

The Boolean Rainbow Ramsey Number of Antichains, Boolean Posets and Chains

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Abstract

Motivated by the paper, Boolean lattices: Ramsey properties and embeddings Order, 34 (2) (2017), of Axenovich and Walzer, we study the Ramsey-type problems on the Boolean lattices. Given posets P and Q , we look for the smallest Boolean lattice \mathcal{B}_N such that any coloring of elements of \mathcal{B}_N must contain a monochromatic P or a rainbow Q as an induced subposet. This number N is called the Boolean rainbow Ramsey number of P and Q in the paper.

Particularly, we determine the exact values of the Boolean rainbow Ramsey number for P and Q being the antichains, the Boolean posets, or the chains. From these results, we also derive some general upper and lower bounds of the Boolean rainbow Ramsey number for general P and Q in terms of the poset parameters.

Mathematics Subject Classifications: 05C55, 05D05, 05D10

1 Introduction

A poset $P = (P, \leq_P)$ is a set P equipped with a partial order \leq_P . For any two elements $a, b \in P$ if either $a \leq_P b$ or $b \leq_P a$ holds, then a and b are *comparable*, otherwise

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they are *incomparable*. In this paper, we study the Ramsey-type problems of the well-known poset, *Boolean lattice*, \mathcal{B}_n whose underlying set is the collection of all subsets of $[n] := \{1, 2, \dots, n\}$ ($[0] = \emptyset$ as a convention) and the partial order is the inclusion relation on sets. The k -th level of \mathcal{B}_n is the collection of all k -subsets of $[n]$, denoted by $\binom{[n]}{k}$. A family \mathcal{F} of subsets is isomorphic to a poset P if there exists an order-preserving bijection ϕ between P and \mathcal{F} , i.e., $P \xleftrightarrow{\phi} \mathcal{F}$ such that for any $x_1, x_2 \in P$, $x_1 <_P x_2$ if and only if $\phi(x_1) \subset \phi(x_2)$. If a family \mathcal{F} contains a subfamily \mathcal{G} isomorphic to P , we will say \mathcal{F} contains P as a *strong (or induced) subposet* (\mathcal{F} contains P , for short), or say the subsets in \mathcal{G} form a copy of P . A *coloring (k-coloring)* of \mathcal{B}_n is a mapping c from \mathcal{B}_n to a set of positive integers (to $[k]$). Given a coloring c of \mathcal{B}_n , we say \mathcal{B}_n contains a *monochromatic P* under c if there is a family of subsets of the same color containing P .

In the literature on Ramsey theory, the Ramsey problems have been greatly studied on the set systems (hypergraphs), the graphs, the planes, and the general posets. For example, see [1, 5, 6, 9, 13, 18]. The following type of Ramsey problems on Boolean lattices has been recently studied by Axenovich and Walzer [2]:

Question 1. Given posets P_1, \dots, P_k , can one find the least integer n such that for any k -coloring of \mathcal{B}_n , it always contains a monochromatic P_i of color i for some i ?

In their paper, they called such a number the poset Ramsey number. Since there are many Ramsey properties investigated on the families of posets with various parameters, to make it more precise, we suggest the name *the Boolean Ramsey number* and use it in this paper. Let us define the term formally below:

Definition 2. Given posets P_1, \dots, P_k , the *Boolean Ramsey number* $R(P_1, \dots, P_k)$ is the minimum integer n such that for any k -coloring of \mathcal{B}_n , it always contains a monochromatic P_i of color i for some i . Moreover, if $P_i = P$ for all $1 \leq i \leq k$, then we use $R_k(P)$ to denote $R(P_1, \dots, P_k)$.

Using a theorem in [16], Axenovich and Walzer proved that the number $R_k(P)$ is finite in [2]:

Theorem 3. (Theorem 6, [2]) *For any poset that is not an antichain, $R_k(P) = \Theta(k)$.*

Including the above theorem, Axenovich and Walzer obtained a series of results on the Boolean Ramsey number for the target posets being the Boolean lattices in [2], which are improved by Lu and Thompson [15] very recently.

Given a coloring c of \mathcal{B}_n , we say \mathcal{B}_n contains a *rainbow P* if it contains P and all the subsets forming P are of distinct colors under c . Imitating the rainbow Ramsey number in graph theory, we give the following definition:

Definition 4. Given two posets P and Q , the *Boolean rainbow Ramsey number* $RR(P, Q)$ is the minimum integer n such that for any coloring c of \mathcal{B}_n , it always contains either a monochromatic P or a rainbow Q as a subposet.

A result by Johnston, Lu, and Milans [12] implies that $RR(P, Q)$ is finite for any P and Q . They proved that for n sufficiently large, any coloring of \mathcal{B}_n contains either a monochromatic *Boolean algebra* $B_{alg(r)}$ or a rainbow $B_{alg(s)}$ for any given positive integers r and s . A *Boolean algebra* $B_{alg(r)}$ in \mathcal{B}_n is such a family $\{S_0 \cup (\cup_{i \in I} S_i) \mid I \subset [r]\}$, where S_0, S_1, \dots, S_r are pairwise disjoint subsets of $[n]$ with $S_i \neq \emptyset$ for $i \geq 1$.

Theorem 5. (Theorem 5, [12]) *For $N \geq r2^{(2r+1)2^{s-1}-2}$, \mathcal{B}_N contains either a rainbow $B_{alg(r)}$ or a monochromatic $B_{alg(s)}$ under any coloring.*

Containing the Boolean algebra $B_{alg(r)}$ implies containing \mathcal{B}_r since $B_{alg(r)}$ is isomorphic to \mathcal{B}_r . So $RR(P, Q)$ is finite when P and Q are both Boolean lattices. In addition, Trotter [17] introduced the *2-dimension* of a poset P , $\dim_2(P)$, the minimum number n for which \mathcal{B}_n contains P and proved that $\dim_2(P)$ is finite for any poset P . As a consequence, $RR(P, Q)$ exists for any P and Q .

