Flag-transitive point-primitive symmetric (v, k, λ) designs with bounded k

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Abstract

In 2012, Tian and Zhou conjectured that a flag-transitive and point-primitive automorphism group of a symmetric (v, k, λ) design must be an affine or almost simple group. In this paper, we study this conjecture and prove that if $k \leqslant 10^3$ and $G \leqslant Aut(\mathcal{D})$ is flag-transitive and point-primitive, then G is affine or almost simple. This supports the conjecture.

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1 Introduction

A symmetric (v, k, λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ consists of a finite set $\mathcal{P}$ of v points, and a family of k-subsets B of $\mathcal{P}$, called blocks $\mathcal{B}$, such that every two points of $\mathcal{P}$ is contained in exactly λ blocks of $\mathcal{B}$, where $|\mathcal{B}| = |\mathcal{P}|$ and $2 < k < v - 2$. The order of symmetric (v, k, λ) design is $n = k - \lambda$.

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A *flag* of $\mathcal{D}$ is an incident point-block pair. The design $\mathcal{D}$ is called *flag-transitive* if $G \leqslant Aut(\mathcal{D})$ acts transitively on the set of flags of $\mathcal{D}$. In 1987, Kantor [11] studied symmetric $(v, k, 1)$ designs $\mathcal{D}$ of order n admitting a flag-transitive automorphism group G and proved that either $\mathcal{D}$ is Desarguesian and $L_3(n) \trianglelefteq G$, or G is a sharply Frobenius group of odd order $(n^2+n+1)(n+1)$, where n^2+n+1 is a prime. Regueiro ([13, 14]), Zhou et al. ([5]) proved that if a non-trivial symmetric (v, k, λ) design with $\lambda \leqslant 4$ admitting a flag-transitive and point-primitive automorphism group G, then it is of affine or almost simple type. In [16], Tian and Zhou extend this result to the case of $\lambda \leqslant 100$ and conjectured that a flag-transitive and point-primitive automorphism group of a symmetric (v, k, λ) design must be an affine or almost simple group. In this paper, we study this conjecture in terms of block size k. The proof of this paper uses some essential ideas of Camina, Gagen [2] and Zieschang [18].

Our main result is as follows:

Theorem 1. *Let $\mathcal{D}$ be a non-trivial symmetric (v, k, λ) design with $k \leqslant 10^3$. If $G \leqslant Aut(\mathcal{D})$ acts flag-transitively and point-primitively on $\mathcal{D}$, then G must be of affine or almost simple type.*

The examples of symmetric (v, k, λ) designs admitting a flag-transitive and point-primitive automorphism group can be seen in [13, 17]. Indeed, there exist many symmetric (v, k, λ) designs admitting a flag-transitive and point-imprimitive automorphism group. In the following, we give two examples of these designs. For more examples of symmetric (v, k, λ) designs, see [13], [9, Section 3.6].

Example 2 (Regueiro [13, Section 1.2.2])**.** There are exactly three non-isomorphic symmetric $(16, 6, 2)$ designs, of which exactly two admit flag-transitive and point-imprimitive groups, and these are $2^4 : S_4$ and $(Z_2 \times Z_8)(S_4.2)$.

Example 3 (Praeger and Zhou [15, Proposition 1.5])**.** The design of points and hyperplane complements of the projective geometry $PG(3, 2)$ is the unique symmetric $(15, 8, 4)$ design admitting a flag-transitive and point-imprimitive automorphism group S_5.

This paper is organized as follows. After this Introduction, in Section 2, we present a rough description of O'Nan-Scott Theorem for finite primitive groups and some well-known results which will be needed in the sequel. In Section 3, we reduce the proof of Theorem 1 to the product action type. In Section 4, we prove that product action type cannot occur by using some technical and complicated methods, such as a very detailed discussion of the structure of blocks of symmetric (v, k, λ) designs. Finally, we give a proof of Theorem 1.

2 Preliminaries

Throughout this paper, a non-abelian simple group will be denoted by T and the socle of G by $Soc(G)$.

Let $G \leqslant Sym(\mathcal{P})$ be a finite primitive group. Then O'Nan-Scott Theorem [12] shows that each finite primitive group G is permutational equivalent to one of the following types:

(i) Affine type, $Soc(G) = Z_p^m \leqslant G \leqslant AGL(m,p)$ and Z_p^m acts regularly on $\mathcal{P}$;

(ii) Almost simple type, $Soc(G) = T \leqslant G \leqslant Aut(T)$ and T is the unique minimal normal subgroup of G;

(iii) Simple diagonal type, $Soc(G) = T^\ell \leqslant G \leqslant T^\ell.(Out(T)\times S_\ell)$, $\ell \geqslant 2$ and $|\mathcal{P}| = |T|^{\ell-1}$;

(iv) Twisted wreath product type, $Soc(G) = T^\ell \leqslant G \leqslant T^\ell : S_\ell$ and T^ℓ acts regularly on $\mathcal{P}$;

(v) Product action type, $Soc(G) = T^\ell \leqslant G \leqslant H \wr S_\ell$, where H with a primitive action of almost simple or simple diagonal type.

Thus, in order to prove Theorem 1, it suffices to show that types (iii)-(v) do not occur.

The following lemmas will be used frequently in the following sections.

Lemma 4. (Ionin and van Trung [10, Remark, 6.10]) *If $\mathcal{D}$ is a symmetric (v,k,λ) design, then $k(k-1) = \lambda(v-1)$.*

Since $1 < k < v-1$, it follows that $k > \lambda + 1$, and so the order of symmetric design $n = k - \lambda \geqslant 2$.

Lemma 5 (Bruck-Ryser-Chowla Theorem [9, Section 2.4])**.** *Let v, k, and λ be integers with $\lambda(v-1) = k(k-1)$ for which there exists a symmetric (v,k,λ) design.*

(i) *If v is even, then $n{=}k{-}\lambda$ is a square.*

(ii) *If v is odd, then the diophantine equation $(k-\lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2{=}z^2$ has a solution in integers x, y, z not all zero.*

Lemma 6 (Feit-Thompson Theorem [6, Theorem])**.** *Every finite group of odd order is solvable.*

Lemma 7 (Huppert and Blackburn [8, Chapter X, Theorem 3.6])**.** *The Suzuki groups $Sz(q)$ are the only non-abelian simple groups of order prime to 3.*

Lemma 8 (Conway, Curtis, Norton and Wilson [3])**.** *Let T be a non-abelian simple group with $|T| < 10^6$. Then one of the following cases holds.*

Case	T	$Out(T)$	$\|T\|$	Case	T	$Out(T)$	$\|T\|$	Case	T	$Out(T)$	$\|T\|$
1	A_5	Z_2	60	12	$U_3(3)$	Z_2	6048	23	$L_2(32)$	Z_5	32736
2	$L_2(7)$	Z_2	168	13	$L_2(23)$	Z_2	6072	24	$U_3(4)$	Z_4	62400
3	A_6	$Z_2 \times Z_2$	360	14	$L_2(25)$	$Z_2 \times Z_2$	7800	25	M_{12}	Z_2	95040
4	$L_2(8)$	Z_3	504	15	M_{11}	1	7920	26	$U_3(5)$	S_3	126000
5	$L_2(11)$	Z_2	660	16	$L_2(27)$	Z_6	9828	27	J_1	1	175560

6	$L_2(13)$	Z_2	1092	17	$L_2(29)$	Z_2	12180	28	A_9	Z_2	181440
7	$L_2(17)$	Z_2	2448	18	$L_2(31)$	Z_2	14880	29	$L_3(5)$	Z_2	372000
8	A_7	Z_2	2520	19	$L_4(2)$	Z_2	20160	30	M_{22}	Z_2	443520
9	$L_2(19)$	Z_2	3420	20	$L_3(4)$	$Z_2 \times S_3$	20160	31	J_2	Z_2	604800
10	$L_2(16)$	Z_4	4080	21	$U_4(2)$	Z_2	25920	32	$S_4(4)$	Z_4	979200
11	$L_3(3)$	Z_2	5616	22	$Sz(8)$	Z_3	29120				

3 Simple diagonal and Twisted wreath product action

Suppose that $G \leqslant T^\ell.(Out(T) \times S_\ell)$ has a simple diagonal action on $\mathcal{P}$. Let $N = Soc(G) = T^\ell$ ($\ell \geqslant 2$) and let $\overline{T} = \{(t, t, \ldots, t) | t \in T\}$ be the diagonal subgroup of N. Then $\overline{T} \cong T$ and $\mathcal{P}$ can be identified with the coset space $N \backslash \overline{T}$. So, $|\mathcal{P}| = |T|^{\ell-1}$ and $G_{\overline{T}} \leqslant Aut(T) \times S_\ell$.

Proposition 9. *If $\mathcal{D}$ is a symmetric (v, k, λ) design with $k \leqslant 10^3$ which admits a flag-transitive and point-primitive automorphism group G, then G is not of simple diagonal type.*

Proof. Since $k(k-1) = \lambda(v-1)$, $k \mid |G_{\overline{T}}|$ yields $k \mid \lambda(|T|^{\ell-1} - 1, \ell!|T||Out(T)|)$. Thus,

$$k \mid \lambda \ell! |Out(T)| \tag{1}$$

which, by $(\lambda v)^{\frac{1}{2}} < k$, implies that $(\lambda |T|^{\ell-1})^{\frac{1}{2}} < \lambda \ell! |Out(T)|$, namely,

$$|T|^{\ell-1} < \lambda (\ell!)^2 |Out(T)|^2. \tag{2}$$

By $k \leqslant 10^3$, $(\lambda v)^{\frac{1}{2}} < k$ and $60 \leqslant |T|$, we have $60^{\frac{\ell-1}{2}} < 10^3$, and so $\ell \leqslant 3$.

