# On the minimum size of hamiltonian saturated hypergraphs


Andrzej Ruciński*

Department of Discrete Mathematics
Adam Mickiewicz University
Poznań, Poland

rucinski@amu.edu.pl

Andrzej Żak†

Faculty of Applied Mathematics
AGH University of Science and Technology
Kraków, Poland

zakandrz@agh.edu.pl



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**Abstract**

For $1 \leqslant \ell < k$, an *$\ell$-overlapping $k$-cycle* is a $k$-uniform hypergraph in which, for some cyclic vertex ordering, every edge consists of $k$ consecutive vertices and every two consecutive edges share exactly $\ell$ vertices. A $k$-uniform hypergraph $H$ is *$\ell$-hamiltonian saturated* if $H$ does not contain an $\ell$-overlapping hamiltonian $k$-cycle but every hypergraph obtained from $H$ by adding one edge does contain such a cycle. Let $\mathrm{sat}(N, k, \ell)$ be the smallest number of edges in an $\ell$-hamiltonian saturated $k$-uniform hypergraph on $N$ vertices. In the case of graphs Clark and Entringer showed in 1983 that $\mathrm{sat}(N, 2, 1) = \lceil \frac{3N}{2} \rceil$. The present authors proved that for $k \geqslant 3$ and $\ell = 1$, as well as for all $0.8k \leqslant \ell \leqslant k-1$, $\mathrm{sat}(N, k, \ell) = \Theta(N^\ell)$. Here we prove that $\mathrm{sat}(N, 2\ell, \ell) = \Theta\left(N^\ell\right)$.

**Mathematics Subject Classifications:** 05C65


## 1 Introduction

A $k$-uniform hypergraph ($k$-*graph* for short) is a pair $H = (V, E)$, where $V$ is a finite set (of vertices) and $E \subseteq \binom{V}{k}$ is a family of $k$-element subsets of $V$ called edges of $H$. We will often identify $H$ with its vertex set $E$. For instance, we will denote by $|H|$ the number of edges in $H$.

Given integers $1 \leqslant \ell < k$, we define an *$\ell$-overlapping $k$-cycle* or, shortly, *$(\ell, k)$-cycle*, as a $k$-graph in which, for some cyclic ordering of its vertices, every edge consists of $k$ consecutive vertices, and every two consecutive edges (in the natural ordering of the

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edges induced by the ordering of the vertices) share exactly $\ell$ vertices. An *$\ell$-overlapping $k$-path* (or *$(\ell,k)$-path*) is defined similarly, that is, with vertices ordered $v_1,\dots,v_s$, the edges of the path are $\{v_1,\dots,v_k\}$, $\{v_{k-\ell+1},\dots,v_{k+\ell}\}$, $\dots$, $\{v_{s-k+1},\dots,v_s\}$. Note that the number of edges of an $(\ell,k)$-cycle with $s$ vertices is $s/(k-\ell)$ (and thus, $s$ is divisible by $k-\ell$). Likewise, it can be easily seen that the number of vertices of an $(\ell,k)$-path equals $\ell$ modulo $k-\ell$.

Given a $k$-graph $H$ and a $k$-element set $e\in H^c$, where $H^c=\binom{V}{k}\setminus H$ is the complement of $H$, we denote by $H+e$ the hypergraph obtained from $H$ by adding $e$ to its edge set. For $1\leqslant \ell\leqslant k-1$, a $k$-graph $H$ is *$\ell$-hamiltonian saturated* (a.k.a. *maximally non-$\ell$-hamiltonian*) if $H$ is not $\ell$-hamiltonian but for every $e\in H^c$ the $k$-graph $H+e$ is such. The largest number of edges in an $\ell$-hamiltonian saturated $k$-graph on $N$ vertices has been determined in [5].

In this paper we are interested in the other extreme. For $N$ divisible by $k-\ell$, let $\mathrm{sat}(N,k,\ell)$ be the *smallest* number of edges in an $\ell$-hamiltonian saturated $k$-graph on $N$ vertices. In the case of graphs, Clark and Entringer proved in 1983 that

$$\mathrm{sat}(N,2,1)=\lceil \tfrac{3N}{2}\rceil \text{ for } N\geqslant 52. \tag{1}$$

For $k$-graphs with $k\geqslant 3$ the problem was first mentioned in [6, 7]. It seems to be quite hard to obtain such precise results as for graphs. Therefore, the emphasis has been put on the order of magnitude of $\mathrm{sat}(N,k,\ell)$. It is quite easy to see that

$$\mathrm{sat}(N,k,\ell)=\Omega(N^\ell),\ \text{ for all } k\geqslant 3, 1\leqslant \ell\leqslant k-1, \tag{2}$$

(see, e.g., Proposition 2.1 in [8]). The present authors proved in [8] that for $k\geqslant 3$ and $\ell=1$, as well as for all $0.8k\leqslant \ell\leqslant k-1$,

$$\mathrm{sat}(N,k,\ell)=\Theta(N^\ell) \tag{3}$$

(see [10] for the case $\ell=k-1$). We also conjectured that (3) holds true for all $1\leqslant\ell\leqslant k-1$. In [9] we proved a weaker general upper bound

$$\mathrm{sat}(N,k,\ell)=O\left(N^{\frac{k+\ell}{2}}\right).$$

In the same paper we improved the above bound in the smallest open case by showing that $\mathrm{sat}(N,4,2)=O\left(N^{\frac{14}{5}}\right)$. In this paper we confirm our conjecture in the middle of the range.

*Theorem* 1. For all $\ell\geqslant 2$ and $N$ divisible by $\ell$, $\mathrm{sat}(N,2\ell,\ell)=\Theta\left(N^\ell\right)$.

Our proof combines two general approaches to this type of problems developed, respectively, in [8] and [10, 9].

## 2 Construction

In this section, after setting some parameters, we will describe our construction and present the proof of Theorem 1 based on two lemmas which will be proved later.

