Abstract

Let $P_n$ be a path graph on $n$ vertices. We say that a graph $G$ is $P_n$-induced-saturated if $G$ contains no induced copy of $P_n$, but deleting any edge of $G$ as well as adding to $G$ any edge of $G^c$ creates such a copy. Martin and Smith (2012) showed that there is no $P_4$-induced-saturated graph. On the other hand, there trivially exist $P_n$-induced-saturated graphs for $n = 2, 3$. Axenovich and Csikós (2019) ask for which integers $n \geq 5$ do there exist $P_n$-induced-saturated graphs. Räty (2019) constructed such a graph for $n = 6$, and Cho, Choi and Park (2019) later constructed such graphs for all $n = 3k$ for $k \geq 2$. We show by a different construction that $P_n$-induced-saturated graphs exist for all $n \geq 6$, leaving only the case $n = 5$ open.
induced subgraph isomorphic to \( H \). Throughout the rest of the note, we will abbreviate a \( H \)-induced-saturated graph as a \( H \)-IS graph.

While for any graph \( H \), there exist \( H \)-saturated graphs, the same is not true for \( H \)-IS graphs. Indeed, for instance for a path on 4 vertices \( P_4 \), Martin and Smith [6] showed that there exists no \( P_4 \)-IS graph.

On the other hand, it is easy to see that there do exist \( P_2 \)-IS and \( P_3 \)-IS graphs. This leads to a question, asked by Axenovich and Csikós [1], for what integers \( n \geq 5 \) do there exist \( P_n \)-IS graphs. Indeed, for instance for a path on 4 vertices \( P_4 \), Martin and Smith [6] showed that there exists no \( P_4 \)-IS graph. On the other hand, it is easy to see that there do exist \( P_2 \)-IS and \( P_3 \)-IS graphs. This leads to a question, asked by Axenovich and Csikós [1], for what integers \( n \geq 5 \) do there exist \( P_n \)-IS graphs. Rätty [7] was the first to make a progress on this question, showing by an algebraic construction that there exists a \( P_6 \)-IS graph. Cho, Choi and Park [4] later showed that in fact for any \( k \geq 2 \), there exists a \( P_{3k} \)-IS graph. We use a different construction to settle the question completely, with the exception of the case \( n = 5 \).

**Theorem 1.** For each \( n \geq 6 \), there is a \( P_n \)-induced-saturated graph.

In Section 2, we describe our construction of a \( P_n \)-IS graph \( G_n \) for each \( n \geq 6 \). Then in Section 3, we check that the graph \( G_n \) is actually \( P_n \)-IS.

### 2 Construction

We will construct, for each \( n \geq 6 \), a \( P_n \)-IS graph \( G_n \). Our construction has been inspired by the observation of Cho, Choi and Park [4] that the Petersen graph is \( P_6 \)-IS. We let

\[
V(G_n) = \{v_1, \ldots, v_{n-1}, w_1, \ldots, w_{n-1}\}
\]

Further, the edge set \( E(G_n) \) of \( G_n \) is defined as follows. For \( 1 \leq i, j \leq n-1 \), we have \( v_i w_j \in E(G_n) \) if and only if \( i = j \). For \( 1 \leq i, j \leq n-1 \), we have \( v_i v_j \in E(G_n) \) if and only if \( i - j \equiv \pm 1 \mod n-1 \). And finally for \( 1 \leq i, j \leq n-1 \), we have \( w_i w_j \in E(G_n) \) if and only if \( i \neq j \) and \( i - j \not\equiv \pm 1 \mod n-1 \).

Note that the graph \( G_6 \) is isomorphic to the Petersen graph. Labelled graph \( G_7 \) is illustrated in the Figure 1 below, and (unlabelled) graphs \( G_6, G_7, G_8 \) are illustrated in the Figure 2 below.

In the rest of the paper, we will prove that for each \( n \geq 6 \), \( G_n \) is \( P_n \)-IS, by checking the three properties that we need by the definition of an induced saturation.

### 3 Proof that the construction works

**Claim 2.** For each \( n \geq 6 \), \( G_n \) contains no induced copy of \( P_n \).

**Proof.** For \( n = 6 \), the result is easy to check by hand. So throughout rest of the proof assume that \( n \geq 7 \). Also assume for contradiction that we have an induced copy of \( P_n \) in \( G_n \).

First we claim that, since \( n \geq 7 \), among any five mutually disjoint vertices of the form \( w_i, w_j, w_k, w_l, w_m \) for some \( 1 \leq i < j < k < l < m \leq n-1 \), some three form a triangle \( K_3 \) in \( G_n \). To see that, note that \( G_n \) necessarily contains at least one of the
edges \( w_iw_j, w_jw_k, w_kw_l, w_lw_m, w_mw_i \) and due to the symmetry, we may without loss of
generality assume that \( G_n \) contains an edge \( w_iw_j \). But then \( w_iw_jw_l \) forms a triangle.

Write \( W \) for \( \{w_1, \ldots, w_{n-1}\} \subset V(G_n) \). Since \( P_n \) is acyclic, we must have at most four
vertices from \( W \) in our induced copy of \( P_n \). We also must have at least one vertex from
\( W \) in our induced copy of \( P_n \), since \( |V(G_n) \setminus W| = n - 1 < n = |V(P_n)| \). We will consider
four cases depending on the number of vertices of \( W \) in our induced copy of \( P_n \).