First assume that $\ell = 3$. Then $(\lambda v)^{\frac{1}{2}} < k$ implies that $|T| < 10^3$. Further, we have $T = A_5$, $L_2(7)$, A_6, $L_2(8)$ or $L_2(11)$ by Lemma 8. Now $|Out(T)|$ divides 4 and (2) yield $|T| < 24\lambda^{\frac{1}{2}}$. Thus, $\lambda^{\frac{1}{2}} |T| < 10^3$ implies that $\frac{|T|}{24} < \frac{10^3}{|T|}$, and so $T = A_5$.

Let $T = A_5$. However, there are no integer solutions to equation $k(k-1) = (|T|^2 - 1)\lambda = 3599\lambda$, and so there are no solutions in this case, contrary to Lemma 4.

Thus, we have $\ell = 2$ and $v = |T|$. We now assume that $G = T \times T$. Then $k \mid |G_{\overline{T}}|$ and $k(k-1) = \lambda(|T| - 1)$ which lead to the contradiction $k \mid \lambda$.

Let $T \times T < G \leqslant T^2.(Out(T) \times S_2)$. Therefore, by (1), we have $k \mid 2\lambda |Out(T)|$. Now $k \leqslant 10^3$ and $(\lambda v)^{\frac{1}{2}} < k$ imply that $|T| < 10^6$ and, by Lemma 8, $|Out(T)| = 1, 2, 3, 4, 5, 6$ or 12.

Since $k \mid 2\lambda |Out(T)|$, there exists some positive integer z such that $k = \frac{2\lambda |Out(T)|}{z}$, where $1 \leqslant z < 2|Out(T)|$.

Now $k(k-1) = \lambda(|T| - 1)$ yields

$$2|Out(T)|(2\lambda |Out(T)| - z) = z^2(|T| - 1). \tag{3}$$

By Lemma 6, we have $2 \mid z$. Then $z < 2|Out(T)|$ and $|Out(T)| \leqslant 4$ imply that $(|Out(T)|, z) = (2, 2)$, $(3, 2)$, $(3, 4)$, $(4, 2)$, $(4, 4)$ or $(4, 6)$. From Lemma 6 and 2 divides $|T|$, we conclude that $(|Out(T)|, z) \neq (2, 2)$, $(4, 2)$, $(4, 4)$, $(4, 6)$.

We will first assume that $(|Out(T)|, z) = (3, 2)$ or $(3, 4)$. Then 3 divides $|T| - 1$ and Lemma 7 imply that $T = Sz(8)$. However, there are no integer solutions to equation $k(k-1) = \lambda(|Sz(8)| - 1)$, and so there are no solutions in this case, contrary to Lemma 4.

Now assume that $|Out(T)| = 5$. Then $T = L_2(32)$, and so (3) implies that $z = 4$, $\lambda = 5238$ and $k = 13095$. However, $n = k - \lambda = 7858$ is not a square, contradicting with Lemma 5 (i).

If $|Out(T)| = 6$, then $T = U_3(5)$ or $L_2(27)$. This together with (3), we have $z = 6$, $k = 2\lambda$ and so 2 divides $|T| - 1$, contradicting with Lemma 6.

Finally, assume that $|Out(T)| = 12$. Then $T = L_3(4)$. Moreover, from (3) and $1 \leqslant z < 2|Out(T)| = 24$, we get that $z = 12$. As above, $2(2\lambda - 1) = |L_3(4)| - 1$ which implies that $(2, |L_3(4)|) = 1$, contrary to Lemma 6. This completes the proof. □

Proposition 10. *If $\mathcal{D}$ is a symmetric (v, k, λ) design with $k \leqslant 10^3$ which admits a flag-transitive and point-primitive automorphism group G, then G is not of twisted wreath product type.*

Proof. Suppose that $G \leqslant T^\ell : S_\ell$ has a twisted wreath product action on $\mathcal{P}$. Here $Soc(G) = T^\ell$ is regular on $\mathcal{P}$ and $\ell \geqslant 6$. Since $(\lambda v)^{\frac{1}{2}} < k$, this leads to the contradiction that $k > |T|^3 \geqslant 60^3 > 10^3$. Thus, G is not of twisted wreath product type. □

4 Product action

Suppose that $G \leqslant H \wr S_\ell = H_1 \times H_2 \times \cdots \times H_\ell : S_\ell$ has a product action on $\mathcal{P} = \Delta^\ell = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_\ell$, where H_i with a primitive action (of almost simple or simple diagonal type) on a set Δ_i of size $\omega \geqslant 5$, $\ell \geqslant 2$, $H_i \cong H$ and $\Delta_i = \Delta$ for $i = 1, 2, \ldots, \ell$. Then, $|\mathcal{P}| = v = \omega^\ell$. Let $Soc(H) = T^d$ and $Soc(G) = T^{d\ell}$, where $d \geqslant 1$.

Lemma 11 ([13, Lemma 4])**.** $k \mid \lambda\ell(\omega - 1)$.

Lemma 12. *The following statements hold.*

(i) *If $\ell = 2$, then $\omega \leqslant 999$.*

(ii) *If $\ell = 3$, then $\omega \leqslant 99$.*

(iii) *If $\ell = 4$, then $\omega \leqslant 31$.*

(iv) *If $\ell \geqslant 5$, then $\omega \leqslant 14$.*

Proof. Using $k(k-1) = \lambda(v-1)$ and $v = \omega^\ell$, we get $\omega^\ell - 1 \leqslant \lambda(\omega^\ell - 1) \leqslant 999000$, and so (i)-(iii) hold. For part (iv), we have $\omega^5 \leqslant \omega^\ell \leqslant 999001$ which implies that $\omega \leqslant 14$. □

First of all, we have

Lemma 13. *H cannot be of simple diagonal type.*

Case	ℓ	v	$Soc(H)$	$Out(T)$	Case	ℓ	v	$Soc(H)$	$Out(T)$
1	3	60^3	$A_5 \times A_5$	2	4	2	360^2	$A_6 \times A_6$	2^2
2	2	60^2	$A_5 \times A_5$	2	5	2	504^2	$L_2(8) \times L_2(8)$	3
3	2	168^2	$L_2(7) \times L_2(7)$	2	6	2	660^2	$L_2(11) \times L_2(11)$	2

Proof. Suppose that H is of simple diagonal type. Here $Soc(H) = T^d$, $d \geqslant 2$ and T is a non-abelian simple group. Then we obtain all possible quadruples $(\ell, v, Soc(H), Out(T))$ by Lemmas 8 and 12, and they are listed in the following table.

Thus, we have $G_\alpha \leqslant (Aut(T) \times S_2) \wr S_\ell$ and therefore, by $k \mid \lambda(v-1)$,

$$k \mid \lambda(2^\ell \ell! |T|^\ell |Out(T)|^\ell, |T|^\ell - 1),$$

that is to say, k divides $\lambda|Out(T)|^\ell$. Then there exists some positive integer z such that $k = \frac{\lambda|Out(T)|^\ell}{z}$, where $1 \leqslant z < |Out(T)|^\ell$. And $k(k-1) = \lambda(|T|^\ell - 1)$, so

$$|Out(T)|^\ell(\lambda|Out(T)|^\ell - z) = z^2(|T|^\ell - 1). \tag{4}$$

Recall that $Soc(H) = T^2 = T \times T$ and $Out(T) = 2$, 3 or 2^2. Now we need to check each tuple $(\ell, Out(T))$ of above cases whether it satisfies (4). Thus, we get that $(\ell, Out(T)) = (3, 2)$, $(2, 2)$, $(2, 3)$ or $(2, 2^2)$.

By Lemma 6, we have $(\ell, Out(T)) = (2, 3)$. It follows (4) that 3 divides $|T|^2 - 1$, this implies that $T = Sz(q)$, contrary to Lemma 12. This completes the proof of Lemma 13. □

Therefore, the following result holds.

Lemma 14. *If G is of product action, then H is an almost simple group with socle T acting transitively on Δ. Moreover, if $\alpha = (\delta, \delta, \ldots, \delta) \in \mathcal{P}$ with $\delta \in \Delta$, then k divides $\frac{\ell! \cdot |Out(T)|^\ell \cdot |T|^\ell}{\omega^\ell} = \ell! \cdot |Out(T)|^\ell \cdot |T_\delta|^\ell$.*

Proof. It follows immediately from Lemma 13 and [16, Lemma 3.10]. □

By Lemma 11, we know that $k = \frac{\lambda \ell(\omega - 1)}{z}$ for some positive integer z. And $\lambda(v-1) < k^2$, so

$$\frac{\omega^{\ell-1} + \omega^{\ell-2} + \cdots + \omega + 1}{\omega - 1} = \frac{\omega^\ell - 1}{(\omega-1)^2} < \frac{\lambda \ell^2}{z^2}.$$

Now we examine the possible parameters in Lemma 12 case by case and by $k(k-1) = \lambda(v-1)$, we obtain all the possible parameters $(\omega, \ell, k, \lambda, z)$ by using the software package GAP[7] and the possible socles for H by [1, 4]. There are 96 cases which listed in Tables 2-5. In particular, cases for $\ell = 2$ and z odd (resp. z even) are listed in Table 2 (resp. Table 3).