## 2.1 Parameters setting

We need to choose the values of some parameters carefully and in doing so a pivotal role is played by the following notion. Given a positive integer $x$, let $C$ and $D$ be two disjoint sets with $|C| = x$ and $|D| = \infty$. Let $\nu(x) = \max_P |V(P)|$, where the maximum is taken over all $(\ell, 2\ell)$-paths $P$ which are subgraphs of the complete $2\ell$-uniform hypergraph with vertex set $C \cup D$ and such that

$$C \subset V(P) \subset C \cup D \qquad \text{and} \qquad |e \cap C| \geqslant \ell + 1 \qquad \text{for all} \qquad e \in P. \tag{4}$$

*Proposition* 2. If $x \geqslant \ell + 1$, then

$$\nu(x) = \begin{cases} x\frac{2\ell}{\ell+1}, & \text{if } (\ell+1)|x\ , \\ \left\lfloor \frac{x}{\ell+1} \right\rfloor 2\ell + \ell, & \text{otherwise.} \end{cases} \tag{5}$$

In particular,

$$\nu(x) \geqslant \frac{2\ell}{\ell+1}x - \ell. \tag{6}$$

*Proof.* Let $x = q(\ell+1) + r$, where $q = \left\lfloor \frac{x}{\ell+1} \right\rfloor$ and $0 \leqslant r \leqslant \ell$. Let $P$ be an $(\ell, 2\ell)$-path with $|V(P)| = \nu(x)$ and $t$ edges satisfying (4). Let $e_1, \ldots, e_t$ be the edges of $P$ in the linear order underlying $P$. Set $s = \left\lfloor \frac{t+1}{2} \right\rfloor$. Clearly, $t \in \{2s-1, 2s\}$. Recall that, by (4), $|e_i \cap C| \geqslant \ell + 1$ for each $i \in \{1, \ldots, 2s-1\}$. Hence, $s \leqslant q$, because $e_1, e_3, \ldots, e_{2s-1}$ are pairwise disjoint. Also by (4), if $t = 2s$, then

$$(e_t \cap C) \setminus \bigcup_{j=1}^{s} e_{2j-1} = (e_t \cap C) \setminus e_{2s-1} \neq \emptyset.$$

Thus, if $r = 0$, then $t = 2s-1$ and $|V(P)| = s \cdot 2\ell$. Otherwise, $t \leqslant 2s$ and $|V(P)| \leqslant s \cdot 2\ell + \ell$, and so the right-hand-side of (5) is the upper bound on $|V(P)|$.

To show equality in (5), let us view $P$ as a binary sequence $Q$, where each vertex of $C$ is represented by a symbol $c$ and each vertex of $V(P) \cap D$ is represented by a symbol $d$. (And the edges of $P$ follow the sequence $Q$ according to the definition of an $(\ell, 2\ell)$-path.) We now construct a sequence $Q$ which yields a path $P$ satisfying (4) and with $|V(P)|$ equal to the R-H-S of (5).

Let $Q$ begin with $\ell - 1$ vertices from $D$ and then traverse a group of $\ell + 1$ vertices from $C$, and so on $q$ times. If $r > 0$, then at the end we add $r$ vertices from $C$ followed by $\ell - r$ vertices from $D$ (see (7) below).

$$\overbrace{\underbrace{d, \ldots, d}_{\ell-1}, \underbrace{c, \ldots, c}_{\ell+1}}^{e_1} \overbrace{\underbrace{d, \ldots, d}_{\ell-1}, \underbrace{c, \ldots, c}_{\ell+1}}^{e_3} \cdots \overbrace{\underbrace{d, \ldots, d}_{\ell-1}, \underbrace{c, \ldots, c}_{\ell+1}}^{e_{2q-1}}, (\underbrace{c, \ldots, c}_{r}, \underbrace{d, \ldots, d}_{\ell-r}) \tag{7}$$

It is easy to check that $P$ satisfies (4). Clearly, $|V(P)| = q \cdot 2\ell$, if $r = 0$, and $|V(P)| = q \cdot 2\ell + \ell$, if $r > 0$. □

The function $\nu(x)$ is non-decreasing, but, as an immediate consequence of Proposition 2, it cannot increase too fast.

*Proposition* 3. For all $x \geqslant 1$ we have $\nu(x-1) \geqslant \nu(x) - \ell$. Moreover, if $x$ or $x-1$ is divisible by $\ell+1$, then $\nu(x-1) = \nu(x) - \ell$.

*Proof.* Let $x = q(\ell+1) + r$ as in the proof of Proposition 2. It is easy to check that, by (5), if $2 \leqslant r \leqslant \ell$, then $\nu(x-1) = \nu(x)$, while in the remaining two cases, $r = 0$ and $r = 1$, we have $\nu(x-1) = \nu(x) - \ell$. □

We now define parameters and sets our construction will rely upon. Let

$$N_0 = 100\ell^5 \tag{8}$$

and let $N \geqslant N_0$ be an integer divisible by $\ell$. Define integers

$$n = \left\lfloor \frac{N + 4\ell^3}{8\ell^3 + 2\ell} \right\rfloor \tag{9}$$

and

$$a = \frac{N + 4\ell^3 - n(8\ell^3 + 2\ell)}{\ell}. \tag{10}$$

Using (8), it is easy to check that

$$n \geqslant 10\ell^2. \tag{11}$$

Moreover, by (9), $n > \frac{N+4\ell^3}{8\ell^3+2\ell} - 1$, which is equivalent to $a < 8\ell^2 + 2$. Consequently, in view of (11), $a \leqslant n-1$. Let

$$x_i = \begin{cases} 4\ell^2(\ell+1) + 2\ell + 1, & i = 1, \ldots, a, \\ 4\ell^2(\ell+1) + 2\ell, & i = a+1, \ldots, n. \end{cases} \tag{12}$$

*Proposition* 4. For each $I \subset \{1, \ldots, n\}$ with $|I| = n-1$

$$2n\ell + \sum_{i \in I} \nu(x_i - 2\ell) + 4\ell^2 + 4\ell < N < (2n+2)\ell + \sum_{i=1}^{n} \nu(x_i - 2\ell) - 4\ell^3. \tag{13}$$

*Proof.* By (5) and (12),

$$\nu(x_i - 2\ell) = \begin{cases} 8\ell^3 + \ell, & i = 1, \ldots, a \\ 8\ell^3, & i = a+1, \ldots, n. \end{cases} \tag{14}$$

By (14) and (10),

$$\sum_{i=1}^{n} \nu(x_i - 2\ell) = a(8\ell^3 + \ell) + (n-a)(8\ell^3) = a\ell + 8n\ell^3 = N + 4\ell^3 - 2n\ell, \tag{15}$$

thus, the second inequality of (13) holds. On the other hand, by (14) and (15),

$$\sum_{i\in I}\nu(x_i-2\ell)\leqslant\sum_{i=1}^{n}\nu(x_i-2\ell)-8\ell^3=N-4\ell^3-2n\ell<N-(4\ell^2+4\ell)-2n\ell,$$

where the last inequality holds, since $\ell\geqslant 2$. Hence, the first inequality of (13) holds too. □