Analogous to [2], Cox and Stolee [7] studied the existence of the monochromatic weak subposets in the Boolean lattices. Here a *weak subposet* means an injection from the poset to the Boolean lattice, which preserves the inclusion relation but not necessary the non-inclusion relation. A very recent paper [4] by the third author and others presents the results on the Ramsey properties of both types of subposets in the Boolean lattices.

In this paper, we study the strong version of the Boolean rainbow Ramsey number and the relations between it and the Boolean Ramsey number. We determine the exact values of $RR(P, Q)$ for specific posets and use the results to derive the upper and lower bounds for $RR(P, Q)$ for general P and Q . We focus on the *antichains* A_n , the *Boolean posets* B_n , and the *chains* C_n , where A_n is a set of n pairwise incomparable elements, B_n is isomorphic to \mathcal{B}_n , and C_n is a set of n mutually comparable elements. In the paper, the scribe \mathcal{B}_n refers to the underlying Boolean lattice we color, and the capital B_n refers to the desired monochromatic or rainbow Boolean posets. Table 1 is the summary of our results on $RR(P, Q)$ when P and Q are one of the three types of posets.

P	Q	$RR(P, Q)$
A_m	A_n	$\min\{N \mid \binom{N}{\lfloor N/2 \rfloor} \geq (m-1)(n-1) + 1\}$.
A_2	C_n	n
B_m	B_1	m
B_1	B_n	$2^n - 1$
C_m	A_n	$\begin{cases} n + 2, & m = 2 \text{ and } n \geq 3 \\ (m-1)(n-1) + 2, & m = n = 2, \text{ or } m \geq 3 \text{ and } n \geq 2 \end{cases}$
C_m	B_n	$(m-1)(2^n - 1)$
C_m	C_n	$(m-1)(n-1)$

Table 1: $RR(P, Q)$ of antichains, Boolean posets, and chains

The organization of the remaining sections in the paper is the following. In Section 2, we first determine the values of the Boolean rainbow Ramsey numbers for antichains or

chains, and deduce some lower bounds from those results. In Section 3, we give an upper bound on $RR(B_m, B_n)$ for general m and n , and determine the values of $RR(B_1, B_n)$ and $RR(B_m, B_1)$ as listed in Table 1. Using the idea of determining $RR(B_1, B_n)$, we manage to solve $RR(C_m, B_n)$, which is also demonstrated in this section. The last section contains discussions for the relations of $R_k(P)$, $RR(P, Q)$, and the forbidden subposet problems.

2 Chains, antichains, and the general lower bounds

When P and Q are both chains or both antichains, the Boolean rainbow Ramsey numbers can be determined by simple arguments.

Proposition 6. *For the chains, we have $RR(C_m, C_n) = (m - 1)(n - 1)$.*

Proof. For $N < (m - 1)(n - 1)$, we give a coloring c to \mathcal{B}_N by coloring the subsets with i in the consecutive $m - 1$ levels $\binom{[N]}{(i-1)(m-1)} \cup \binom{[N]}{(i-1)(m-1)+1} \cdots \cup \binom{[N]}{i(m-1)-1}$ for $1 \leq i \leq \lceil \frac{N}{m-1} \rceil$. Since a chain C_m in the Boolean lattices consists of m subsets of distinct sizes, \mathcal{B}_N does not contain a monochromatic C_m under c . In the coloring c , the number of color classes is at most $n - 1$, so \mathcal{B}_N does not contain a rainbow C_n as well. When $N \geq (m - 1)(n - 1)$, consider any coloring of the chain $\emptyset \subset [1] \subset [2] \subset \cdots \subset [N]$ in \mathcal{B}_N . By the pigeonhole principle, it contains at least n subsets of distinct colors or at least m subsets of the same color, as desired. \square

Let $N_{m,n}$ be the minimum integer such that $\binom{N_{m,n}}{\lfloor N_{m,n}/2 \rfloor} \geq (m - 1)(n - 1) + 1$.

Proposition 7. *For the antichains, we have $RR(A_m, A_n) = N_{m,n}$.*

Proof. For $N < N_{m,n}$, partition \mathcal{B}_N into $\binom{N}{\lfloor N/2 \rfloor}$ chains, $\mathcal{C}_1, \dots, \mathcal{C}_{\binom{N}{\lfloor N/2 \rfloor}}$ using the symmetric chain decompositions [3, 10]. Then color every $m - 1$ chains $\mathcal{C}_{(i-1)(m-1)+1}, \dots, \mathcal{C}_{i(m-1)}$ with i . Hence any antichain in a color class has size at most $m - 1$. Also, the coloring contains at most $n - 1$ colors. Thus, there is no rainbow A_n . For $N \geq N_{m,n}$, the level $\binom{[N]}{\lfloor N/2 \rfloor}$ contains at least $(m - 1)(n - 1) + 1$ subsets. Then any coloring of \mathcal{B}_N contains at least m subsets of the same color or n subsets of distinct colors in the level $\binom{[N]}{\lfloor N/2 \rfloor}$. \square

The *height* and *width* of a poset P , $h(P)$ and $w(P)$, are the sizes of a maximum chain and antichain of P , respectively. It is clear $RR(P, Q) \geq RR(P', Q')$ if P' and Q' are subposets of P and Q , respectively. The following bounds on $RR(P, Q)$ for general P and Q are the consequences of Proposition 6 and Proposition 7.