Table 2: Cases for ℓ=2 and z odd

Case	ω	(v,k,λ)	z	$Soc(H)$	Stabilizer in $Soc(H)$
1	5	$(5^2,16,10)$	5	A_5	A_4
2	9	$(9^2,16,3)$	3	A_9, $L_2(8)$	A_8, $2^3:7$
3	13	$(13^2,64,24)$	9	A_{13}, $L_3(3)$	A_{12}, $3^2:2S_4$
4	13	$(13^2,120,85)$	17	A_{13}, $L_3(3)$	A_{12}, $3^2:2S_4$
5	17	$(17^2,64,14)$	7	A_{17}, $L_2(16)$	A_{16}, $2^4:15$
6	21	$(21^2,56,7)$	5	A_{21}, $L_3(4)$, A_7, $L_2(7).2$	A_{20}, $2^4:A_5$, S_5, D_{16}
7	21	$(21^2,320,232)$	29	A_{21}, $L_3(4)$, A_7, $L_2(7).2$	A_{20}, $2^4:A_5$, S_5, D_{16}
8	25	$(25^2,144,33)$	11	A_{25}	A_{24}
9	25	$(25^2,352,198)$	27	A_{25}	A_{24}
10	29	$(29^2,616,451)$	41	A_{29}	A_{28}
11	29	$(29^2,736,644)$	49	A_{29}	A_{28}
12	33	$(33^2,256,60)$	15	A_{33}, $L_2(32)$	A_{32}, $2^5:31$
13	37	$(37^2,856,535)$	45	A_{37}	A_{36}
14	41	$(41^2,400,95)$	19	A_{41}	A_{40}
15	41	$(41^2,736,322)$	35	A_{41}	A_{40}
16	45	$(45^2,760,285)$	33	A_{45}, $A_6.2$, A_{10}, $U_4(2)$	A_{44}, D_{16}, S_8, $2.(A_4\times A_4).2$
17	49	$(49^2,576,138)$	23	A_{49}	A_{48}
18	53	$(53^2,352,44)$	13	A_{53}	A_{52}
19	57	$(57^2,784,189)$	27	A_{57}, $L_3(7)$, $L_2(19)$	A_{56}, $7^2:2L_2(7):2$, A_5
20	61	$(61^2,280,21)$	9	A_{61}	A_{60}
21	85	$(85^2,904,113)$	21	$A_{85}, S_4(4), L_4(4)$	$A_{84}, 2^6:(3\times A_5), 2^6:GL_3(4)$
22	89	$(89^2,496,31)$	11	A_{89}	A_{88}

Table 3: Cases for ℓ=2 and z even

Case	ω	(v,k,λ)	z	$Soc(H)$	Stabilizer in $Soc(H)$
23	6	$(6^2,15,6)$	4	A_6, A_5	$A_5, 5:2$
24	7	$(7^2,33,22)$	8	A_7, $L_2(7)$	A_6, S_4
25	8	$(8^2,28,12)$	6	A_8, $L_2(7)$	A_7, $7:3$
26	10	$(10^2,45,20)$	8	A_{10}, A_5, A_6	A_9, S_3, $3^2:4$
27	11	$(11^2,25,5)$	4	$A_{11}, L_2(11), M_{11}$	A_{10}, A_5, M_{10}
28	12	$(12^2,66,30)$	10	$A_{12}, M_{11}, M_{12}, L_2(11)$	$A_{11}, L_2(11), M_{11}, 11:5$
29	13	$(13^2,57,19)$	8	A_{13}, $L_3(3)$	A_{12}, $3^2:2S_4$
30	14	$(14^2,91,42)$	12	$A_{14}, L_2(13)$	$A_{13}, 13:6$
31	15	$(15^2,161,115)$	20	A_{15}, A_6, A_7, A_8	A_{14}, S_4, $L_2(7)$, $2^3:L_3(2)$
32	16	$(16^2,120,56)$	14	A_{16}	A_{15}
33	16	$(16^2,171,114)$	20	A_{16}	A_{15}
34	16	$(16^2,205,164)$	24	A_{16}	A_{15}
35	18	$(18^2,153,72)$	16	A_{18}	A_{17}
36	19	$(19^2,81,18)$	8	A_{19}	A_{18}
37	20	$(20^2,190,90)$	18	$A_{20}, L_2(19)$	$A_{19}, 19:9$
38	21	$(21^2,265,159)$	24	A_{21}, $L_3(4)$, A_7, $L_2(7).2$	A_{20}, $2^4:A_5$, S_5, D_{16}
39	22	$(22^2,70,10)$	6	A_{22}, M_{22}	$A_{21}, L_3(4)$

40	22	$(22^2, 162, 54)$	14	A_{22}, M_{22}	$A_{21}, L_3(4)$
41	22	$(22^2, 231, 110)$	20	A_{22}, M_{22}	$A_{21}, L_3(4)$
42	23	$(23^2, 385, 280)$	32	A_{23}, M_{23}	A_{22}, M_{22}
43	24	$(24^2, 276, 132)$	22	$A_{24}, M_{24}, L_2(23)$	$A_{23}, M_{23}, 23:11$
44	25	$(25^2, 417, 278)$	32	A_{25}	A_{24}
45	26	$(26^2, 325, 156)$	24	$A_{26}, L_2(25)$	$A_{25}, 5^2:12$
46	27	$(27^2, 169, 39)$	12	$A_{27},\ U_4(2)$	$A_{26},\ 2^4:L_2(4)$
47	28	$(28^2, 378, 182)$	26	$A_{28},\ A_8,\ L_2(8),\ L_2(27),$ $L_2(7).2,\ U_3(3),\ S_6(2)$	$A_{27},\ S_6,\ D_{18},\ 3^3:13,$ $D_{12},\ 3_+^{1+2}:8,\ U_4(2):2$
48	29	$(29^2, 721, 618)$	48	A_{29}	A_{28}
49	30	$(30^2, 435, 210)$	28	$A_{30}, L_2(29)$	$A_{29}, 29:14$
50	31	$(31^2, 321, 107)$	20	$A_{31}, L_3(5), L_5(2)$	$A_{30}, 5^2:GL_2(5), 2^4:L_4(2)$
51	31	$(31^2, 385, 154)$	24	$A_{31}, L_3(5), L_5(2)$	$A_{30}, 5^2:GL_2(5), 2^4:L_4(2)$
52	31	$(31^2, 705, 517)$	44	$A_{31}, L_3(5), L_5(2)$	$A_{30}, 5^2:GL_2(5), 2^4:L_4(2)$
53	32	$(32^2, 496, 240)$	30	$A_{32}, L_2(31)$	$A_{31}, 31:15$
54	34	$(34^2, 561, 272)$	32	A_{34}	A_{33}
55	34	$(34^2, 771, 514)$	44	A_{34}	A_{33}
56	34	$(34^2, 946, 774)$	54	A_{34}	A_{33}
57	35	$(35^2, 289, 68)$	16	A_{35}, A_7, A_8	$A_{34}, (A_4{\times}3):2; 2^4:(S_3{\times}S_3)$
58	36	$(36^2, 260, 52)$	14	$A_{36}, L_2(8), M_{10}, PGL_2(9),$ $A_9, U_3(3), S_4(3), S_6(2)$	$A_{35}, D_{14}, D_{20}, D_{20},$ $S_7, L_2(7), S_6, S_8$
59	36	$(36^2, 371, 106)$	20	$A_{36}, L_2(8), M_{10}, PGL_2(9),$ $A_9, U_3(3), S_4(3), S_6(2)$	$A_{35}, D_{14}, D_{20}, D_{20},$ $S_7, L_2(7), S_6, S_8$
60	36	$(36^2, 630, 306)$	34	$A_{36}, L_2(8), M_{10}, PGL_2(9),$ $A_9, U_3(3), S_4(3), S_6(2)$	$A_{35}, D_{14}, D_{20}, D_{20},$ $S_7, L_2(7), S_6, S_8$
61	37	$(37^2, 153, 17)$	8	A_{37}	A_{36}
62	38	$(38^2, 703, 342)$	36	$A_{38}, L_2(37)$	$A_{37}, 37:18$
63	40	$(40^2, 247, 38)$	12	$A_{40}, L_4(3), S_4(3)$	$A_{49}, 3^3:L_3(3), 3_+^{1+2}:2A_4$
64	40	$(40^2, 534, 178)$	26	$A_{40}, L_4(3), S_4(3)$	$A_{49}, 3^3:L_3(3), 3_+^{1+2}:2A_4$
65	40	$(40^2, 780, 380)$	38	$A_{40}, L_4(3), S_4(3)$	$A_{49}, 3^3:L_3(3), 3_+^{1+2}:2A_4$
66	42	$(42^2, 861, 420)$	40	$A_{42}, L_2(41)$	$A_{41}, 41:20$
67	43	$(43^2, 441, 105)$	20	A_{43}	A_{42}
68	44	$(44^2, 946, 462)$	42	$A_{44}, L_2(43)$	$A_{43}, 43:21$
69	45	$(45^2, 737, 268)$	32	$A_{45},\ A_6.2,\ A_{10},\ U_4(2)$	$A_{44},\ D_{16},\ S_8,\ 2.(A_4\times A_4).2$
70	49	$(49^2, 801, 267)$	32	A_{49}	A_{48}
71	51	$(51^2, 625, 150)$	24	A_{51}	A_{50}
72	56	$(56^2, 286, 26)$	10	$A_{56}, L_3(4), A_8$	$A_{55}, A_6, (A_5\times 3):2$
73	58	$(58^2, 532, 84)$	18	A_{58}	A_{57}
74	59	$(59^2, 841, 203)$	28	A_{59}	A_{58}
75	61	$(61^2, 745, 149)$	24	A_{61}	A_{60}
76	69	$(69^2, 561, 66)$	16	A_{69}	A_{68}
77	71	$(71^2, 721, 103)$	20	A_{71}	A_{70}
78	76	$(76^2, 925, 148)$	24	A_{76}	A_{75}
79	79	$(79^2, 481, 37)$	12	A_{79}	A_{78}
80	106	$(106^2, 750, 50)$	14	A_{106}	A_{105}

