( m \) edges, which is \( d \)-regular and \( td \) \((=k) \)-uniform.

**Corollary 4.10.** Let \( H = (\mathbb{Z}_m; \{1, 2, \ldots, d-1\}) \), where \( m \) is odd and \( d \) is even such that \( \gcd(d, m) = 1 \). Then \( t \circ H \) is minimal non-odd-bipartite with \( m \) edges, which is \( d \)-regular and \( td \)-uniform.

Note that in Corollary 4.10, if \( d = 2 \), then \( H \) is an odd cycle \( C_m \), and \( t \circ C_m = C_m^{2t, t} \) (a generalized power hypergraph), both of which are minimal non-odd-bipartite.

Thirdly we use a projective plane \((X, \mathcal{B})\) of order \( q \) to construct a regular minimal non-odd-bipartite hypergraph. Recall a projective plane of order \( q \) consists of a set \( X \) of \( q^2 + q + 1 \) elements called points, and a set \( \mathcal{B} \) of \((q+1)\)-subsets of \( X \) called lines, such that any two points lie on a unique line. It can be derived from the definition that any point lies on \( q + 1 \) lines, and two lines meet in a unique point, and there are \( q^2 + q + 1 \) lines. Now define a hypergraph based on \((X, \mathcal{B})\), denoted by \( G = (X, \mathcal{B}) \), whose vertices are the points of \( X \) and edges are the lines of \( \mathcal{B} \). Then \( G = (X, \mathcal{B}) \) is a \((q + 1)\)-regular \((q + 1)\)-uniform hypergraph with \( q^2 + q + 1 \) vertices.
Theorem 4.11. Let \((X, \mathcal{B})\) be a projective plane of order \(q\), and let \(G = (X, \mathcal{B})\) be a hypergraph defined as in the above. If \(q\) is odd, then \(G = (X, \mathcal{B})\) is minimal non-odd-bipartite.

Proof. Let \(e\) be an edge of \(G = (X, \mathcal{B})\) or a line of \((X, \mathcal{B})\). Then
\[
  B_{G-e}B_{G-e}^\top = qI + J,
\]
where \(I\) is the identity matrix, and \(J\) is an all-ones matrix, both of size \(q^2 + q\). So
\[
  \det(B_{G-e}B_{G-e}^\top) = \det(qI + J) = (q^2 + 2q)q^{q^2+q-1} \equiv 1 \pmod{2},
\]
implies that rank \(B_G = m - 1\) over \(\mathbb{Z}_2\), where \(m\) is the number of edges of \(G\). The result follows by Theorem 3.4(3).

It is known that if \(q\) is an odd prime power, then there always exists a projective plane of order \(q\) by using the vector space \(\mathbb{F}_q^3\). By Lemma 4.9 and Theorem 4.11, we easily get the following result.

Corollary 4.12. Let \(q\) be an odd prime power. There exists a \((q + 1)\)-regular \((q + 1)\)-uniform minimal non-odd-bipartite hypergraph with \(q^2 + q + 1\) edges. For any positive integer \(t > 1\), there exists a \((q + 1)\)-regular \(t(q + 1)\)-uniform minimal non-odd-bipartite hypergraph with \(q^2 + q + 1\) edges.

Remark 4.13. From Corollary 4.8 and Corollary 4.12, the minimal non-odd-bipartite hypergraphs \(G\) have degree \(d\) and edge number \(m\) such that gcd\((d, m) = 1\). (Note that gcd\((q + 1, q^2 + q + 1) = 1\).) As gcd\((d, m) = 1\), from the equality \(nd = mk\), we have \(d \mid k\), where \(n, k\) are the number of vertices and the uniformity of \(G\) respectively.

In fact, there exist \(d\)-regular minimal non-odd-bipartite hypergraphs with \(m\) edges such that gcd\((d, m) > 1\). For example, let \(G\) be a 6-uniform 6-regular hypergraph with 9 edges below:
\[
\{1, 2, 3, 4, 5, 6\}, \quad \{1, 4, 5, 6, 7, 9\}, \quad \{1, 3, 5, 6, 7, 8\}, \quad \{1, 2, 4, 6, 7, 8\}, \quad \{1, 3, 5, 7, 8, 9\}, \quad \{1, 2, 6, 7, 8, 9\}, \quad \{2, 3, 4, 5, 7, 9\}, \quad \{2, 3, 4, 5, 8, 9\}, \quad \{2, 3, 4, 6, 8, 9\}.
\]

By Theorem 3.4, it is easy to verify that \(G\) is minimal non-odd-bipartite.