Corollary 8. *For any posets P and Q , we have $RR(P, Q) \geq (h(P) - 1)(h(Q) - 1)$ and $RR(P, Q) \geq N_{w(P), w(Q)}$.*

For P and Q being different types of posets, the problem becomes much harder and more interesting. we manage to determine $RR(C_m, A_n)$ in Theorem 9 and give an upper bound for $RR(A_m, C_n)$ in Theorem 11.

Theorem 9. For the chains and antichains, we have

$$RR(C_m, A_n) = \begin{cases} n + 2, & m = 2 \text{ and } n \geq 3, \\ (m - 1)(n - 1) + 2, & m = n = 2, \text{ or } m \geq 3 \text{ and } n \geq 2. \end{cases}$$

To prove Theorem 9, we need more preparations. Two families \mathcal{F} and \mathcal{G} of subsets are said to be *incomparable* if for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$, neither $F \subseteq G$ nor $G \subseteq F$. The following structure in the Boolean lattices will help us to determine $RR(C_m, A_n)$.

Lemma 10. Let $m \geq 2$ and $n \geq 2$ be integers. For $m \geq 3$, we can find n pairwise incomparable chains $\mathcal{C}_1, \dots, \mathcal{C}_n$ with $|\mathcal{C}_i| = (m - 1)(i - 1) + 1$ in $\mathcal{B}_{(m-1)(n-1)+2}$. When $m = 2$, this also holds for $n = 2$. For $m = 2$ and $n \geq 3$, we can find such a collection of chains in \mathcal{B}_{n+2} .

Proof. We prove the theorem by induction on n and constructing the families of chains recursively. Fix $m \geq 3$. For $n = 2$, we take the two chains $\mathcal{C}_1 = \{\{m\}\}$ and $\mathcal{C} = \{\{m + 1\} \cup [j] \mid 0 \leq j \leq m - 1\}$ contained in \mathcal{B}_{m+1} . Suppose we already have pairwise incomparable chains $\mathcal{C}_1, \dots, \mathcal{C}_n$ in $\mathcal{B}_{(m-1)(n-1)+2}$ for some $n \geq 2$. In $\mathcal{B}_{(m-1)n+2}$, take $\mathcal{C}'_i = \{[(m - 1)n + 1] - X \mid X \in \mathcal{C}_i\}$ for $1 \leq i \leq n$ and $\mathcal{C}'_{n+1} = \{\{(m - 1)n + 2\} \cup [j] \mid 0 \leq j \leq (m - 1)n\}$. If \mathcal{C}'_i and \mathcal{C}'_j are not incomparable for some $i < j \leq n$, say $Y_1 = [(m - 1)n + 1] - X_1$ in \mathcal{C}'_i is a subset of $Y_2 = [(m - 1)n + 1] - X_2$ in \mathcal{C}'_j , then X_2 in \mathcal{C}_j is a subset of X_1 in \mathcal{C}_i . This contradicts the incomparability of \mathcal{C}_i and \mathcal{C}_j . Note that $(m - 1)n + 1 > (m - 1)(n - 1) + 2$ if $m \geq 3$. So each set in \mathcal{C}'_i for $1 \leq i \leq n$ contains $(m - 1)n + 1$ which is not in any set in \mathcal{C}'_{n+1} . Also, $(m - 1)n + 2$ is in every set in \mathcal{C}'_{n+1} but not in any set in \mathcal{C}'_i for $1 \leq i \leq n$. Therefore, $\mathcal{C}'_1, \dots, \mathcal{C}'_{n+1}$ are incomparable. For $m = n = 2$, the above construction consists of $\{\{2\}\}$ and $\{\{3\}, \{1, 3\}\}$, which are incomparable.

Now for $m = 2$ and $n \geq 3$, we first take $\mathcal{C}_1 = \{\{1, 2\}\}$ and $\mathcal{C}_2 = \{\{4\}, \{1, 4\}\}$ as the desired incomparable chains in \mathcal{B}_4 . Suppose we already have pairwise incomparable chains $\mathcal{C}_1, \dots, \mathcal{C}_n$ in \mathcal{B}_{n+2} for some $n \geq 3$. In \mathcal{B}_{n+3} , take $\mathcal{C}'_i = \{[n + 2] - X \mid X \in \mathcal{C}_i\}$, for $1 \leq i \leq n$, and $\mathcal{C}'_{n+1} = \{\{n + 3\} \cup [j] \mid 0 \leq j \leq n\}$. Similar to the case of $m \geq 3$, $\mathcal{C}'_1, \dots, \mathcal{C}'_n$ are pairwise incomparable. The argument of incomparability of \mathcal{C}'_{n+1} and \mathcal{C}'_i , $i \leq n$ is slightly different from the case of $m \geq 3$: we have $n + 2$ in each set in \mathcal{C}'_i for $1 \leq i \leq n - 1$ but not in any set in \mathcal{C}'_{n+1} , $n + 1$ in each set in \mathcal{C}'_n but not in any set in \mathcal{C}'_{n+1} , and $n + 3$ in each set in \mathcal{C}'_{n+1} but not in any set in \mathcal{C}'_i for $1 \leq i \leq n$. Hence $\mathcal{C}'_1, \dots, \mathcal{C}'_{n+1}$ are pairwise incomparable. \square

Proof of Theorem 9. We use the constructions in Lemma 10 to show $RR(C_2, A_n) \leq n + 2$ for $n \geq 3$, and $RR(C_m, A_n) \leq (m - 1)(n - 1) + 2$ for $m = n = 2$, or for $m \geq 3$.