Table 4: Cases for ℓ=3

Case	ω	(v, k, λ)	$Soc(H)$	Stabilizer in $Soc(H)$
81	5	$(5^3, 32, 8)$	A_5	A_4
82	6	$(6^3, 130, 78)$	A_6, A_5	$A_5, 5:2$
83	7	$(7^3, 153, 68)$	$A_7, L_2(7)$	A_6, S_4
84	7	$(7^3, 172, 86)$	$A_7, L_2(7)$	A_6, S_4
85	7	$(7^3, 324, 306)$	$A_7, L_2(7)$	A_6, S_4
86	8	$(8^3, 147, 42)$	$A_8, L_2(7)$	A_7, $7:3$
87	9	$(9^3, 456, 285)$	$A_9, L_2(8)$	A_8, $2^3:7$
88	10	$(10^3, 297, 88)$	A_{10}, A_5, A_6	A_9, S_3, $3^2:4$
89	11	$(11^3, 400, 120)$	$A_{11}, L_2(11), M_{11}$	A_{10}, A_5, M_{10}
90	11	$(11^3, 666, 333)$	$A_{11}, L_2(11), M_{11}$	A_{10}, A_5, M_{10}
91	14	$(14^3, 845, 260)$	$A_{14}, L_2(13)$	$A_{13}, 13:6$
92	15	$(15^3, 483, 69)$	A_{15}, A_6, A_7, A_8	A_{14}, S_4, $L_2(7)$, $2^3:L_3(2)$
93	16	$(16^3, 820, 164)$	A_{16}	A_{15}

Table 5: Cases for ℓ=4

Case	ω	(v, k, λ)	$Soc(H)$	Stabilizer in $Soc(H)$
94	5	$(5^4, 352, 198)$	A_5	A_4
95	6	$(6^4, 260, 52)$	A_6, A_5	$A_5, 5:2$
96	7	$(7^4, 801, 267)$	$A_7, L_2(7)$	A_6, S_4

In the following, we begin to deal with the possible cases of Tables 2-5 one by one. First of all, we deal with the possible cases of Tables 2 and 3. It should be noted that $\ell = 2$.

Let $\delta \in \Delta$ and let T_δ be the stabilizer of δ in T. We first use Lemma 14 to rule out some possibilities of T in column 5 of Tables 2, 3 with $k \nmid 2|T_\delta|^2|Out(T)|^2$, and listed in Table 6. So we have the following result.

Lemma 15. *The cases of Tables 6 cannot occur. Thus, for the remaining T in Tables 2, 3, we get that T acts 2-transitively on Δ and so does H.*

Table 6: Cases of Lemma 15

Case	(v, k, λ)	T	T_δ	$\lvert T_\delta\rvert$	$Out(T)$
6	$(21^2, 56, 7)$	A_7	S_5	120	2
6	$(21^2, 56, 7)$	$L_2(7).2$	D_{16}	16	2
7	$(21^2, 320, 232)$	A_7	S_5	120	2
16	$(45^2, 760, 285)$	M_{10}	D_{16}	16	2

16	$(45^2, 760, 285)$	A_{10}	S_8	40320	2
16	$(45^2, 760, 285)$	$S_4(3)$	$2.(A_4 \times A_4).2$	576	2
19	$(57^2, 784, 189)$	$L_2(19)$	A_5	60	2
21	$(85^2, 904, 113)$	$S_4(4)$	$2^6 : (3 \times A_5)$	11520	4
26	$(10^2, 45, 20)$	A_5	S_3	6	2
31	$(15^2, 161, 115)$	A_6	S_4	24	2^2
46	$(27^2, 169, 39)$	$U_4(2)$	$2^4 : L_2(4)$	960	2
47	$(28^2, 378, 182)$	A_8	S_6	720	2
47	$(28^2, 378, 182)$	$L_2(7).2$	D_{12}	12	2
47	$(28^2, 378, 182)$	$U_3(3)$	$3_+^{1+2} : 8$	216	2
57	$(35^2, 289, 68)$	A_7	$(A_4 \times 3) : 2$	72	2
57	$(35^2, 289, 68)$	A_8	$2^4 : (S_3 \times S_3)$	576	2
58	$(36^2, 260, 52)$	$L_2(8)$	D_{14}	14	3
58	$(36^2, 260, 52)$	M_{10}	D_{20}	20	2^2
58	$(36^2, 260, 52)$	A_9	S_7	5040	2
58	$(36^2, 260, 52)$	$PGL(2, 9)$	D_{20}	20	2^2
58	$(36^2, 260, 52)$	$U_3(3)$	$L_2(7)$	168	2
58	$(36^2, 260, 52)$	$S_4(3)$	S_6	720	2
59	$(36^2, 371, 106)$	$L_2(8)$	D_{14}	14	3
59	$(36^2, 371, 106)$	M_{10}	D_{20}	20	2^2
59	$(36^2, 371, 106)$	A_9	S_7	5040	2
59	$(36^2, 371, 106)$	$PGL(2, 9)$	D_{20}	20	2^2
59	$(36^2, 371, 106)$	$U_3(3)$	$L_2(7)$	168	2
59	$(36^2, 371, 106)$	$S_4(3)$	S_6	720	2
60	$(36^2, 630, 306)$	$L_2(8)$	D_{14}	14	3
60	$(36^2, 630, 306)$	M_{10}	D_{20}	20	2^2
60	$(36^2, 630, 306)$	A_9	S_7	5040	2
60	$(36^2, 630, 306)$	$PGL(2, 9)$	D_{20}	20	2^2
60	$(36^2, 630, 306)$	$U_3(3)$	$L_2(7)$	168	2
60	$(36^2, 630, 306)$	$S_4(3)$	S_6	720	2
63	$(40^2, 247, 38)$	$S_4(3)$	$3_+^{1+2} : 2A_4$	648	2
64	$(40^2, 534, 178)$	$S_4(3)$	$3_+^{1+2} : 2A_4$	648	2
65	$(40^2, 780, 380)$	$S_4(3)$	$3_+^{1+2} : 2A_4$	648	2
69	$(45^2, 737, 268)$	$A_6.2$	D_{16}	16	2^2
69	$(45^2, 737, 268)$	$U_4(2)$	$2.(A_4 \times A_4).2$	576	2
69	$(45^2, 737, 268)$	A_{10}	S_8	576	2
72	$(56^2, 286, 26)$	$L_3(4)$	A_6	360	$2 \times S_3$
72	$(56^2, 286, 26)$	A_8	$(A_5 \times 3) : 2$	360	2

Proof. We only need to check each possible case of Tables 2, 3 one by one. The last statement follows from $T \trianglelefteq H$. □

Let $Soc(H_1) = T_1$ and $Soc(H_2) = T_2$. Clearly, $T_1 \cong T_2 \cong T$ and $Soc(G) = T_1 \times T_2$. Now we begin to deal with the remaining cases of Tables 2, 3. Let $\alpha = (\delta, \delta) \in \mathcal{P} = \Delta_1 \times \Delta_2$. Recall that T_i acts 2-transitively on Δ_i for $i = 1, 2$. Thus, we get that

Lemma 16. *$(T_1 \times T_2)_\alpha \cong (T_1)_\delta \times (T_2)_\delta$ and $G_\alpha \leqslant (H_1)_\delta \times (H_2)_\delta : S_2$ acting on $\mathcal{P}$ has three orbits $\Theta_1 = \{(\delta, \delta)\}$, $\Theta_2 = (\delta, \delta^*)^{G_\alpha} = \{(\delta^{*t}, \delta)|\ t \in (T_1)_\delta\} \cup \{(\delta, \delta^{*t})|\ t \in (T_2)_\delta\}$ and $\Theta_3 = (\delta^*, \delta^*)^{G_\alpha} = \{(\delta^{*t_1}, \delta^{*t_2})| t_1 \in (T_1)_\delta$ and $t_2 \in (T_2)_\delta\}$, where $\delta^* \in \Delta \setminus \{\delta\}$. Furthermore, $|\Theta_1| = 1$, $|\Theta_2| = 2(\omega - 1)$ and $|\Theta_3| = (\omega - 1)^2$.*