Let $A_i$ and $B_i$, $i=1,\dots,2n$, be a family of $4n$ pairwise disjoint sets with sizes:

$$|A_i|=\begin{cases}3\ell-1 \text{ for } i=1,\dots,n\\ 2\ell-1 \text{ for } i=n+1,\dots,2n,\end{cases}\tag{16}$$

and

$$|B_i|=\begin{cases}x_i-3\ell+1 \text{ for } i=1,\dots,n\\ b_i \qquad\qquad \text{for } i=n+1,\dots,2n,\end{cases}\tag{17}$$

where the $b_i$'s differ from each other by at most one and are chosen in such a way that

$$\sum_{i=1}^{2n}(|A_i|+|B_i|)=N.\tag{18}$$

Observe that $b_i$'s are well defined and positive. Indeed, by (16), (17), (12), and (10), using also the inequality $4\ell n(\ell^2+\ell+1)\leqslant 8\ell^3+2\ell n-4\ell^3$, which, due to (11), is valid for $\ell\geqslant 2$,

$$\begin{aligned}\sum_{i=1}^{2n}|A_i|+\sum_{i=1}^{n}|B_i|&=n(2\ell-1)+\sum_{i=1}^{n}x_i=n(2\ell-1)+a+n(4\ell^2(\ell+1)+2\ell)\\&<a\ell+4\ell n(\ell^2+\ell+1)-n\leqslant N-n.\end{aligned}$$

Finally, since the $b_i$'s differ from each other by at most one, we have that, by the R-H-S of (13) and by (14), for $i=n+1,\dots,2n$,

$$\begin{aligned}|A_i|+|B_i|&\leqslant\left\lceil\frac{N}{n}\right\rceil<\frac{N}{n}+1<\frac{n\cdot\max_i\nu(x_i-2\ell)+2n\ell}{n}+1\\&<\frac{n(8\ell^3+\ell)+2n\ell}{n}+1=8\ell^3+3\ell+1<10\ell^3.\end{aligned}\tag{19}$$

## 2.2 Main construction

Our construction stems from a base graph $G$ which consists of a maximally non-hamiltonian graph $G_1$ on $n$ vertices $\{1,\dots,n\}$ with bounded degree to which $n$ pendant vertices $\{n+1,\dots,2n\}$ have been added, so that for each $i=1,\dots,n$, the pair $\{i,n+i\}$ is an edge of $G$. By analyzing the constructions in [2, 3, 4] one can see that the hamiltonian saturated graphs obtained there do have bounded maximum degree. An alternative way is by combining (1) with a result of Bondy [1] (cf. [8]).

Fix $\ell \geqslant 2$. The desired $2\ell$-graph $H$ will be defined on an $N$-vertex set

$$V = \bigcup_{i=1}^{2n} U_i,$$

where $U_i = A_i \cup B_i$ and $A_i$, $B_i$ are given in the previous subsection. Note that, by (12), for each $i = 1, \dots, n$, we have $|A_i \cup B_i| = x_i \leqslant 10\ell^3$. This and (19) imply together that for all $i = 1, \dots, 2n$,

$$|U_i| \leqslant 10\ell^3. \tag{20}$$

Before defining the edge set of $H$, we need some more terminology and notation, which will be illustrated by an example. For a graph $F$ and a set $U \subset V(F)$, denote by $F[U]$ the subgraph of $F$ induced by $U$. For $S \subset V$, set

$$tr(S) = \{i : S \cap U_i \neq \emptyset\} \text{ and } \min(S) = \min\{i \in tr(S)\}$$

(The set $tr(S)$ is often called *the trace of* $S$, but we will not use this name here.)

*Example* 1. In Fig. 1 we have $tr(e_1) = \{1, 2\}$, $tr(e_2) = \{1, 3, 2n\}$, $tr(e_3) = \{2, 3\}$ and $tr(e_4) = \{3, n+1\}$ and thus, $\min(e_1) = 1$, $\min(e_2) = 1$, $\min(e_3) = 2$ and $\min(e_4) = 3$.

Further, let $c(S)$ be the number of connected components of $G^3[tr(S)]$, where $G^3$ is the third power of $G$, that is, the graph with the same vertex set as $G$, but with edges joining all pairs of distinct vertices at distance at most three in $G$.

The role of the third power can be explained as follows. In order to find a hamiltonian $(\ell, 2\ell)$-cycle in $H + e$, we will look for a hamiltonian path between two non-adjacent vertices of $G_1$, selected from the vertices of $tr(e)$ or their neighbors. In the worst case, $tr(e) \subset \{n+1, \dots, 2n\}$ and we will be forced to find a hamiltonian path between the neighbors $u, v$ of some vertices $n+u$ and $n+v$. Our construction will yield $c(e) \geqslant \ell+1 \geqslant 2$ which allows us to select $n+u$ and $n+v$ so that they are non-adjacent in $G^3$. Consequently, $u$ and $v$ will be non-adjacent in $G_1$, which, by the choice of $G_1$, guarantees the existence (in $G_1$) of a hamiltonian path between $u$ and $v$.

We define the ultimate $2\ell$-graph $H$ via three other hypergraphs. Let

$$H_1 = \left\{ e \in \binom{V}{2\ell} : tr(e) \in G \text{ and } |A_i \cap e| = \ell \text{ for both } i \in tr(e) \right\}.$$

We split $H_1 = H_1^1 \cup H_1^2$, where $H_1^1 = \{e \in H_1 : tr(e) \in G_1\}$. Further, let

$$H_2 = \left\{ e \in \binom{V}{2\ell} : \left|e \cap U_{\min(e)}\right| \geqslant \ell + 1 \right\}.$$

*Example* 2. Recall that, in Fig.1, $tr(e_1) = \{1, 2\}$. Moreover, $|e \cap A_1| = |e \cap A_2| = 3 = \ell$. Thus, if $\{1, 2\}$ is an edge of $G$, then $e_1 \in H_1$ (more precisely, $e_1 \in H_1^1$). Furthermore, $tr(e_2) = \{1, 3, 2n\}$ and $\min(e_2) = 1$. Since $|e_2 \cap U_1| = 4 = \ell + 1$, we have $e_2 \in H_2$.