Given $m, n \geq 2$, we consider an arbitrary coloring on \mathcal{B}_N with $N = (m - 1)(n - 1) + 2$ if $m \geq 3$ or $m = n = 2$, or with $N = n + 2$ if $m = 2$ and $n \geq 3$. We have a family of pairwise incomparable chains $\mathcal{C}_1, \dots, \mathcal{C}_n$ in \mathcal{B}_n as described in Lemma 10. In both cases of N , for $i \geq 2$, there exist at least i colors on the subsets in \mathcal{C}_i , otherwise it contains a monochromatic C_m . However, if there are i colors on the subsets in \mathcal{C}_i , then we can pick a set from each chain $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ in order so that all sets are of distinct colors. The n sets form a rainbow A_n since all the chains are incomparable.

Let us establish the lower bound $RR(C_2, A_n) > n + 1$ for $n \geq 3$. For $0 \leq i \leq n + 1$, we color every subset in $\binom{[n+1]}{i}$ with i . Clearly, there is no monochromatic C_2 under this coloring. If it contains rainbow A_n , then neither \emptyset nor $[n + 1]$ can be in the rainbow antichain. So the rainbow A_n must contain a set $X_i \in \binom{[n+1]}{i}$ for $1 \leq i \leq n$. However, if $X_1 \not\subseteq X_n$, then $X_n = [n + 1] - X_1$ and any other X_i either contains X_1 or is contained in X_n . As a consequence, the rainbow A_n does not exist.

Next, we show $RR(C_m, A_n) > (m - 1)(n - 1) + 1$ for $m = n = 2$, or for $m \geq 3$ and $n \geq 2$. Let $N = (m - 1)(n - 1) + 1$. For each $X \in \mathcal{B}_N$, color X with $1 + \lceil \frac{|X|}{m-1} \rceil$. This gives an $(n + 1)$ -coloring of \mathcal{B}_N such that no color appears on m subsets of distinct sizes. So, there is no monochromatic C_m under the coloring. Meanwhile, colors 1 and $n + 1$ appear only on \emptyset and $[N]$, respectively. Any family containing subsets of n distinct colors must contain at least one of the two sets, and thus cannot be an antichain A_n . \square

Recall $N_{m,n}$ is the minimum integer such that $\binom{N_{m,n}}{\lfloor N_{m,n}/2 \rfloor} \geq (m - 1)(n - 1) + 1$.

Theorem 11. *Given $m \geq 2$, we have*

$$RR(A_m, C_n) \leq (m - 1)(n - 2) + N_{m,2}$$

for all $n \geq 2$. Moreover, the equality holds for $n = 2$ or $m = 2$.

Proof. Fix $m \geq 2$. For $n = 2$, to avoid a rainbow C_2 in the Boolean lattices \mathcal{B}_N means that each subset has the same color as the empty set, and thus \mathcal{B}_N is monochromatic. So, \mathcal{B}_N contains a monochromatic A_m for $N \geq N_{m,2}$. On the other hand, for $N \leq N_{m,2} - 1$, a monochromatic \mathcal{B}_N contains neither monochromatic A_m nor rainbow C_2 . Hence $RR(A_m, C_2) = N_{m,2}$. Note that this shows that the equality holds when $n = 2$ as the claimed statement.

Claim: $RR(A_m, C_{n+1}) \leq RR(A_m, C_n) + m - 1$ for any $n \geq 2$.

Suppose we already have $RR(A_m, C_n) = k$ for some $n \geq 2$. Consider any coloring c of \mathcal{B}_{k+m-1} . Assume that there is no monochromatic A_m under c . Then we show there is a rainbow C_{n+1} . Assume $c([k + m - 1]) = 1$ and let \mathcal{F}_1 be the family of nonempty subsets of color 1. Since the maximum size of an antichain in \mathcal{F}_1 is at most $m - 1$, by the well-known Dilworth's theorem [8], we can partition \mathcal{F}_1 into at most $m - 1$ disjoint chains, say $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_\ell$ with $\ell \leq m - 1$. For each \mathcal{C}_i , let x_i be a common element in each subset in \mathcal{C}_i . Let $S = [k + m - 1] \setminus \{x_1, \dots, x_\ell\}$. Observe that $|S| \geq k$ and the color of each subset of S , except for \emptyset , is not 1. We define a new coloring c' of \mathcal{B}_S , the Boolean lattices formed by all subsets of S , by letting $c'(X) = c(X)$ for all nonempty $X \subseteq S$ and $c'(\emptyset) = c(S)$. Because $|S| \geq k$ and there is no monochromatic A_m under c' , we can find a rainbow chain C_n in \mathcal{B}_S . Note that a rainbow chain cannot contain both S and \emptyset . If this chain does not contain \emptyset , then it together with $[k + m - 1]$ form a rainbow C_{n+1} under c . Else the chain contains \emptyset , then replace \emptyset with S . The new chain together with $[k + m - 1]$ form a rainbow C_{n+1} under c . Therefore $RR(A_m, C_{n+1}) \leq k + m - 1 = RR(A_m, C_n) + m - 1$ as we claimed. Repeatedly using the last inequality $n - 2$ times and the fact $RR(A_m, C_2) = N_{m,2}$ we obtain the inequality in the theorem.

Finally, let us show the equality for $m = 2$. By simple calculation we have $(m - 1)(n - 1) + N_{m,2} = n$ when $m = 2$. It suffices to show $RR(A_2, C_n) \geq n$. For $N < n$, we color the subsets \emptyset and $[N]$ in \mathcal{B}_N with the same color, and each of the remaining subsets with distinct colors. Clearly, there does not exist monochromatic A_2 , and every maximal rainbow chain has size $N \leq n - 1$. Therefore, the lower bound holds and the proof is completed. \square

Since the exact value of $RR(C_m, A_n)$ is known, we also have the following lower bound on $RR(P, Q)$ in terms of $h(P)$ and $w(Q)$.