Set $\Gamma_i = \alpha^{T_i}$ for $i = 1, 2$, where $T_1 \cong T_1 \times 1$ and $T_2 \cong 1 \times T_2$. In particular, $\Gamma_1 = \alpha^{T_1 \times 1} = \{(\delta^{t_1}, \delta)|\ t_1 \in T_1\}$, $\Gamma_2 = \alpha^{1 \times T_2} = \{(\delta, \delta^{t_2})|\ t_2 \in T_2\}$. Then the transitivity of T_i on Δ_i implies that

Lemma 17. *$|\Gamma_1| = |\Gamma_2| = \omega$, $|\Gamma_1 \cap \Gamma_2| = 1$ and $|\Theta_2 \cap \Gamma_1| = |\Theta_2 \cap \Gamma_2| = \omega - 1$.*

Lemma 18. *Let $c = |\Theta_2 \cap B|$, where $\alpha \in B$. Then the following hold:*

(i) *c is independent of the choice of the block through α;*

(ii) *$kc = \lambda|\Theta_2| = 2\lambda(\omega - 1)$ and $c = z$ is independent of the choice of α.*

Proof. (i) Let B^* be a block such that $\alpha \in B^*$. The flag-transitivity of G implies that there exists $g \in G_\alpha$ such that $B^g = B^*$. Then $(\Theta_2 \cap B)^g = \Theta_2 \cap B^*$ by $\Theta_2^g = \Theta_2$. Thus, $|\Theta_2 \cap B| = |\Theta_2 \cap B^*| = c$ and c is independent of the choice of the block through α.

(ii) Counting in two ways the flags (β, B) of $\mathcal{D}$ such that $\beta \in \Theta_2$ and $\alpha \in B$, we have $kc = \lambda|\Theta_2| = 2\lambda(\omega - 1)$. The last statement follows from $z = \frac{2\lambda(\omega-1)}{k}$. □

The following result will play an important role in this section.

Lemma 19. *Let $M = N_G(T_1) \cap N_G(T_2)$ and $\alpha \in B$. Then $|G : M| = 2$, and one of the following holds:*

(i) *if $G_{\alpha B}$ cannot interchange T_1 and T_2, then $M_{\alpha B} = G_{\alpha B}$ and $(\alpha^{M_B}, B^M) = (k, \frac{b}{2})$ or $(\frac{k}{2}, b)$;*

(ii) *if $G_{\alpha B}$ can interchange T_1 and T_2, then $|G_{\alpha B} : M_{\alpha B}| = 2$ and $(\alpha^{M_B}, B^M) = (k, b)$, that is to say, M acts flag-transitively on $\mathcal{D}$.*

Proof. By the primitivity of G on $\mathcal{P}$, we have $Soc(G) = T_1 \times T_2$ is a minimal normal subgroup of G which implies that G acts transitively on $\{T_1, T_2\}$ by conjugation. Note that M is the stabilizer of T_1 in G. Then $|G : M| = 2$.

(i) If $G_{\alpha B}$ cannot interchange T_1 and T_2, then $G_{\alpha B} \leqslant M_{\alpha B}$. Thus, $M \leqslant G$ implies that $G_{\alpha B} = M_{\alpha B}$.

By the flag-transitivity of G, we have

$$|G : G_{\alpha B}| = |G : M_{\alpha B}| = |G : M| \cdot |M : M_B| \cdot |M_B : M_{\alpha B}|$$

which, by the primitivity of G and $M \trianglelefteq G$, implies that

$$bk = |G : M| \cdot |B^M| \cdot |\alpha^{M_B}|.$$

At this point, $|G : M| = 2$ yields $|\alpha^{M_B}| = k$ or $\frac{k}{2}$.

(ii) The second statement follows from

$$|G : G_{\alpha B}| = \frac{1}{2}|G : M_{\alpha B}| = \frac{1}{2}|G : M| \cdot |M : M_{\alpha B}| = |M : M_B| \cdot |M_B : M_{\alpha B}| = |B^M| \cdot |\alpha^{M_B}|$$

and $|\alpha^{M_B}|$ divides k. □

Lemma 20. *$Fix_{\mathcal{P}}((T_1)_\alpha) = \Gamma_2$ and $Fix_{\mathcal{P}}((T_2)_\alpha) = \Gamma_1$.*

Proof. We only need to prove the first assertion. Clearly, $(T_1)_\alpha \cong (T_1)_\delta \times 1$ and $\Gamma_2 \subseteq Fix_{\mathcal{P}}((T_1)_\alpha)$, where $\alpha = (\delta, \delta) \in \mathcal{P}$. On the other hand, choose an element $(\varepsilon_1, \varepsilon_2)$ in $Fix_{\mathcal{P}}((T_1)_\alpha)$. By the 2-transitivity of T, we have $\varepsilon_1 = \delta$ which implies that $Fix_{\mathcal{P}}((T_1)_\alpha) \subseteq \Gamma_2$. Thus, $Fix_{\mathcal{P}}((T_1)_\alpha) = \Gamma_2$. □

Let $\beta \in \Gamma_2$ and denote by $\mathcal{J}$ the set of blocks of $\mathcal{D}$ through α and β. Clearly, $|\mathcal{J}| = \lambda$ and $\mathcal{J}^{(T_1)_\alpha} = \mathcal{J}$.

Lemma 21. *If $(T_1)_\alpha \leqslant (T_1)_B$ for some $B \in \mathcal{J}$, then $M_B \leqslant N_M((T_1)_\alpha)$. Furthermore, $\alpha^{M_B} \subseteq B \cap \Gamma_2$.*

Proof. Recall that $(T_1)_\alpha \cong (T_1)_\delta \times 1$. By the 2-transitivity of T_1 on Δ_1, we have $(T_1)_\alpha$ is a maximal subgroup of $T_1 \cong T_1 \times 1$. If $(T_1)_B = T_1$, then

$$(\alpha, B)^{(T_1)_B} = (\alpha^{(T_1)_B}, B) = (\alpha^{T_1}, B) = (\Gamma_1, B),$$

in other words, $\Gamma_1 \subseteq B$. By the transitivity of G_B on B, we have ω divides k. However, there is no case of Tables 2 and 3 which can satisfy the condition of $\omega \mid k$, a contradiction. Thus, $(T_1)_B = (T_1)_\alpha$ and, by $(T_1)_B \trianglelefteq M_B$, $M_B \leqslant N_M((T_1)_\alpha)$.

Let $\gamma \in \alpha^{M_B}$. Then there exists an element $t \in M_B$ such that $\gamma = \alpha^t$. Thus, $(T_1)_\alpha^t = (T_1)_\alpha$ and $\gamma^{(T_1)_\alpha} = \gamma^{(T_1)_\alpha^t} = \alpha^{t(T_1)_\alpha^t} = \alpha^t = \gamma$ by $M_B \leqslant N_M((T_1)_\alpha)$, namely, $\gamma \in Fix_B((T_1)_\alpha)$. By Lemma 20, we have $Fix_{\mathcal{P}}((T_1)_\alpha) = \Gamma_2$ which implies that $\alpha^{M_B} \subseteq B \cap \Gamma_2$. □

Lemma 22. *$|\Gamma_1 \cap B| + |\Gamma_2 \cap B| = c + 2$.*

Proof. From $\{\Gamma_1 \cup \Gamma_2\} \setminus \{\alpha\} = \Theta_2$ and $\{(\Gamma_1 \cap B) \cup (\Gamma_2 \cap B)\} \setminus \{\alpha\} = \Theta_2 \cap B$, we conclude that

$$\{\Gamma_1 \cap B \setminus \{\alpha\}\} \cup \{\Gamma_2 \cap B \setminus \{\alpha\}\} = \Theta_2 \cap B.$$

On the other hand, $\{\Gamma_1 \cap B \setminus \{\alpha\}\} \cap \{\Gamma_2 \cap B \setminus \{\alpha\}\} = \varnothing$ and Lemma 18 imply that

$$|\Gamma_1 \cap B| - 1 + |\Gamma_2 \cap B| - 1 = |\Theta_2 \cap B| = c.$$

Thus, $|\Gamma_1 \cap B| + |\Gamma_2 \cap B| = c + 2$. □

Table 7: Cases of Lemma 23

Case	(v,k,λ)	T	T_δ	$(T_\delta)_{min}$	Case	(v,k,λ)	T	T_δ	$(T_\delta)_{min}$
2	$(9^2,16,3)$	A_9	A_8	8	39	$(22^2,70,10)$	A_{22}	A_{21}	21
2	$(9^2,16,3)$	$L_2(8)$	$2^3:7$	7	39	$(22^2,70,10)$	M_{22}	$L_3(4)$	21
5	$(17^2,64,14)$	A_{17}	A_{16}	16	61	$(37^2,153,17)$	A_{37}	A_{36}	36
6	$(21^2,56,7)$	A_{21}	A_{20}	20	63	$(40^2,247,38)$	A_{40}	A_{39}	39
6	$(21^2,56,7)$	$L_3(4)$	$2^4:A_5$	5	72	$(56^2,286,26)$	A_{56}	A_{55}	55
18	$(53^2,352,44)$	A_{53}	A_{52}	52	76	$(69^2,561,66)$	A_{69}	A_{68}	68
20	$(61^2,280,21)$	A_{61}	A_{60}	60	79	$(79^2,481,37)$	A_{79}	A_{78}	78
22	$(89^2,496,31)$	A_{89}	A_{88}	88	80	$(106^2,750,50)$	A_{106}	A_{105}	105
27	$(11^2,25,5)$	A_{11}	A_{10}	10					