Similarly, $|e_4 \cap U_3| = 5 \geqslant \ell + 1$, so $e_4 \in H_2$ too. Finally, $|e_3 \cap U_3| \geqslant \ell + 1$, but $\min(e_3) = 2$ and $|e_3 \cap U_2| = 1$. Hence $e_3 \notin H_2$. Since $e_3 \not\subset A_2 \cup A_3$, $e_3 \notin H_1$ either, regardless of whether $\{2, 3\}$ is an edge of $G$ or not.

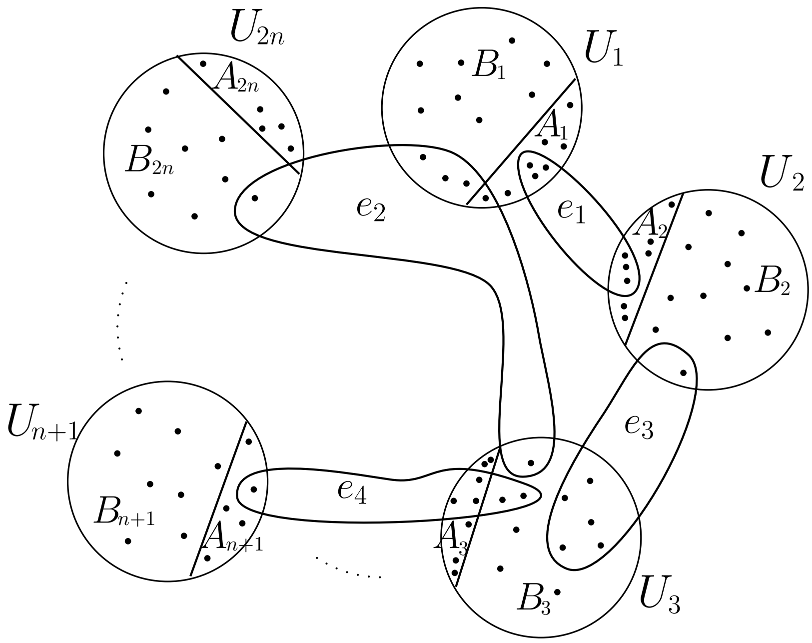


Figure 1: An illustration to construction: $\ell = 3$.

Note that if $P$ is an $(\ell, 2\ell)$-path in $H_2$, then there is an index $i$ such that every edge of $P$ draws at least $\ell + 1$ vertices from $U_i$. Indeed, let $e, e' \in P$ with $|e \cap e'| = \ell$. Let $i = \min(e)$. Since $|e \cap U_i| \geqslant \ell + 1$, $|e' \cap U_i| \geqslant 1$. Hence, $i \in tr(e')$ and so $\min(e') \leqslant \min(e)$. By symmetry, $\min(e) \leqslant \min(e')$. Thus $\min(e') = \min(e) = i$. By transitivity, $\min(f) = i$ for every $f \in P$.

The third element of the construction is

$$H_3 = \left\{ e \in \binom{V}{2\ell} : c(e) \leqslant \ell \right\}.$$

Note that

$$H_1 \cup H_2 \subseteq H_3, \tag{21}$$

where $H_1 \cup H_2$ is a $2\ell$-graph with vertex set $V$ whose edge set is the union of the edge sets of $H_1$ and $H_2$. Indeed, if $e \in H_1$, then $tr(e) \in G_1$ and so $c(e) = 1 \leqslant \ell$. If $e \in H_2$, then $|e \cap U_{\min(e)}| \geqslant \ell + 1$ and, consequently, $|tr(e)| \leqslant 1 + (\ell - 1) = \ell$. Clearly, $c(e) \leqslant |tr(e)|$, hence (21) follows.

We are going to show (cf. Lemma 5 in Section 3) that $H_1 \cup H_2$ is non-$\ell$-hamiltonian. Finally, we define $H$ as a non-$\ell$-hamiltonian $2\ell$-graph satisfying the containments

$$H_1 \cup H_2 \subseteq H \subseteq H_3$$

and such that $H + e$ is $\ell$-hamiltonian for every $e \in H_3 \setminus H$. (If $H_3$ is non-$\ell$-hamiltonian itself, we set $H = H_3$.)

### 2.3 Proof of Theorem 1

In [8] we proved the following result. Let $comp(F)$ denote the number of connected components of a graph $F$.

*Claim* 1. Let $r$, $\ell$, and $\Delta$ be constants. If $\Delta(G) \leqslant \Delta$, then the number of $r$-element subsets $T \subseteq V(G)$ with $comp(G[T]) \leqslant \ell$ is $O(n^\ell)$.

Theorem 1 is an consequence of Claim 1, our construction presented in the previous subsection, and the following two lemmas the proofs of which are deferred to the later sections. Lemma 5 guarantees that the definition of $H$ is not vacuous.

*Lemma* 5. $H_1 \cup H_2$ is non-$\ell$-hamiltonian.

Lemma 6 implies quickly that $H$ is indeed $\ell$-hamiltonian saturated (see the proof of Theorem 1 below.)

*Lemma* 6. For every $e \in \binom{V}{2\ell} \setminus H_3$, the $2\ell$-graph $H_1 \cup H_2 + e$ is $\ell$-hamiltonian.

*Proof of Theorem 1.* By (2), $\mathrm{sat}(N, 2\ell, \ell) = \Omega(N^\ell)$. In order to prove the upper bound, we begin by showing that $|H| = O(N^\ell)$. Observe that

$$H_3 = \bigcup_{T \subset V(G)} \left\{ e \in \binom{V}{2\ell} \,:\, tr(e) = T \right\},$$

where the sum is over all subsets $T$ of $V(G)$ of size at most $2\ell$ with $comp(G^3[T]) \leqslant \ell$. Since $G_1$ has bounded degree, so does $G$ and $G^3$. Thus, by Claim 1 with $r \leqslant 2\ell$, the number of such subsets $T$ is $O(n^\ell)$. Moreover, given $T$,

$$\left| \left\{ e \in \binom{V}{2\ell} \,:\, tr(e) = T \right\} \right| \leqslant \binom{\sum_{i \in T} |U_i|}{2\ell} \leqslant (|T| \cdot 10\ell^3)^{2\ell} = O(1),$$

by (20). Consequently, $|H_3| = O(n^\ell) = O(N^\ell)$ and, thus, also $|H| = O(N^\ell)$.