Corollary 12. *For any posets P and Q , we have*

$$RR(P, Q) \geq \begin{cases} w(Q) + 2, & h(P) = 2 \text{ and } w(Q) \geq 3, \\ (h(P) - 1)(w(Q) - 1) + 2, & h(P) = w(Q) = 2, \text{ or} \\ h(P) \geq 3 \text{ and } w(Q) \geq 2. \end{cases}$$

3 Results related to Boolean posets

Recall that the 2-dimension of a poset, $\dim_2(P)$, is the minimum number d such that the Boolean lattice \mathcal{B}_d contains P as a subposet. Thus, the following upper bound on $RR(P, Q)$ for general P and Q is immediate.

Fact 13. *For any poset P and Q , we have*

$$RR(P, Q) \leq RR(\mathcal{B}_{\dim_2(P)}, \mathcal{B}_{\dim_2(Q)}).$$

From the above inequality, it turns out that determining $RR(B_m, B_n)$ is a more fundamental task. In the following, we first give an upper bound on $RR(B_m, B_n)$ using the ordinary Boolean Ramsey numbers $R_k(B_m)$, and then present the exact value of $RR(B_m, B_n)$ for $m = 1$ or $n = 1$.

Theorem 14. *For the Boolean posets B_m and B_n , we have*

$$RR(B_m, B_n) \leq \sum_{k=1}^{2^n-1} R_k(B_m).$$

Proof. Let $N = \sum_{k=1}^{2^n-1} R_k(B_m)$. Consider a coloring c on \mathcal{B}_N . We assume that \mathcal{B}_N does not contain a monochromatic B_m under c , and then prove there exists a rainbow B_n .

Let us arrange the nonempty subsets of $[n]$ in the order I_1, \dots, I_{2^n-1} such that $|I_k| \leq |I_{k'}|$ if $k < k'$. Partition $[N]$ into $2^n - 1$ disjoint subsets X_k 's such that $|X_k| = R_k(B_m)$. For $X, Y \subseteq [N]$, denote $[Y, X] = \{Z \mid Y \subseteq Z \subseteq X\}$. Note that $[Y, X]$ is isomorphic to $\mathcal{B}_{|X \setminus Y|}$. Define $Y_0 = \emptyset$. Since $[Y_0, X_1]$ is isomorphic to \mathcal{B}_m , there exists some set in $[Y_0, X_1]$ whose color is different from $c(Y_0)$. We pick such a set and call it Y_1 . Similarly, we can find a set in $[Y_0, X_2]$ whose color is different from $c(Y_0)$ and $c(Y_1)$, otherwise the sets in $[Y_0, X_2]$ are colored by only two colors, which implies $[Y_0, X_2]$ contains a monochromatic

B_m . We pick such a set and call it Y_2 . Since $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$, we have $Y_1 \cap Y_2 = \emptyset$. Imitating the same operations, we can find pairwise disjoint Y_0, \dots, Y_n , $Y_i \subseteq X_i$ for $1 \leq i \leq n$, of all distinct colors. These sets play the roles of I_0, I_1, \dots, I_n of the rainbow B_n . For $n+1 \leq k \leq n + \binom{n}{2}$, we have $|I_k| = 2$. Suppose $I_{n+1} = \{i, j\}$. Then $[Y_i \cup Y_j, (Y_i \cup Y_j) \cup X_{n+1}]$ is isomorphic to $\mathcal{B}_{R_{n+1}(B_m)}$. As before, we can find a set whose color is different from $c(Y_0), \dots, c(Y_n)$ in $[Y_i \cup Y_j, (Y_i \cup Y_j) \cup X_{n+1}]$, and choose it as the set Y_{n+1} . In general, we can find Y_k in $[\cup_{i: I_i \subset I_k} Y_i, (\cup_{i: I_i \subset I_k} Y_i) \cup X_k]$, whose color is different from $c(Y_0), c(Y_1), \dots, c(Y_{k-1})$, for $k \leq 2^n - 1$. It is clear that if $I_i \subset I_j$, then $Y_i \subset Y_j$. Now if I_i and I_j are incomparable, then there exist some $i' \in I_i \setminus I_j$ and $j' \in I_j \setminus I_i$. Thus, $Y_{i'} \subset (Y_i \setminus Y_j)$ and $Y_{j'} \subset (Y_j \setminus Y_i)$, which implies that Y_i and Y_j are incomparable as well. As a conclusion, these sets Y_0, \dots, Y_{2^n-1} form a rainbow B_n in \mathcal{B}_N . \square

Remark. By Theorem 3 (Theorem 5, [2]), $RR(P) \leq C \cdot k$, where C is a constant determined by P only. Thus, Theorem 14 is a proof of the existence of $RR(P, Q)$ without using Theorem 5. We will have more discussions on this aspect in next section.

Now we determine $RR(B_m, B_n)$ for $m = 1$ or $n = 1$.

Theorem 15. *Let $m \geq 1$ and $n \geq 1$. For the Boolean posets, we have*

$$RR(B_m, B_n) = \begin{cases} m, & \text{if } n = 1, \\ 2^n - 1, & \text{if } m = 1. \end{cases}$$

Proof. First, we show $RR(B_m, B_1) = m$ for $m \geq 1$. For $N < m$, we color every set in \mathcal{B}_N with the same color to avoid a monochromatic B_m and a rainbow B_1 . On the other hand, the inequality in Theorem 14 shows that $RR(B_m, B_1) \leq R_1(B_m) = m$.