Lemma 23. *If $(T_1)_\alpha \leqslant (T_1)_B$ for some $B \in \mathcal{J}$, then $k \leqslant 2(c+1)$. Therefore, the cases of Table 7 cannot occur.*

Proof. Lemmas 19, 21 and 22 imply that $\frac{k}{2} \leqslant |\alpha^{M_B}| \leqslant |B\cap\Gamma_2| \leqslant c+1$, and so $k \leqslant 2(c+1)$. Let $(T_\delta)_{min}$ denote the minimal degree of T_δ. Recall that $(T_1)_\alpha \cong (T_1)_\delta \times 1$. In each case of Table 7, $(T_\delta)_{min} > \lambda$ implies that there exists $B \in \mathcal{J}$ such that $(T_1)_\alpha \leqslant (T_1)_B$. However, there is no case of Table 7 which can satisfy the condition of $k \leqslant 2(c+1)$. Thus, the cases of Table 7 cannot occur. For the values of $(T_\delta)_{min}$, we only need to consider the indexes of maximal subgroups of T_δ. □

From now on we begin to deal with the remaining cases of Table 2. It should be noted that c is odd. First, we have

Lemma 24. *$G_{\alpha,B}$ cannot interchange T_1, T_2.*

Proof. Set $x = |B \cap \Gamma_1 \cap \Theta_2|$. If $G_{\alpha,B}$ interchange T_1 and T_2, then there exists an element $g \in G_{\alpha,B}$ such that $T_1^g = T_2$. So, $(B \cap \Gamma_1 \cap \Theta_2)^g = B \cap \Gamma_2 \cap \Theta_2$ implies that $|B \cap \Gamma_1 \cap \Theta_2| = |B \cap \Gamma_2 \cap \Theta_2| = x$ and $|B \cap \Theta_2| = c = 2x$, this leads to the contradiction that $2 \mid c$. □

By Lemmas 19 and 24, one of the following holds:

(I) $|\alpha^{M_B}| = k$ and $|B^M| = \frac{b}{2}$;

(II) $|\alpha^{M_B}| = \frac{k}{2}$ and $|B^M| = b$.

Now we begin to deal with the above two cases one by one.
Case (I): $|\alpha^{M_B}| = k$ and $|B^M| = \frac{b}{2}$.
Without loss of generality, we may assume that $|B \cap \Gamma_1| < |B \cap \Gamma_2|$.

Lemma 25. *Let $|B \cap \Gamma_1| \leqslant \frac{c+1}{2}$ and $|B \cap \Gamma_2| \geqslant \frac{c+3}{2}$. Then the following hold:*

(i) $k > \frac{(c-1)\omega}{2}$;

(ii) $|B \cap \Gamma_1| = \frac{c+1}{2}$ *and* $|B \cap \Gamma_2| = \frac{c+3}{2}$. *Furthermore,* $\frac{(c+1)(c+3)}{4}$ *divides* k.

Proof. The statement (i) follows immediately from the remaining cases of Table 2. For (ii), since M_B acts transitively on B, we have $k \leqslant |B \cap \Gamma_1|\omega$ and, by $|B \cap \Gamma_1| \leqslant \frac{c+1}{2}$ and (i), $|B \cap \Gamma_1| = \frac{c+1}{2}$ and $|B \cap \Gamma_2| = \frac{c+3}{2}$. The last assertion follows from the transitivity of M_B on B. □

Theorem 26. *If M_B is transitive on B, then the remaining cases of Table 2 cannot occur.*

Proof. We only need to check each possible case of Table 3 whether it satisfies the condition of $\frac{(c+1)(c+3)}{4}$ divides k. □

Case (II): $|\alpha^{M_B}| = \frac{k}{2}$ and $|B^M| = b$.

First of all, we have

Lemma 27. $|G_B : M_B| = 2$ *and there exists* $t \in G_B \setminus M_B$ *such that* $G_B = \langle M_B, t\rangle$. *Further,* $B = \alpha^{M_B} \cup (\alpha^{M_B})^t = \alpha^{M_B} \cup (\alpha^t)^{M_B}$.

Proof. Since $|G_B : M_B| = |G_B : G_B \cap M| = |G_BM : M|$, $|B^M| = b$ implies that $|G_B : M_B| = 2$. From $|\alpha^{G_B}| = k$ and $|\alpha^{M_B}| = \frac{k}{2}$, we conclude that $\alpha^{M_B} \cap (\alpha^{M_B})^t = \alpha^{M_B} \cap (\alpha^t)^{M_B} = \varnothing$. □

Let $C_1 = \alpha^{M_B}$, $C_2 = (\alpha^t)^{M_B}$ and let $C_{11} = \{\delta_1|(\delta_1,\delta_2) \in C_1\}$, $C_{12} = \{\delta_2|(\delta_1,\delta_2) \in C_1\}$, $C_{21} = \{\epsilon_1|(\epsilon_1,\epsilon_2) \in C_2\}$, $C_{22} = \{\epsilon_2|(\epsilon_1,\epsilon_2) \in C_2\}$.

Lemma 28. *The following hold:*

(i) $|C_{11}| = |C_{22}|$ *and* $|C_{12}| = |C_{21}|$;

(ii) $B \cap \Gamma_1 \subseteq C_1$ *and* $B \cap \Gamma_2 \subseteq C_2$.

Proof. (i) Let $\beta = \alpha^t$. Then $\beta \in C_2$. Thus, by $T_1^t = T_2$ and $T_2^t = T_1$,

$$(B \cap \Gamma_1)^t = B \cap \alpha^{T_1 t} = B \cap \alpha^{tT_2} = B \cap \beta^{T_2}$$

and

$$(B \cap \Gamma_2)^t = B \cap \alpha^{T_2 t} = B \cap \alpha^{tT_1} = B \cap \beta^{T_1}.$$

Now M_B acts transitively on C_i and $|C_1| = |C_2|$ imply that $|C_{11}| = |C_{22}|$ and $|C_{12}| = |C_{21}|$, where $i = 1, 2$.

(ii) We first assume that $\Gamma_1 \cap C_2 \neq \varnothing$ and $\Gamma_2 \cap C_2 \neq \varnothing$. Thus, by the fact that M_B acts transitively on C_i for $i = 1, 2$, $C_2 = C_1^t$ and $|C_1| = |C_2|$, we have $|\Gamma_1 \cap C_1| = |\Gamma_1 \cap C_2| = |\Gamma_2 \cap C_1| = |\Gamma_2 \cap C_1|$. At this point, $|B \cap \Gamma_i| = |(C_1 \cup C_2) \cap \Gamma_i| = |C_1 \cap \Gamma_i| + |C_2 \cap \Gamma_i|$ implies that $|B \cap \Gamma_1| + |B \cap \Gamma_2| = c + 2$ is even, contrary to the fact that c is odd.

Without loss of generality, we may assume that $\Gamma_1 \cap C_2 \neq \varnothing$ and $\Gamma_2 \cap C_2 = \varnothing$. Then, by the fact that M_B acts transitively on C_i and $C_2 = (C_1)^t$, $|C_{11}| = |C_{12}| = |C_{21}| = |C_{22}|$. Further more, $C_{11} = C_{21}$, $2|C_{11}| < \omega$ and $|B \cap \Gamma_2| = \frac{|B\cap\Gamma_1|}{2} = \frac{c+2}{3}$. Thus, we have $|B \cap \Gamma_2| \cdot |C_{11}| = \frac{k}{2}$ which implies that $\frac{c+2}{3}$ divides $\frac{k}{2}$ and $\frac{3k}{c+2} < \omega$. However, there is no such a case of Table 2 satisfying the above two conditions. Thus, $B \cap \Gamma_1 \subseteq C_1$ and $B \cap \Gamma_2 \subseteq C_2$. □

Lemma 29. *There exist two positive integers x, y such that $|B \cap \Gamma_1|x = |B \cap \Gamma_2|y = \frac{k}{2}$ and $x + y \leqslant \omega$.*

Proof. The conclusion follows from Lemmas 22, 27 and the fact that M_B acts transitively on C_i for $i = 1, 2$. The last statement follows from $|C_{11}| + |C_{12}| \leqslant \omega$. □

Theorem 30. *The cases of Table 2 cannot occur with the possible exception of $(Case, T) = (20, A_{61})$. For the exceptional case, $|B \cap \Gamma_1| = 4$, $|B \cap \Gamma_2| = 7$, $x = 35$ and $y = 20$.*

Proof. We only need to check each case of Table 2 whether it satisfies the following system of equations:

$$\begin{cases} |B \cap \Gamma_1| + |B \cap \Gamma_2| = c + 2; \\ |B \cap \Gamma_1|x = |B \cap \Gamma_2|y = \frac{k}{2}; \\ |B \cap \Gamma_1| \leqslant |B \cap \Gamma_2|; \\ x + y \leqslant \omega. \end{cases}$$

It should be noted that $|B \cap \Gamma_1|$ and $|B \cap \Gamma_2|$ in the first equation are both unknowns. □

Set $B_1 = \{\delta_1 | (\delta_1, \delta_2) \in B\}$ and $B_2 = \{\delta_2 | (\delta_1, \delta_2) \in B\}$. Clearly, $B_1 = C_{11} \cup C_{21}$ and $B_2 = C_{12} \cup C_{22}$ and $|B_1| = |B_2|$ by Lemma 28.