It remains to show that $H$ is $\ell$-hamiltonian saturated. Recall that, by construction (and Lemma 5), $H$ is non-$\ell$-hamiltonian. Let $e \in \binom{V}{2\ell} \setminus H$. If $e \in H_3$ then, by the definition of $H$, $H + e$ is $\ell$-hamiltonian. On the other hand, if $e \in \binom{V}{2\ell} \setminus H_3$, then $H + e \supseteq H_1 \cup H_2 + e$ is $\ell$-hamiltonian by Lemma 6. This shows that $H$ is, indeed, $\ell$-hamiltonian saturated and thus, the proof of Theorem 1 is completed. □

## 3 Proof of Lemma 5.

### 3.1 $(\ell, 2\ell)$-paths in $H_1 \cup H_2$

Before turning to the actual proof, we first prove a result about $(\ell, 2\ell)$-paths in $H_1 \cup H_2$.

*Proposition* 7. Let $m \geqslant 1$ and $P = (e, e_1, \dots, e_m, e')$ be an $(\ell, 2\ell)$-path in $H_1 \cup H_2$ such that $e, e' \in H_1^1$ and $e_i \in H_1^2 \cup H_2$, $i = 1, \dots, m$. The following hold:

(a) $P$ does not contain an edge $f \in H_1^2$ disjoint from $e \cup e'$;

(b) $P$ does not contain two disjoint edges $f, f' \in H_1^2$;

(c) $\min(e_i) \in tr(e) \cap tr(e')$, $i = 1, \dots, m$.

In the proof of Proposition 7, we will need the following result.

*Claim* 2. Let $m \geqslant 1$ and let $P = (e, e_1, \dots, e_m, e')$ be an $(\ell, 2\ell)$-path such that $e, e' \in H_1$ and $e_i \in H_2$, $i = 1, \dots, m$. Then $\min(e_1) = \dots = \min(e_m) \in tr(e) \cap tr(e')$, $i = 1, \dots, m$.

*Proof.* Let $\alpha = \min(e_1)$. Then, by the definition of $H_2$ and the fact that $|e_1 \setminus e_2| = \ell < \ell+1$, we have $\alpha \in tr(e_2)$. Hence, $\min(e_2) \leqslant \alpha = \min(e_1)$. By symmetry, $\min(e_1) \leqslant \min(e_2)$. Thus, $\min(e_1) = \min(e_2)$. By transitivity, $\min(e_i) = \alpha$ for every $i = 1, \dots, m$. By the same token, $\alpha \in tr(e)$ and $\alpha \in tr(e')$. □

*Proof of Proposition 7.* Since $m \geqslant 1$, we have $e \cap e' = \emptyset$. If $P$ does not contain any edge of $H_1^2$, then the statements (a) and (b) are vacuous, while (c) follows from Claim 2. Assume that $H_1^2 \cap P = \{f_1, \dots, f_t\}$ where $t \geqslant 1$ and $f_j$, $j = 1, \dots, t$, are listed in order of appearance on $P$. Let $tr(f_1) = \{\alpha, n + \alpha\}$. Furthermore, let $f_0 = e$ and $f_{t+1} = e'$.

If $f_j \cap f_{j+1} \neq \emptyset$ then, trivially,

$$tr(f_j) \cap tr(f_{j+1}) \neq \emptyset \quad j = 0, 1, \dots, t. \tag{22}$$

Otherwise, (22) holds by Claim 2. It follows by the structure of $G$ that $tr(f_j) = \{\alpha, n+\alpha\}$, $j = 1, \dots, t$, and $\alpha \in tr(f_j)$, $j \in \{0, t+1\}$, that is, $\alpha \in tr(e) \cap tr(e')$.

Since $e \cap e' = \emptyset$ and $|A_\alpha \cap f_j| = \ell$ for every $j \in \{0, \dots, t+1\}$, (a) holds by the first part of (16), while (b) holds by the second part of (16). Note that it follows that (c) holds for every edge $f_j$, $j = 1, \dots, t$, that is, for every edge $e_i \in H_1^2$.

Let us now consider $e'' \in P \cap H_2$. If $m \geqslant 3$, then, by (a) and (b), the only edge in $\{e_1, \dots, e_m\} \cap H_1^2$ is either $\{e_1\}$ or $\{e_m\}$. Without loss of generality assume that $e_1 \in H_1^2$ and $e_m \in H_2$. (For $m = 2$, we may assume the same with $e'' = e_m$.) By Claim 2 applied to the path from $e_1$ to $e'$, we conclude that $\min(e'') \in tr(e_1) = \{\alpha, n + \alpha\}$, as well as, $\min(e'') \in tr(e') \subset \{1, \dots, n\}$. Hence, $\min(e'') = \alpha \in tr(e) \cap tr(e')$ and (c) holds. □

### 3.2 Proof of Lemma 5.

In this subsection we complete the proof of Lemma 5.

*Proof of Lemma 5.* Suppose $C$ is a hamiltonian $(\ell, 2\ell)$-cycle in $H_1 \cup H_2$. We are going show that $|V(C)| < N$ which will be a contradiction. Our proof at some point (cf. proof of Claim 4) relies on the assumption that the graph $G_1$ is not hamiltonian. Let $M = \{e_1, \dots, e_m\}$ be a maximal set of pairwise disjoint edges of $C \cap H_1^1$, listed in the order of appearance on $C$. Further, for $i = 1, \dots, m$, let $P_i$ be the $(\ell, 2\ell)$-path in $C$ joining the last $\ell$ vertices of $e_i$ with the first $\ell$ vertices of $e_{i+1}$, where $e_{m+1} := e_1$. Notice that

$$C \setminus M = \bigcup_{i=1}^{m} P_i, \tag{23}$$

where all $P_i$'s are vertex disjoint, see Fig. 2.

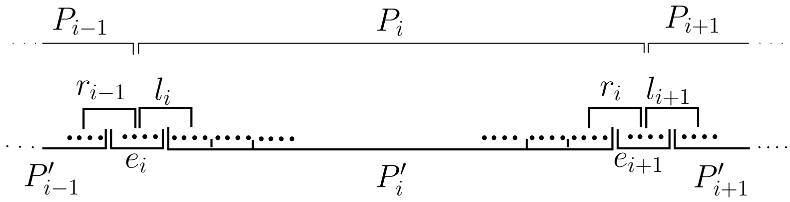


Figure 2: Fragment of $C$

Let $l_i$ be the first edge of $P_i$ and $r_i$ be the last edge of $P_i$ (note that they may coincide). We also define $P'_i$ to be the $(\ell, 2\ell)$-path arising from $P_i$ by removing both $l_i$ and $r_i$. Note that, by the definition of $M$,

$$P'_i \subset H_1^2 \cup H_2. \tag{24}$$

We call $P'_i$ *trivial* if $P'_i \subset H_1^2$. We further define

$$P''_i = P'_i \cap H_2. \tag{25}$$

Note that $P''_i$ is an $(\ell, 2\ell)$-path, too. Indeed, by Proposition 7a), every edge in $P'_i \cap H_1^2$ intersects $l_i$ or $r_i$ (and thus, is the first or the last edge of $P'_i$).