The next is $RR(B_1, B_n) = 2^n - 1$ for $n \geq 1$. For $N < 2^n - 1$, we can color all subsets in the level $\binom{[N]}{i}$ with i for $1 \leq i \leq N$. Then every color class is an antichain. Moreover, to have a rainbow B_n , we need at least 2^n different colors, which is greater than the number of colors in the coloring. Therefore, no rainbow B_n under this coloring, and we have $RR(B_1, B_n) \geq 2^n - 1$.

By Theorem 14, $RR(B_1, B_n) \leq 2^{n-1}(2^n - 1)$, which is far from the lower bound $2^n - 1$ in the last paragraph. We will use an idea analogous to the proof of Theorem 14 to obtain the desired upper bound. For $N \geq 2^n - 1$, let S be the union of two disjoint sets $\{x_1, \dots, x_n\}$ and $[N - n]$. Since \mathcal{B}_S is isomorphic to \mathcal{B}_N , we can consider colorings of \mathcal{B}_S . Assume \mathcal{B}_S does not contain a monochromatic B_1 under a given coloring c . Then we will prove that it contains a rainbow B_n under c . Our strategy is to inspect the colors of the subsets of $\{x_1, \dots, x_n\}$ one by one. If all the subsets of $\{x_1, \dots, x_n\}$ are of different colors, then we are done. Once the color of a set $X \subset \{x_1, \dots, x_n\}$ already appeared before, then we look for a new color in a chain which contains X as the minimal element.

Let us arrange all subsets of $\{x_1, \dots, x_n\}$ in the order $X_0, X_1, \dots, X_{2^n-1}$ such that $|X_{k'}| \leq |X_k|$ if $k' < k$. First define $Y_0 = X_0 = \emptyset$ and $m_0 = 0$. For $k \geq 1$, let us define $m_k \leq N - n$ the smallest integer such that $m_k \geq m_{k'}$ for any $X_{k'} \subset X_k$ and $c(X_k \cup [m_k])$ is different from any color in $\{c(Y_0), c(Y_1), \dots, c(Y_{k-1})\}$. Once such an integer exists, then

define $Y_k = X_k \cup [m_k]$. By the definition of Y_k and m_k , $Y_{k'} \subset Y_k$ if and only if $X_{k'} \subset X_k$. Hence the sets $Y_0, Y_1, \dots, Y_{2^n-1}$ form a rainbow B_n .

Claim: For $0 \leq k \leq 2^n - 1$, m_k exists.

Suppose, on the contrary, some m_k does not exist. Let k_0 be the smallest k such that m_k does not exist. For any $k < k_0$, we call a chain \mathcal{C}_k in \mathcal{B}_S a *principal chain* of Y_k if it consists of $|Y_k| + 1$ sets and the largest one is Y_k such that all the colors of the sets in \mathcal{C}_k are in $\{c(Y_0), c(Y_1), \dots, c(Y_k)\}$. We show the principal chain exists for all Y_k with $k < k_0$. It is clear $\mathcal{C}_0 = \{\emptyset\}$ is the principal chain of Y_0 . For Y_k with $k \geq 1$, let $k^* = \max\{k' \mid X_{k'} \subset X_k\}$. By induction, the principal chain \mathcal{C}_{k^*} of Y_{k^*} exists, and all the colors of the sets in \mathcal{C}_{k^*} are in $\{c(Y_0), c(Y_1), \dots, c(Y_{k^*})\}$. Moreover, by the definition of m_k , the colors $c(X_k \cup [i])$, $m_{k^*} \leq i \leq m_k - 1$, have appeared in $\{c(Y_0), c(Y_0), \dots, c(Y_{k-1})\}$. Note that \mathcal{C}_{k^*} is the principal chain of $Y_{k^*} = X_{k^*} \cup [m_{k^*}]$ and we have $|X_{k^*}| = |X_k| - 1$ by the choice of k^* . We can concatenate \mathcal{C}_{k^*} to the chain $\{X_k \cup [i] \mid m_{k^*} \leq i \leq m_k\}$ to get a longer chain with

$$\begin{aligned} & |Y_{k^*}| + 1 + (m_k - m_{k^*} + 1) \\ = & (|X_{k^*}| + m_{k^*}) + 1 + (m_k - m_{k^*} + 1) \\ = & |X_k| + m_k + 1 \\ = & |Y_k| + 1 \end{aligned}$$

sets. This is the principal chain of Y_k . Now since m_{k_0} does not exist, it means the colors $c(X_{k_0} \cup [i])$, $m_{k^{**}} \leq i \leq N - n$, are all in $\{c(Y_0), c(Y_1), \dots, c(Y_{k_0-1})\}$, where $k^{**} = \max\{k' \mid X_{k'} \subset X_{k_0}\}$. In other words, the number of the colors of the subsets in the chain $\mathcal{C}_{k^{**}} \cup \{X_{k_0} \cup [i] \mid m_{k^{**}} \leq i \leq N - n\}$ is at most k_0 . Note that any chain under the coloring c is a rainbow chain since we assume there is no monochromatic B_1 . So we have $|Y_{k^{**}}| + 1 + (N - n - m_{k^{**}} + 1) \leq k_0$. Since $|Y_{k^{**}}| = |X_{k^{**}}| + m_{k^{**}}$, we have $N \leq n + k_0 - |X_{k^{**}}| - 2$. By the choice of k^{**} , we also have $|X_{k^{**}}| = |X_{k_0}| - 1$. In addition, $k_0 \leq \sum_{i=1}^{|X_{k_0}|} \binom{n}{i}$ according to the ordering of the X_i 's. Then

$$N \leq n + \sum_{i=1}^{|X_{k_0}|} \binom{n}{i} - |X_{k_0}| - 1 < 2^n - 1.$$

This contradicts our assumption of N . As a conclusion, we have $RR(B_1, B_n) \leq 2^n - 1$. \square