There also exist \(d\)-regular \(k\)-uniform minimal non-odd-bipartite hypergraphs such that \(d \nmid k\). For example, let \(G\) be a 6-regular 8-uniform hypergraph with 9 edges below:
\[
\{1, 2, 3, 4, 5, 6, 7, 8\}, \quad \{1, 3, 4, 5, 6, 7, 9, 11\}, \quad \{1, 4, 5, 6, 7, 8, 9, 10\}, \quad \{1, 5, 7, 8, 9, 10, 11, 12\}, \quad \{1, 2, 3, 6, 7, 9, 10, 12\}, \quad \{1, 2, 4, 5, 8, 10, 11, 12\}, \quad \{2, 3, 4, 6, 8, 10, 11, 12\}, \quad \{2, 3, 4, 5, 8, 9, 11, 12\}, \quad \{2, 3, 6, 7, 9, 10, 11, 12\}.
\]

By Theorem 3.4, it is also easy to verify that \(G\) is minimal non-odd-bipartite.

Example 4.14. The minimal non-odd-bipartite uniform hypergraph with fewest edges. By Theorem 3.4, if \(G\) is a \(k\)-uniform minimal non-odd-bipartite hypergraph with \(n\) vertices and \(m\) edges, then \(m\) is odd. If \(m = 1\), \(G\) is surely odd-bipartite. So, \(m \geq 3\), and hence
the maximum degree is at most 3 if \( m = 3 \). Assume that \( m = 3 \). By Theorem 3.4, each vertex has an even degree, implying that \( G \) is 2-regular. So \( 2n = 3k \); and \( 3 | n \). Letting \( n = 3l \), we have \( k = 2l \). So \( G = C_3^{2l} \), which is the unique example of minimal non-odd-bipartite hypergraph with 3 edges. It is consistent with the fact that \( C_3 \) is the unique minimal non-bipartite simple graph with 3 edges by taking \( k = 2 \).

**Example 4.15.** The minimal non-odd-bipartite uniform hypergraph with fewest vertices. If \( G \) is a \( k \)-uniform minimal non-odd-bipartite hypergraph with \( n \) vertices and \( m \) edges. Then \( n \geq k + 1 \), as an edge is odd-bipartite. Assume that \( n = k + 1 \). Then \( m \leq \binom{k + 1}{k + 1} = k + 1 \), with equality if and only if \( G \) is a \(( k + 1 )\)-simplex denoted by \( K_{k+1}^{[k]} \) ([8]), i.e. any \( k \) vertices of \( K_{k+1}^{[k]} \) forms an edge. Let \( \Delta \) be the maximum degree of \( G \), which is even by Theorem 3.4. As \( m \) is odd by Theorem 3.4, we have
\[
m - 1 \geq \Delta \geq \frac{km}{k + 1} = m - \frac{m}{k + 1} \geq m - 1,
\]
which implies that \( m = k + 1 \) and \( k \) is even. So, \( K_{k+1}^{[k]} \) is the unique example of \( k \)-uniform minimal non-odd-bipartite hypergraph with \( k + 1 \) vertices by Theorem 3.4. If taking \( k = 2 \), then \( C_3 \) is the unique minimal non-bipartite simple graph with 3 vertices.

**Example 4.16.** Example of non-regular minimal non-odd-bipartite hypergraph. Let \( G \) be a 4-uniform hypergraph on vertices \( 1, 2, \ldots, 9 \) with edges
\[
\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 6, 7\}, \{1, 5, 8, 9\}, \{6, 7, 8, 9\}.
\]
It is easy to verify that \( G \) is non-regular minimal non-odd-bipartite by Theorem 3.4.

**Remark 4.17.** Example of minimal non-odd-bipartite hypergraph with cut edges. Let \( G \) be a 4-uniform hypergraph \( G \) with vertices \( 1, 2, \ldots, 18 \) and 9 edges:
\[
\{1, 3, 4, 5\}, \{2, 3, 4, 6\}, \{5, 7, 8, 9\}, \{6, 7, 8, 9\}, \{1, 2, 10, 11\}, \{10, 12, 13, 14\}, \{11, 12, 13, 15\}, \{14, 16, 17, 18\}, \{15, 16, 17, 18\},
\]
where \{1, 2, 10, 11\} is a cut edge of \( G \). By Theorem 3.4, \( G \) is minimal non-odd-bipartite.