If $P''_i$ is non-empty, then let

$$\alpha_i = \min(f) \text{ for every } f \in P''_i \tag{26}$$

By Claim 2, $\alpha_i$ is well defined.

Observe that each edge $e \in (H_1^1 \cap C) \setminus M$ intersects some $e_i \in M$, so $e = l_i$ or $e = r_{i-1}$. We call an edge $l_i$ (or $r_i$) *bad* if it belongs to $H_1^1$, $|P_i| \geqslant 2$, and $tr(l_i) \neq tr(e_i)$ ($tr(r_i) \neq tr(e_{i+1})$, resp.). We call $P_i$ *problematic* if either $l_i$ or $r_i$ is bad or $P'_i$ contains an edge from $H_1^2$. Otherwise, we call $P_i$ *nice.*

Let $Tr(M) = \{tr(e) : e \in M\}$ be a graph defined by the traces of edges in $M$. Clearly, $|Tr(M)| = m$. Since, for each $e \in M$ and $j \in tr(e)$, $|e \cap A_j| = \ell$,

$$\Delta(Tr(M)) \leqslant 2, \tag{27}$$

by (16). In particular

$$m \leqslant n. \tag{28}$$

We need, however, better bounds on $m$. Let $q$ be the number of problematic $(\ell, 2\ell)$-paths among $P_1, \ldots, P_m$.

*Claim* 3.

$$m \leqslant \left\lfloor n - \frac{q}{2} \right\rfloor \tag{29}$$

*Proof.* Let $P$ be problematic. Suppose $e$ is a bad edge in $P$. If $e = l_i$ then since $tr(l_i) \neq tr(e_i)$, there exists $\beta \in tr(e)$ such that $|(e \cap A_\beta) \setminus e_i| = \ell$. Since $|P| \geqslant 2$,

$$|(e \cap A_\beta) \setminus (e_i \cup e_{i+1})| = \ell, \tag{30}$$

as well. By symmetry, the same holds if $e = r_i$. If $P'$ contains an edge $e$ which belongs to $H_1^2$, then (30) is also true, since $e$ does not intersect any edge of $M$. To sum up, for each $i = 1, \ldots, m$, there exists $\beta_i \in tr(P_i)$ such that

$$\left| (V(P_i) \cap A_{\beta_i}) \setminus \bigcup_{j=1}^{m} e_j \right| \geqslant \begin{cases} \ell \text{ if } P_i \text{ is problematic} \\ 0 \text{ otherwise.} \end{cases} \tag{31}$$

Note that $\beta_i$'s need not be different. Since $|A_{\beta_i}| \leqslant 3\ell - 1$, (31) implies that $deg_{Tr(M)}(\beta_i) \leqslant 1$ if $P_i$ is problematic (and $deg_{Tr(M)}(\beta_i) \leqslant 2$ if not). If two problematic $P_i$'s yield the same $\beta_i$ as above, then we conclude that $deg_{Tr(M)}(\beta_i) = 0$. Thus,

$$\sum_{i=1}^{n} deg_{Tr(M)}(\beta_i) \leqslant 2n - q.$$

Therefore,

$$m = |Tr(M)| \leqslant \left\lfloor \frac{2n - q}{2} \right\rfloor. \qquad \square$$

*Claim* 4. Suppose that $P_i' \neq \emptyset$ for every $i = 1, \ldots, m$. Then

$$m \leqslant n - 1 \tag{32}$$

*Proof.* If $q \geqslant 1$, then the claim follows by Claim 3. Assume that $q = 0$ and $|Tr(M)| = m = n$. Then, by (27), $Tr(M)$ is a 2-regular spanning subgraph of $G_1$. Since $q = 0$, each $P_i$ is nice and so

$$P_i' \subset H_2, \tag{33}$$

by (24). Let $f_i$ be any edge of $P_i'$. Recall that $\alpha_i = \min(f_i)$, see (26) and because $P_i' = P_i''$ by (33). If $l_i \in H_1^2 \cup H_2$, then $\alpha_i \in tr(e_i)$ by Proposition 7(c) applied to $P + e_i + e_{i+1}$. Otherwise, if $l_i \in H_1^1$, then $\alpha_i \in tr(l_i)$, again by Proposition 7(c), this time applied to $P_i$. Since $P_i$ is nice, $l_i$ is not bad and so, $tr(e_i) = tr(l_i)$. Hence, $\alpha_i \in tr(e_i)$, as before. By symmetry, $\alpha_i \in tr(e_{i+1})$, too. Thus, $Tr(M)$ is connected and, consequently, $Tr(M)$ is a hamiltonian cycle in $G_1$, a contradiction. $\square$

*Claim* 5. If $P_i$ is nice, then

$$|V(P_i')| \leqslant \nu(x_{\alpha_i} - 2\ell).$$

*Proof.* Since $P_i$ is nice, $P_i' = P_i'' \subset H_2$ by (24). If $P_i' = \emptyset$, then the claim trivially holds. Assume that $f_i \in P_i'$. Then $\alpha_i = \min(f_i)$. Similarly, as in the proof of Claim 4, we infer that $\alpha_i \in tr(e_i)$ and $\alpha_i \in tr(e_{i+1})$. In particular, since $e_i, e_{i+1} \in H_1^1$, $\alpha_i \leqslant n$. Thus, $|A_{\alpha_i} \cap e_i| = \ell$ and $|A_{\alpha_i} \cap e_{i+1}| \geqslant \ell$, which implies that $|V(P_i') \cap U_{\alpha_i}| \leqslant x_{\alpha_i} - 2\ell$. Therefore, the claim follows by the definitions of $H_2$ and $\nu$. □