Theorem 31. *The possible exception of Lemma 30 cannot occur.*

Proof. Now $|B_1| = |B_2| = |C_{11}| + |C_{12}| = x + y = 55$. By $Soc(G) = T_1 \times T_2 \trianglelefteq M$ and $(T_1 \times T_2)_B \leqslant (T_1)_{B_1} \times (T_2)_{B_2}$, we have

$$|B^M| \geqslant |B^{T_1 \times T_2}| = |T_1 \times T_2 : (T_1 \times T_2)_B| \geqslant |B_1^{T_1}| \cdot |B_2^{T_2}| \geqslant \binom{61}{55}^2.$$

Since M is transitive on $\mathcal{B}$, this leads to the contradiction that $|\mathcal{B}| > |\mathcal{P}|$. □

Lemma 32. *If M_B is intransitive on B, then the remaining cases of Table 2 cannot occur.*

Proof. It follows immediately from Lemmas 30 and 31. □

In the following, we begin to deal with the possible remaining cases of Table 3. It should be noted that c is even. By Lemma 19 and c is even, one of the following cases holds.

(I) $G_{\alpha B} = M_{\alpha B}$ and one of the following holds.

- (i) $|\alpha^{M_B}| = k$ and $|B^M| = \frac{b}{2}$;
- (ii) $|\alpha^{M_B}| = \frac{k}{2}$ and $|B^M| = b$.

(II) $|G_{\alpha B} : M_{\alpha B}| = 2$ and the following holds.

- (i) $|\alpha^{M_B}| = k$ and $|B^M| = b$.

Now we begin to deal with the above three cases one by one.

Case (I) (i): $G_{\alpha,B} = M_{\alpha,B}$, $|\alpha^{M_B}| = k$ and $|B^M| = \frac{b}{2}$.

Without loss generality, we may assume that $|B \cap \Gamma_1| \leqslant |B \cap \Gamma_2|$. In particular, $|B \cap \Gamma_1| + |B \cap \Gamma_2| = c + 2$ is even.

Lemma 33. *Suppose that M_B acts transitively on B. Then the following statements hold:*

(i) $\frac{z\omega}{2} < k$;

(ii) $|B \cap \Gamma_1| = |B \cap \Gamma_2| = \frac{c+2}{2}$.

Proof. For (i), we only need to check each possible case of Table 3 one by one. The second statement (ii) follows from (i), $k \leqslant |B \cap \Gamma_1|\omega$ and $|B \cap \Gamma_1| \leqslant \frac{c+2}{2}$. □

Lemma 34. *The cases of Table 3 cannot occur with the possible exceptions of* $(Case, \omega) = (23, 6), (25, 8), (26, 10), (28, 12), (30, 14), (32, 16), (35, 18), (37, 20), (41, 22), (43, 24), (45, 26), (47, 28), (49, 30), (53, 32), (54, 34), (60, 36), (62, 38), (65, 40), (66, 42), (68, 44)$. *In particular, with the above possible exceptions, we always have* $|B_1| = |B_2| = \frac{|B|}{|B \cap \Gamma_1|} = \omega - 1$.

Proof. We only need to check each possible case of Table 3 whether it satisfies the condition of $\frac{c+2}{2} \mid k$. □

Lemma 35. *The possible exceptions of Lemma 34 cannot occur.*

Proof. Since $Soc(G) = T_1 \times T_2 \trianglelefteq M$, $|B^M| \geqslant |B^{T_1 \times T_2}| \geqslant |{B_1}^{T_1}| \cdot |{B_2}^{T_2}|$. Therefore, $|B_1| = |B_2| = \frac{|B|}{|B \cap \Gamma_1|} = \omega - 1$ and the transitivity of M_B on B imply that

$$b = 2|B^M| \geqslant 2|B^{T_1 \times T_2}| \geqslant 2|B_1^{T_1}| \cdot |B_2^{T_2}| \geqslant 2\omega^2 = 2v,$$

the desired contradiction. □

Theorem 36. *Suppose that $G_{\alpha,B}$ cannot interchange T_1, T_2 and M_B is transitive on B. Then the remaining cases of Table 3 cannot occur.*

Proof. It follows immediately from Lemmas 34 and 35. □

Case (I) (ii): $G_{\alpha,B} = M_{\alpha,B}$, $|\alpha^{M_B}| = \frac{k}{2}$ and $|B^M| = b$.

Here $B^M = \mathcal{B}$ and there exists $t \in G_B \setminus M_B$ such that $B = \alpha^{M_B} \cup (\alpha^t)^{M_B} = C_1 \cup C_2$.

Lemma 37. *If M_B is intransitive on B, then one of following holds:*

(i) $\Gamma_1 \cap C_2 \neq \varnothing$ *and* $\Gamma_2 \cap C_2 \neq \varnothing$;

(ii) $\Gamma_1 \cap C_2 = \varnothing$ *and* $\Gamma_2 \cap C_2 = \varnothing$.

Proof. Without loss of generality, we may assume that $\Gamma_1 \cap C_2 \neq \varnothing$ and $\Gamma_2 \cap C_2 = \varnothing$. Then $C_{12} \cap C_{22} \neq \varnothing$, and so the transitivity of M_B on C_i implies that $C_{12} = C_{22}$, where $i = 1, 2$. Note that $C_{11} \cap C_{21} = \varnothing$. Let $\beta = \alpha^t \in C_2$. By $T_1^t = T_2$ and $T_2^t = T_1$, we have $|C_{11}| = |C_{12}| = |C_{21}| = |C_{22}|$ and $|B \cap \Gamma_1| = 2|B \cap \Gamma_2|$. Thus, by Lemma 22, $|B \cap \Gamma_2| = \frac{c+2}{3}$ which implies that $\frac{c+2}{3}$ divides $\frac{k}{2}$. However, there is no case of Table 3 which can satisfy the condition of $\frac{c+2}{3} \mid \frac{k}{2}$. ☐

Lemma 38. *Suppose that $\Gamma_1 \cap C_2 \neq \varnothing$ and $\Gamma_2 \cap C_2 \neq \varnothing$. Then $|B \cap \Gamma_1| = |B \cap \Gamma_2| = \frac{c+2}{2}$ and $\frac{c+2}{4}$ divides $\frac{k}{2}$.*

Proof. Since $\Gamma_1 \cap C_2 \neq \varnothing$ and $\Gamma_2 \cap C_2 \neq \varnothing$, $C_{11} \cap C_{21} \neq \varnothing$ and $C_{12} \cap C_{22} \neq \varnothing$. The transitivity of M_B on C_i yields $C_{11} = C_{21}$ and $C_{12} = C_{22}$, where $i = 1, 2$. Together with $C_1^t = C_2$, $C_2^t = C_1$, we get that $|C_{11}| = |C_{22}| = |C_{12}| = |C_{21}|$. By Lemma 22, we have $|B \cap \Gamma_1| = |B \cap \Gamma_2| = \frac{c+2}{2}$ and $|C_i \cap \Gamma_1| = |C_i \cap \Gamma_2| = \frac{c+2}{4}$ for $i = 1, 2$. The last statement follows immediately from the transitivity of M_B on C_i, where $i = 1, 2$. ☐

Lemma 39. *Suppose that $\Gamma_1 \cap C_2 \neq \varnothing$ and $\Gamma_2 \cap C_2 \neq \varnothing$. Then the cases of Table 3 cannot occur with the possible exceptions of $(Case, \omega) = (25, 8)$, $(28, 12)$, $(32, 16)$, $(37, 20)$, $(43, 24)$, $(47, 28)$, $(53, 32)$, $(60, 36)$, $(65, 40)$ or $(68, 44)$. Furthermore, with the above possible exceptions, we always have $|C_{11}| = |C_{12}| = |C_{21}| = |C_{22}| = \omega - 1$ and $|B_1| = |B_2| = \omega - 1$.*

Proof. We only need to check each case of Table 3 for the condition $\frac{c+2}{4} \mid \frac{k}{2}$. ☐

Lemma 40. *The possible exceptions of Lemma 39 cannot occur.*

Proof. Clearly, we have $(T_1 \times T_2)_B \leqslant (T_1)_{B_1} \times (T_2)_{B_2}$ and $|B_1^{T_1}| = |B_2^{T_2}| = \omega - 1$. Since $Soc(G) = T_1 \times T_2$, $|B^M| \geqslant |B^{T_1 \times T_2}|$ and $|B_1^{T_1}| \cdot |B_2^{T_2}| = |T_1 : (T_1)_{B_1}| \cdot |T_2 : (T_2)_{B_2}|$,

$$|B^{T_1 \times T_2}| = |T_1 \times T_2 : (T_1 \times T_2)_B| \geqslant |T_1 \times T_2 : ((T_1)_{B_1} \times (T_2)_{B_2})| = |T_1 : (T_1)_{B_1}| \cdot |T_2 : (T_2)_{B_2}|$$

and Lemma 39 imply that $|B^M| = |B^{T_1 \times T_2}| = |B_1^{T_1}| \cdot |B_2^{T_2}| = \omega^2$. Thus, we have

$$(T_1 \times T_2)_B = (T_1)_{B_1} \times (T_2)_{B_2} = (T_1)_{C_{11}} \times (T_2)_{C_{12}} = (T_1)_{C_{21}} \times (T_2)_{C_{22}}.$$