5 Least H-eigenvalue of minimal non-odd-bipartite hypergraphs

Let \( G \) be a \( k \)-uniform minimal hypergraph. Let \( x \in \mathbb{C}^{V(G)} \) whose entries are indexed by the vertices of \( G \). For a subset \( U \) of \( V(G) \), denote \( x^U := \Pi_{v \in U} x_v \). Then we have
\[
\mathcal{A}(G)x^k = \sum_{e \in E(G)} kx^e,
\]
(5.1)

**Theorem 5.1.** Let \( G \) be a \( k \)-uniform minimal non-odd-bipartite hypergraph with \( n \) vertices and \( m \) edges, where \( k \) is even. Then

\[
\begin{align*}
(1) \quad \lambda_{\min}(G) & \leq -\rho(G) + \frac{2k}{n\sqrt{r}}. \\
(2) \quad \lambda_{\min}(G) & \leq -(1 - \frac{2}{m})\rho(G).
\end{align*}
\]
Proof. As $G$ is connected by Lemma 3.1, by Perron-Frobenius theorem, there exists a positive eigenvector $x$ of $A(G)$ associated with the spectral radius $\rho(G)$. We may assume $\|x\|_k = 1$. Then

$$\rho(G) = A(G)x^k = \sum_{e \in E(G)} kx^e .$$  \hfill (5.2)

Observe that there exists a vertex $u$ such that $x_u \leq \frac{1}{n^{1/k}}$. Let $\bar{e}$ be an edge of $G$ containing $u$. Then

$$x^{\bar{e}} = x_u \prod_{v \in \bar{e}, v \neq u} x_v \leq \frac{1}{n^{1/k}} .$$

By the definition, $G - \bar{e}$ is odd-bipartite with an odd-bipartition $\{U, U^c\}$. Now define a vector $y$ on the vertices of $G - \bar{e}$ such that $y_v = x_v$ if $v \in U$ and $y_v = -x_v$ otherwise. Note that $\bar{e}$ intersects $U^c$ in an even number of vertices as $G$ is non-odd-bipartite, which implies that $y^{\bar{e}} = x^{\bar{e}} > 0$. By Lemma 2.3(2) and Eq. (5.2),

$$\lambda_{\min}(G) \leq A(G)y^k = -A(G)x^k + 2kx^{\bar{e}} \leq -\rho(G) + \frac{2k}{n^{1/k}} .$$

For the second result, from Eq. (5.2), there exists one edge $\hat{e}$ such that $kx^{\hat{e}}$ is not greater than the average of the summands $kx^e$ over all $m$ edges $e$ of $G$, that is,

$$kx^{\hat{e}} \leq \frac{\rho(G)}{m} .$$

Note that $G - \hat{e}$ is also odd-bipartite with an odd-bipartition say $\{W, W^c\}$. Now define a vector $z$ on the vertices of $G - \hat{e}$ such that $z_v = x_v$ if $v \in W$ and $z_v = -x_v$ otherwise. By a similar discussion as the above, we have

$$\lambda_{\min}(G) \leq A(G)z^k = -A(G)x^k + 2kx^{\hat{e}} \leq -(1 - \frac{2}{m})\rho(G) .$$

\hfill \Box

**Corollary 5.2.** Let $k$ be a positive even integer. For any $\epsilon > 0$, for any $k$-uniform minimal non-odd-bipartite hypergraph $G$ with sufficiently larger number of vertices or edges,

1. $-\rho(G) < \lambda_{\min}(G) < -\rho(G) + \epsilon ,$

2. $-1 < \lambda_{\min}(G)/\rho(G) < -1 + \epsilon .$

For a connected $k$-uniform hypergraph $G$, where $k$ is even, if we denote

$$\alpha(G) := \rho(G) + \lambda_{\min}(G), \quad \beta(G) := -\lambda_{\min}(G)/\rho(G) ,$$

then by Lemma 2.4, $\alpha(G) \geq 0$, with equality if $G$ is odd-bipartite; and $0 < \beta(G) \leq 1$, with right equality if and only if $G$ is odd-bipartite. So we can use $\alpha(G)$ and $\beta(G)$ to measure the non-odd-bipartiteness of an even uniform hypergraph.

Furthermore, by Theorem 5.1 and Corollary 5.2, if $G$ is minimal non-odd-bipartite, then $\alpha(G) \to 0$ and $\beta(G) \to 1$ when the number of vertices or edges of $G$ goes to infinity. So, the minimal non-odd-bipartite hypergraphs are very close to be odd-bipartite in this sense.
References