*Claim* 6. If $P_i$ is problematic, then

$$|V(P_i')| \leqslant \nu(x_{\alpha_i}) + \ell.$$

*Proof.* By Proposition 7(a),(b) and by the choice of $M$, $P_i'$ contains at most one edge, say $f_i$, from $H_1^2$. Moreover, this edge is the first or the last edge of $P_i'$. The rest of $P_i'$ (i.e., $P_i'$ minus the first or the last $\ell$ vertices) is contained in $H_2$. Hence, by Claim 2, $\alpha_i \in tr(e_i)$ or $\alpha_i \in tr(e_{i+1})$. In particular, $\alpha_i \leqslant n$. Thus, the claim follows by the definition of $\nu$. □

We are now in the position to finish the proof of Lemma 5. Suppose that there are exactly $q$ problematic paths among the $P_i$'s. Let $I' \subset [1, n]$ be the set of those indices $i$ for which $P_i$ is problematic, and $I'' = [1, m] \setminus I'$. By (23), Claims 5 and 6, and Proposition 3 (applied $2\ell$ times),

$$\begin{aligned}
|V(C)| &= 2m\ell + \sum_{i=1}^{m} |V(P_i')| \\
&\leqslant 2m\ell + \sum_{i \in I'} (\nu(x_{\alpha_i}) + \ell) + \sum_{i \in I''} \nu(x_{\alpha_i} - 2\ell) \\
&\leqslant 2m\ell + \sum_{i \in I'} (\nu(x_{\alpha_i} - 2\ell) + 2\ell^2 + \ell) + \sum_{i \in I''} \nu(x_{\alpha_i} - 2\ell) \\
&= 2m\ell + \sum_{i=1}^{m} \nu(x_{\alpha_i} - 2\ell) + (2\ell^2 + \ell)q.
\end{aligned}$$

If $q \geqslant 1$, then (since $\nu(x_{\alpha_i} - 2\ell) \geqslant x_{\alpha_i} - 2\ell > 4\ell^2 + 2\ell$) the maximum is attained for $m = n - 1$ and $q = 2$, by Claim 3. Hence,

$$|V(C)| \leqslant 2n\ell + \sum_{i \in I} \nu(x_{\alpha_i} - 2\ell) + 2(2\ell^2 + \ell), \tag{34}$$

where $I \subset [1, n]$ with $|I| \leqslant n - 1$. Otherwise, by Claim 4, either $m \leqslant n - 1$ or $m \leqslant n$ and $P_i' = \emptyset$ for some $i \in \{1, \ldots, m\}$. In both these cases (34) holds as well. Therefore, by (13), $|V(C)| < N$, and so $C$ cannot be a hamiltonian $(\ell, 2\ell)$-cycle, a contradiction. □

# 4 Proof of Lemma 6.

## 4.1 The idea of the proof

One can easily construct $n$ disjoint $(\ell, 2\ell)$-paths $P_1, \dots P_n$ in $H_2$. Each such path $P_j$, however, is relatively short. Indeed, recall that by the definition of $H_2$, every edge of $P_j$ draws at least $\ell+1$ vertices from some fixed set $U_{i_j}$.

Edges from $H_1$ will serve as bridges joining the paths $P_j$. We have seen in the proof of Lemma 5 that, since $G_1$ is not Hamiltonian, we can use at most $n-1$ bridges. Fortunately, the new edge $e \notin H$ will play the role of an additional bridge in $H$, that, together with original $n-1$ edges of $M$, will 'glue' all paths $P_1, \dots, P_n$ into a hamiltonian $(\ell, 2\ell)$-cycle in $H$.

The use of $H_3$ is crucial for the argument. It allows us, when proving the existence of a hamiltonian $(\ell, 2\ell)$-cycle in $H+e$, to restrict only to $e \in \binom{V}{2\ell} \setminus H_3$, for which we know that $c(e) \geqslant \ell + 1$. The remaining edges (i.e. those in $H_3 \setminus H$), which are relatively rare but cumbersome, can be ignored just by the definition of $H$.

## 4.2 Proof of Lemma 6

The forthcoming proof will be illustrated by some diagrams in which we apply the following notation.

- $I$ denotes a vertex from $A_i$
- $I, I, \dots, I$ denotes a sequence of different vertices from $A_i$
- $i$ denotes a vertex from $U_i$ (we do not exclude $A_i$)
- $i, i, \dots, i$ denotes a sequence of different vertices from $U_i$
- $*$ denotes a vertex from $V$
- $*, *, \dots, *$ denotes a sequence of different vertices from $V$

*Proof of Lemma 6.* Let $e \in \binom{V}{2\ell} \setminus H_3$. Recall that, by the definition of $H_3$, $c(e) \geqslant \ell+1$. For a subset $Z \subseteq tr(e)$ let $e(Z) = \{u \in e : tr(u) \in Z\}$. Let $X$ be the vertex set of the component of $G^3[tr(e)]$ which contains vertex $i = \min(e)$ and let $Y = tr(e) \setminus X$. Note that, since $c(e) \geqslant \ell + 1$,

$$|e(X)| \leqslant \ell. \tag{35}$$

If for some $s \in Y$ we have $|e \cap U_s| \geqslant \ell$, then let $j = s$. Otherwise, let $j = \min(e(Y))$. By the choice of $j$

$$|U_t \cap e| \leqslant \ell - 1 \text{ for all } t \notin \{i, j\}. \tag{36}$$

Also, as $i$ and $j$ are in different components of $G^3[tr(e)]$, they do not form an edge of $G$. Even more, if $i = n + i'$ or $j = n + j'$ for some $1 \leqslant i', j' \leqslant n$, then, as $i$ and $j$ are in different components of $G^3[tr(e)]$, we have $ij', i'j, i'j' \notin G_1$ either.

Suppose first that $i, j \in \{1, \ldots, n\}$. Let $P_0$ be a 3-edge $(\ell, 2\ell)$-path with the edge $e$ in the middle and two edges $e'$ and $e''$ from $H_2$. The first $\ell$ vertices of $e$ belong to $e(Y)$ and the first one of them must be from $U_j$. The last $\ell$ vertices of $e$ contain $e(X)$ and the last of them must be from $U_i$. The first edge of $P_0$, $e'$, begins with $\ell$ vertices of $U_j$, the last (third) edge of $P_0$, $e''$, ends with $\ell$ vertices of $U_i$ (see the diagram below).

$$\underbrace{jj\ldots j}_{\ell}\underbrace{\overbrace{j**}^{e(Y)}\overbrace{*i}^{e(X)}}_{e}\underbrace{ii\ldots i}_{\ell} \tag{37}$$

Due to this deliberate construction and the choice of $j$, we have $\min(e') = j$ and $|e' \cap U_j| \geqslant \ell + 1$, so that indeed $e' \in H_2$. Similarly, $e'' \in H_2$. As observed above, $ij \notin G_1$.