The idea of the principal chains in the proof of Theorem 15 can be applied to determine the Boolean rainbow Ramsey number for mixed types of posets below:

Theorem 16. *For the chains and Boolean posets, we have $RR(C_m, B_n) = (m-1)(2^n - 1)$.*

Proof. For $N < (m-1)(2^n - 1)$, we give a coloring of \mathcal{B}_N by assigning color i to the sets in the consecutive $m-1$ levels $\binom{[N]}{(i-1)(m-1)} \cup \binom{[N]}{(i-1)(m-1)+1} \cup \dots \cup \binom{[N]}{i(m-1)-1}$ for $1 \leq i \leq \lceil \frac{N}{m-1} \rceil$. On the one hand, since a color class contains at most $m-1$ subsets of different sizes, \mathcal{B}_N does not contain a monochromatic C_m . On the other hand, the number of colors is at most $2^n - 1$. Hence it does not contain a rainbow B_n .

Now for $N \geq (m-1)(2^n - 1)$ we define $\mathcal{B}_S = \{x_1, x_2, \dots, x_n\} \cup [N - n]$ as in the proof of Theorem 15, and study the colorings of \mathcal{B}_S . For a coloring c of \mathcal{B}_S , we prove that once there is no monochromatic C_n , then there is a rainbow B_n in \mathcal{B}_S under c . Again, we arrange all subsets of $\{x_1, x_2, \dots, x_n\}$ as X_0, \dots, X_{2^n-1} with $|X_{k'}| \leq |X_k|$ if and only if $k' \leq k$. Let $Y_0 = X_0 = \emptyset$ and $m_0 = 0$. Then for all $1 \leq k \leq 2^n - 1$, define $Y_k = X_k \cup [m_k]$, where $m_k \leq N - n$ is the least integer such that $m_k \geq m_{k'}$ whenever $X_{k'} \subset X_k$ and $c(X_k \cup [m_k])$ is a color different from $c(Y_0), \dots, c(Y_{k-1})$. Now it suffices to show all Y_k 's exist as before. If some Y_k does not exist, and $Y_{k'}$ exists for all $k' < k$, then we build up the principal chains $\mathcal{C}_{k'}$ of $Y_{k'}$ with the properties: $|\mathcal{C}_{k'}| = |Y_{k'}| + 1$, the maximal set is $Y_{k'}$, and the color of each set in $\mathcal{C}_{Y_{k'}}$ is in $\{c(Y_0), c(Y_1), \dots, c(Y_{k'})\}$. The only difference is now the principal chain is not necessarily rainbow. Nevertheless, the number of subsets of the same color in each $\mathcal{C}_{k'}$ is at most $m - 1$. Thus, if Y_k does not exist, then the chain $\mathcal{C}_{k^*} \cup \{X_k \cup [m_{k^*}], X_k \cup [m_{k^*} + 1], \dots, X_k \cup [N - n]\}$ contains at most $(m - 1)k$ subsets, where $m_{k^*} = \max\{m_{k'} \mid X_{k'} \subset X_k\}$. Hence, we have $|Y_{k^*}| + 1 + N - n - m_{k^*} \leq (m - 1)k$, and

$$\begin{aligned} N &\leq (m - 1)k - |Y_{k^*}| - 1 + n + m_{k^*} \\ &\leq (m - 1) \sum_{i=1}^{|X_k|} \binom{n}{i} - [(|X_k| - 1) + m_{k^*} + 1] - 1 + n + m_{k^*} \\ &\leq (m - 1) \sum_{i=1}^{|X_k|} \binom{n}{i} + n - |X_k| - 1 \\ &< (m - 1)(2^n - 1). \end{aligned}$$

This is again a contradiction. So \mathcal{B}_S must contain a rainbow B_n under c . □

4 Discussions

We gave an upper bound for $RR(A_m, C_n)$ in Theorem 11. A lower bound can be deduced in the following way: Recall that $N_{m,n}$ is the minimum integer N such that $\binom{N}{\lfloor N/2 \rfloor} \geq (m-1)(n-1) + 1$. Now consider the set S that is a union of disjoint sets $X = \{x_1, \dots, x_{N_{m,2}-1}\}$ and $Y = \{y_1, \dots, y_{n-2}\}$. For each subset $Z \subseteq S$, color it according to $Z \cap Y$. That is, we color Z_1 and Z_2 with the same color if and only if $Z_1 \cap Y = Z_2 \cap Y$. Thus, each color class is isomorphic to a Boolean poset $B_{N_{m,2}-1}$ which does not contain A_m . On the other hand, if Z_1, \dots, Z_k form a rainbow chain, then $Z_1 \cap Y, \dots, Z_k \cap Y$ also form a rainbow chain. However, any rainbow chain formed by subsets of Y has length at most $n - 1$. So there does not exist a rainbow C_n . Therefore, we have $RR(A_m, C_n) \geq n - 2 + N_{m,2}$. Combing the upper bound in Theorem 11, we have

$$n - 2 + N_{m,2} \leq RR(A_m, C_n) \leq (m - 1)(n - 2) + N_{m,2}.$$

Obviously, there is a huge gap between the lower and upper bounds. It would be interesting if one can tighten the gap.