Recall that T_i is 2-transitive on Δ_i for $i = 1, 2$. Thus, $(T_1)_{C_{11}}$ (resp. $(T_2)_{C_{12}}$) is transitive on C_{11} (resp. C_{12}). Let $(\delta_1, \delta_2) \in C_1$ and $(\delta_1', \delta_2') \in C_{11} \times C_{12}$. Then $(T_1 \times T_2)_B = (T_1)_{C_{11}} \times (T_2)_{C_{12}}$ implies that there exist $(t_1, t_2) \in (T_1)_{C_{11}} \times (T_2)_{C_{12}}$ such that $(\delta_1', \delta_2') = (\delta_1, \delta_2)^{(t_1, t_2)}$. In other words, $C_1 = C_{11} \times C_{12}$ and so $\frac{k}{2} = (\omega - 1)^2$. However, there is no exception of Lemma 39 which can satisfy the above condition. ☐

In the following, by Lemma 37, we only need to consider the case where $\Gamma_1 \cap C_2 = \varnothing$ and $\Gamma_2 \cap C_2 = \varnothing$.

Lemma 41. *Suppose that $\Gamma_1 \cap C_2 = \varnothing$ and $\Gamma_2 \cap C_2 = \varnothing$. Then there exist two positive integers x, y such that $|B \cap \Gamma_1| x = |B \cap \Gamma_2| y = \frac{k}{2}$ and $x + y \leqslant \omega$.*

Proof. This can be proved as Lemma 29. ☐

Lemma 42. *The possible remaining cases of Table 3 do not occur with the possible exceptions of* $(Case, \omega) = (32, 16)$ *or* $(60, 36)$*.*

(i) *If* $(Case, \omega) = (32, 16)$ *holds, then* $|B \cap \Gamma_1| = 6$*,* $|B \cap \Gamma_2| = 10$*,* $x = 10$ *and* $y = 6$*.*

(ii) *If* $(Case, \omega) = (60, 36)$ *holds, then* $|B \cap \Gamma_1| = 15$*,* $|B \cap \Gamma_2| = 21$*,* $x = 21$ *and* $y = 15$*.*

Further, with the above two possible exceptions, we have $\alpha^{M_B} = C_{11} \times C_{12} = \{(\delta_1, \delta_2) | \delta_1 \in C_{11}$ *and* $\delta_2 \in C_{12}\}$ *where* $\alpha \in B$ *and* $x + y = \omega$*.*

Proof. This can be proved as Lemma 30. □

Lemma 43. *The two possible exceptions of Lemma 42 cannot occur.*

Proof. Assume that $(Case, \omega) = (32, 16)$. Then $Soc(G) = A_{16} \times A_{16}$. Since $A_{16} \times A_{16} \trianglelefteq M$, it follows that $|\mathcal{B}| = |B^M| \geqslant |B^{A_{16} \times A_{16}}| \geqslant \binom{16}{6} > 16^2 = |\mathcal{P}|$, a contradiction. Assume that $(Case, \omega) = (60, 36)$. Then $Soc(G) = A_{36} \times A_{36}$ or $S_6(2) \times S_6(2)$. If $Soc(G) = A_{36} \times A_{36}$, then $|\mathcal{B}| = |B^M| \geqslant |B^{A_{36} \times A_{36}}| \geqslant \binom{36}{15} > 36^2 = |\mathcal{P}|$, a contradiction. If $Soc(G) = S_6(2) \times S_6(2)$, then $G = S_6(2) \wr S_2$. Note that G only has one conjugacy class of subgroups with index 1296, say G_B. However, G_B has no orbit of length 630, contradicting with the fact that the flag-transitivity of G. This completes the proof of Lemma 43. □

Theorem 44. *Suppose that* M_B *is intransitive on* B *and* $G_{\alpha,B}$ *cannot interchange* T_1*,* T_2*. Then the remaining cases of Table 3 cannot occur.*

Proof. It follows immediately from Lemmas 42 and 43. □

Case (II) (i): $|G_{\alpha,B} : M_{\alpha,B}| = 2$, $|\alpha^{M_B}| = k$ and $|B^M| = b$.

By the transitivity of M_B on B, we have $|B \cap \Gamma_1| = |B \cap \Gamma_2| = \frac{c+2}{2}$ and $\frac{c+2}{2}$ divides k (this can be proved as Lemma 33). Furthermore, we can get the result which is the same as Lemma 34. At this point, $|B_1| = |B_2| = \omega - 1$.

Lemma 45. *If* $|B_1| = |B_2| = \omega - 1$*, then* $k \geqslant (\omega - 1)^2$*.*

Proof. Let $\alpha = (\delta_1, \delta_2) \in B$. By the 2-transitivity of T_i on Δ_i, $(T_1 \times T_2)_\alpha = (T_1)_{\delta_1} \times (T_2)_{\delta_2}$ and $|B_1| = |B_2| = \omega - 1$,

$$|B^{M_\alpha}| = |M_\alpha : M_{\alpha B}| \geqslant |B^{(T_1 \times T_2)_\alpha}| = |B^{(T_1)_{\delta_1} \times (T_2)_{\delta_2}}| \geqslant |B_1^{(T_1)_{\delta_1}}| \cdot |B_2^{(T_2)_{\delta_2}}|$$

implies that $k \geqslant (\omega - 1)^2$. □

By checking each remaining case of Table 3, we prove that the cases of Table 3 cannot satisfy the conditions of $\frac{c+2}{2} \mid k$ and $k \geqslant (\omega - 1)^2$. Therefore, the following holds.

Lemma 46. *Suppose that* $G_{\alpha,B}$ *can interchange* T_1*,* T_2*. Then the remaining cases of Table 3 cannot occur.*

To sum up, we have

Lemma 47. *the possible cases of Tables 2,3 cannot occur.*

Proof. It follows from Lemmas 15, 23, 26, 32, 36, 44 and 46. □

In the following, we begin to deal with the cases of Tables 4, 5. Recall that $\ell = 3$ or 4.

Lemma 48. *The possible cases of Table 4 do not occur with the possible exceptions of* $(Case, \omega) = (85, 7)$ *or* $(89, 11)$.

Proof. It follows from Lemmas 5 (i), 14. □

Lemma 49. $(Case, \omega) = (85, 7)$ *cannot occur.*

Proof. Assume that $Soc(G) = L_2(7) \times L_2(7) \times L_2(7)$. Then $G = L_2(7) \wr Z_3$ or $L_2(7) \wr S_3$. Note that G has two conjugacy classes of subgroups with index 343, say G_{B_1} and G_{B_2}. However, G_{B_1} or G_{B_2} has no orbit of length 324, contradicting with the fact that the flag-transitivity of G. Assume that $Soc(G) = A_7 \times A_7 \times A_7$. Then $G = A_7 \wr Z_3$, $(A_7)^3.6$, $A_7 \wr S_3$, $(A_7)^3.A_4$, $A_7 \wr D_{12}$, $(A_7)^3.S_4$, $S_7 \wr Z_3$ or $S_7 \wr S_3$. Note that G only has one conjugacy class of subgroups with index 343, denoted by G_B. However, the lengthes of orbits of G_B are 1, 18, 108 and 216, contradicting with the fact that $|B| = 324$. □

Lemma 50. $(Case, \omega) = (89, 11)$ *cannot occur.*

Proof. Suppose there exists a symmetric design with parameters $(v, k, \lambda) = (11^3, 400, 120)$. Then the diophantine equation $280x^2 - 120y^2 = z^2$ has a solution in integers x, y, z not all zero by Lemma 5 (ii). From $20 \mid z$ and $z = 20z_0$ for some integer z_0, we conclude that $7x^2 = 3y^2 + 10z_0^2$. Without loss of generality, we may assume that $Gcd(x, y, z_0) = 1$. However, $3y^2 \equiv 0, 3, 5, 6 \pmod 7$, $-10z_0^2 \equiv 0, 1, 2, 4 \pmod 7$ and $3y^2 \equiv -10z_0^2 \pmod 7$ lead to the contradiction that $7 \mid Gcd(x, y, z_0)$. □

Theorem 51. *The cases of Table 4 cannot occur.*

Proof. It follows from Lemmas 48, 49, 50. □

Lemma 52. *Cases 94-96 of Table 5 cannot occur.*

Proof. It follows from Lemmas 5 (i), 14. □

Proposition 53. *If $\mathcal{D}$ is a symmetric (v, k, λ) design with $k \leqslant 10^3$ which admits a flag-transitive, point-primitive automorphism group G, then G is not of product action type.*

Proof. It follows from Lemmas 47, 51 and 52. □

Proof of Theorem 1. It follows from Propositions 9, 10 and 53. This completes the proof of the Theorem 1. □

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