If $i = n + i'$ and $j = n + j'$, then $P_0$ is, if possible, of the form

$$\underbrace{J'\ldots J'}_{\ell}\underbrace{J\ldots J}_{\ell}\underbrace{\overbrace{j**}^{e(Y)}\overbrace{*i}^{e(X)}}_{e}\underbrace{I\ldots I}_{\ell}\underbrace{I'\ldots I'}_{\ell} \tag{38}$$

In this case the first and the last edge of $P_0$ belong to $H_1^2$, and the second and the penultimate – to $H_2$. However, by (16), this construction is not feasible if $|e \cap A_i| = \ell$ or $|e \cap A_j| = \ell$. In such cases we modify $P_0$ as follows (let, say, $|e \cap A_i| = \ell$)

$$\underbrace{J'\ldots J'}_{\ell}\underbrace{J\ldots J}_{\ell}\underbrace{\overbrace{j**}^{e(Y)}\overbrace{I\ldots I}^{e(X)}}_{e}\underbrace{I'\ldots I'}_{\ell} \tag{39}$$

As observed above, $i'j' \notin G_1$. If $i \leqslant n$ and $j = n+j'$, then the right-hand side of $P_0$ is like in diagram (37), while the left-hand side is like in diagram (38) or (39). The construction for $i = n + i'$ and $j \leqslant n$ is analogous.

Since $G_1$ is maximally non-hamiltonian, it contains a hamiltonian path $v_1 v_2 \cdots v_{n-1} v_n$, where $v_1 \in \{i, i'\}$ and $v_n \in \{j, j'\}$, depending on the case. Based on this hamiltonian path, we are building a hamiltonian $(\ell, 2\ell)$-cycle in $H$ as follows.

Note that by (35) and by the construction of $P_0$

$$|U_t \cap P_0| \leqslant \begin{cases} 2\ell - 1 \text{ for } t \in \{v_1, v_n\} \\ \ell - 1 \text{ for } t \in \{v_2, \ldots, v_{n-1}\}. \end{cases} \tag{40}$$

First, we construct $n-1$ pairwise disjoint edges, $e_1 \ldots, e_{n-1} \in H_1$, such that they are also disjoint from $e$ and for each $t = 1, \ldots, n-1$, $e_t$ contains $\ell$ vertices from $A_{v_t}$ followed by $\ell$ vertices from $A_{v_{t+1}}$ (see the diagram below)

$$\underbrace{V_t, \ldots, V_t}_{\ell}\underbrace{V_{t+1}, \ldots, V_{t+1}}_{\ell}.$$

By (40) and (16), this construction is possible.

Next, we construct $n$ $(\ell, 2\ell)$-paths $P_t \subseteq H_2$, $t = 1, \dots, n$, such that $P_t$ consists of all vertices from $U_{v_t} \setminus \big(V(P_0) \cup \bigcup_{t=1}^{n-1} e_t\big)$ and some vertices from $\bigcup_{j=n+1}^{2n} U_j$, and $|V(P_t)|$ is as large as possible. We will do it in two stages. First, instead of $\bigcup_{j=n+1}^{2n} U_j$ we use vertices from some (abstract) infinite set $B$ and denote the resulting $(\ell, 2\ell)$-paths by $P'_t$.

Recall that $|U_i| = x_i$ and that the set $e_{i-1} \cup e_i$ contains already $2\ell$ vertices of $U_i$. Hence, if $U_{v_i} \cap e = \emptyset$, then we still have to use $x - 2\ell$ vertices from $U_{v_i}$ and so, recalling the definition of $\nu(x_i)$,

$$|V(P'_i)| = \nu(x_i - 2\ell). \tag{41}$$

Otherwise, quite roughly,

$$|V(P'_i)| \geqslant \nu(x_i - 4\ell) \geqslant \nu(x_i - 2\ell) - 2\ell^2, \tag{42}$$

by Proposition 3. Note that since $|e| \leqslant 2\ell$, $e$ intersects at most $2\ell$ sets $U_t$. Bearing this in mind, we now estimate from below the total number $N'$ of vertices appearing in all so far constructed elements:

$$\begin{aligned} N' &= |P_0| + \sum_{t=1}^{n-1} |e_t| + \sum_{t=1}^{n} |P'_t| \\ &\geqslant 2(n+1)\ell + \sum_{t, U_t \cap e = \emptyset} \nu(x_t - 2\ell) + \sum_{t, U_t \cap e \neq \emptyset} (\nu(x_t - 2\ell) - 2\ell^2) \\ &\geqslant 2(n+1)\ell + \sum_{t=1}^{n} \nu(x_t - 2\ell) - 4\ell^3 > N, \end{aligned}$$

where the last inequality holds by (13). Note that both, $N'$ and $N$, are divisible by $\ell$. We remove $N' - N$ vertices of $B$ from the paths $P'_1, \dots, P'_t$ in such a way that each path $P'_t$ gets shorter by a multiple of $\ell$ vertices and the vertices removed from each $P'_t$ are the first vertices of $V(P'_t \cap B)$ according to the order of appearance on $P'_t$. Treating the remaining vertices as consecutive, we thus obtain a collection of paths $P''_t$ such that each edge of $P''_t$ still has at least $\ell + 1$ vertices of $U_{v_t}$. Now we arbitrarily replace the remaining vertices of $B$ by the vertices of $\bigcup_{j=n+1}^{2n} U_j$, obtaining the desired paths $P_t \in H_2$.

Finally, note that the sequence

$$S = P_0, P_1, e_1, P_2, e_2, P_3, \dots, e_{n-1}, P_n.$$

spans a hamiltonian $(\ell, 2\ell)$-cycle in $H_1 \cup H_2 + e$. Indeed, the last $\ell$ vertices of $P_0$ and the first $\ell$ vertices of $P_1$ together contain at least $\ell + 1$ vertices of $U_{v_1}$ and thus they form an edge of $H_2$. So do the last $\ell$ vertices of $P_1$ and the first $\ell$ vertices of $e_1$, etc. Finally, the last $\ell$ vertices of $P_n$ and the first $\ell$ vertices of $P_0$ together contain at least $\ell + 1$ vertices of $U_{v_n}$ and so they also form an edge of $H_2$. □

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