One of our main interests is to estimate $RR(B_m, B_n)$. Theorem 5 provides us an upper bound $RR(B_m, B_n) \leq n2^{(2n+1)2^{m-1}-2}$. We also have $RR(B_m, B_n) \leq \sum_{i=1}^{2^n-1} R_i(B_m)$. In the following, we give an estimation of $\sum_{i=1}^{2^n-1} R_i(B_m)$ in terms of m and n . Recall that in the Remark after Theorem 14, we mentioned that $R_k(P) \leq C \cdot k$, where C is a constant determined by P . Indeed, this is a consequence from a result of M eroueh in [16]. Let us introduce the Lubell functions and forbidden subposet problems studied intensively recently. The *Lubell function* $\bar{h}_n(\mathcal{F})$ for a family \mathcal{F} of subsets of $[n]$ is defined as

$$\bar{h}_n(\mathcal{F}) := \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

It is obvious that $\bar{h}_n(\mathcal{B}_n) = n+1$. For the background of the Lubell functions and forbidden subposet problems, we refer the readers to a recent survey [11]. In the literature, Lu and Milans [14] proposed the conjecture:

Conjecture 17. [14] Let $\lambda_n(P)$ be the maximum value of the Lubell function for families of subsets of $[n]$ not containing P as a subposet. Then $\limsup \lambda_n(P)$ is finite.

M eroueh [16] verified this conjecture by showing that if $\bar{h}_n(\mathcal{F}) > 1000m^716^m$, then the family \mathcal{F} must contain B_m , as well as any poset whose 2-dimension is at most m . Thus, when $N \geq 1000m^716^mk$, any k -coloring of \mathcal{B}_N must contain one color class \mathcal{F}_i of subsets of color i with $\bar{h}(\mathcal{F}_i) > 1000m^716^m$, and hence a monochromatic B_m of color i . In other words, $R_k(B_m) \leq 1000m^716^mk$. Using his result, we conclude that

$$RR(B_m, B_n) \leq \sum_{i=1}^{2^n-1} R_i(B_m) \leq 2^{n-1}(2^n - 1)1000m^716^m < m^72^{2n+4m+9}.$$

This upper bound is a little better than that derived from Theorem 5 under certain circumstances. It will be a challenging problem to determine or give a better estimation of $RR(B_m, B_n)$ for both $m \geq 2$ and $n \geq 2$.

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References

- [1] M. AXENOVICH, A. GY ARF AS, H. LIU, AND D. MUBAYI, Multicolor Ramsey numbers for triple systems. *Discrete Mathematics*, **322** (2014) 69–77.

- [2] M. AXENOVICH AND S. WALZER, Boolean lattices: Ramsey properties and embeddings. *Order*, **34**(2) (2017) 287–298.
- [3] N. G. DE BRUIJN, C. VAN T. EBBENHORST, AND D. KRUYSWIJK, On the set of divisors of a number. *Nieuw Arch. Wiskunde* (2), **23** (1951) 191–193.
- [4] F.-H. CHANG, D. GERBNER, W.-T. LI, A. METHUKU, D. T. NAGY, B. PATKÓS, AND M. VIZER, Rainbow Ramsey problems for the Boolean lattice. [arXiv:1809.08629](https://arxiv.org/abs/1809.08629).
- [5] D. CONLON, J. FOX AND B. SUDAKOV, Hypergraph Ramsey numbers. *Journal of the American Mathematical Society*, **23** (1) (2010) 247–266.
- [6] D. CONLON AND J. FOX, B. SUDAKOV, Recent developments in graph Ramsey theory. *Surveys in Combinatorics 2015*, 49–118.
- [7] C. COX, D. STOLEE, Ramsey numbers for partially-ordered sets. *Order*, **35**(3) (2017) 557–579.
- [8] R. P. DILWORTH, A Decomposition Theorem for Partially Ordered Sets. *Annals of Mathematics*, **51**(1) (1950) 161–166.
- [9] S. FUJITA, C. MAGNANT, AND K. OZEKI, Rainbow generalizations of Ramsey theory—a dynamic survey. *Theory and Application of Graphs*, (2014).
- [10] C. GREENE AND D. J. KLEITMAN, Strong versions of Sperner’s theorem. *Journal of Combinatorial Theory, Series A*, **20** (1976), 80–88.
- [11] J. R. GRIGGS AND W.-T. LI, Progress on poset-free families of subsets, in: *Recent Trends in Combinatorics*, (2016), 317–338.
- [12] T. JOHNSTON, L. LU, AND K. G. MILANS, Boolean algebras and Lubell functions, *Journal of Combinatorial Theory, Series A*, **136** (2015) 174–183.
- [13] H. A. KIERSTEAD AND W. T. TROTTER, A Ramsey theoretic problem for finite ordered sets. *Discrete Mathematics*, **63**(2-3) (1987) 217–223.
- [14] L. LU AND K. G. MILANS, Set families with forbidden subposets. *Journal of Combinatorial Theory, Series A*, **136** (2015) 126–142.
- [15] L. LU AND J. C. THOMPSON, Poset Ramsey Numbers for Boolean Lattices. [arXiv:1909.08680](https://arxiv.org/abs/1909.08680) (2019).
- [16] A. MÉROUEH, A LYM inequality for induced posets. *Journal of Combinatorial Theory, Series A*, **155** (2018) 398–417.
- [17] W. T. TROTTER, Embedding finite posets in cubes. *Discrete Mathematics* **12** (2) (1975) 165–172.
- [18] W. T. TROTTER, Ramsey theory and partially ordered sets. *Contemporary Trends in Discrete Mathematics*, RL Graham, et al., eds., DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 49, (1999) 337–347.