If we have one vertex from \( W \) in our induced copy of \( P_n \), then we know our induced copy
contains all of the vertices \( v_1, \ldots, v_{n-1} \), but these form a cycle, which gives a contradiction.

If we have two vertices from \( W \) in our induced copy of \( P_n \), we may without loss of
generality assume that our induced copy contains all of the vertices \( v_1, \ldots, v_{n-2} \), but not
the vertex \( v_{n-1} \). Since \( P_n \) contains no vertex of degree more than two, we know our copy of
\( P_n \) can not contain any of the vertices \( w_2, \ldots, w_{n-3} \). But looking at all three two-element
subsets of the set \( \{w_{n-2}, w_{n-1}, w_1\} \), we see that adding none of these subsets to the set
\( \{v_1, v_2, \ldots, v_{n-2}\} \) will create an induced copy of \( P_n \).

Next assume we have three vertices \( w_i, w_j, w_k \) from \( W \) in our induced copy of \( P_n \).
Since \( P_n \) is acyclic, we know we must have at least one of the relations \( i - j \equiv \pm 1 \)
mod $n - 1, i - k \equiv \pm 1 \mod n - 1, j - k \equiv \pm 1 \mod n - 1$ to hold, else $w_i, w_j, w_k$ would form a triangle. Consider two subcases depending on if one or two of the relations above hold (since $n > 4$, we know all three can not hold simultaneously).

If two of the relations above hold, we may without loss of generality assume that we have precisely the vertices $w_1, w_2, w_3$ from $W$ in our induced copy of $P_n$. Then note that we must have $v_2$ in our copy of $P_n$ too, else the degree of $w_2$ in this copy would be zero. Also, $w_2$ has degree one in our copy of $P_n$, hence it forms one of the endpoints of $P_n$. Since $P_n$ is connected, our copy of it must also contain one of the vertices $v_1$ or $v_3$, due to the symmetry we may without loss of generality assume it contains $v_1$. But then it can not contain $v_3$, else it would contain a cycle $v_1v_2v_3w_3w_1$, hence $v_3$ also has degree one in our copy of $P_n$ and it forms another of the endpoints of $P_n$. But then we conclude $n \leq 5$, since distance of the endpoints of $P_n$ in our copy of it is at most four as we have a path $w_3w_1v_1v_2w_2$ connecting them, giving us a desired contradiction.

If just one of the relations above holds, we may without loss of generality assume that we have precisely the vertices $w_1, w_2, w_j$ for some $j$ such that $4 \leq j \leq n - 2$ from $W$ in our induced copy of $P_n$. We can not have $v_j$ in our copy, else $w_j$ would have degree three in the copy. As $P_n$ is connected and $n > 3$, we must have either $v_1$ or $v_2$ in our copy, and we can not have both, as then it would contain a cycle $v_1v_2w_2w_jv_1$. If we have $v_1$ but not $v_2$ in our copy of $P_n$, we can easily see that as $P_n$ is connected and $j \geq 4$, it can contain none of the vertices $v_2, v_3, \ldots, v_j$, and hence contains at most $n - 1$ vertices, giving us a contradiction. If we have $v_2$ but not $v_1$ in our copy, we conclude analogously by noting our copy of $P_n$ contains none of the vertices $v_1, v_{n-1}, \ldots, v_j$ and $j \leq n - 2$.

Finally assume we have four vertices $w_1, w_j, w_k, w_l$ from $W$ in our induced copy of $P_n$. If one of these four vertices is connected to all of the others, we must have a triangle in our copy of $P_n$ (since at least one of the three pairs of the other three vertices is connected too) and reach a contradiction. So due to this observation and the symmetry, it is enough to consider configurations $w_1, w_2, w_l, w_{l+1}$ where $3 \leq l \leq n - 2$.

First consider the case $l = 3$ (the case $l = n - 2$ is analogous). In that case, we can not have $v_1$ or $v_2$ included in our copy of $P_n$, since that would mean degree of $w_1$ or $w_4$ respectively in the copy would be at least three. But then as $P_n$ is connected, none of the vertices $v_4, v_5, \ldots, v_{n-2}, v_{n-1}, v_1$ can be in the copy, so our path $P_n$ has at most six vertices and hence $n \leq 6$, which is a contradiction.

Finally consider the case $3 < l < n - 2$. In this case $w_1w_lw_{l+1}$ is a cycle, contradicting that $P_n$ is acyclic.

\begin{claim}
For each $n \geq 6$, deleting any edge of $G_n$ creates an induced copy of $P_n$.
\end{claim}

\begin{proof}
The edge we delete can be one of three types: $v_iv_j, v_iw_i$ or $w_iw_j$ for some $1 \leq i, j \leq n - 1$; we consider these cases separately.

First assume we delete an edge of the form $v_iv_j$. Then we must have $i - j \equiv \pm 1 \mod n - 1$ and due to the symmetry, we may without loss of generality assume that the edge we deleted was $v_1v_{n-1}$.

Then for $S_1 = \{w_1\} \cup \{v_i : 1 \leq i \leq n - 1\}$, $G_n[S_1]$ is isomorphic to $P_n$. 

\end{proof}
Next assume we delete an edge of the form $v_iv_i$. Then due to the symmetry, we may without loss of generality assume that the edge we deleted was $w_1$. Then for $S_2 = \{w_1, w_{n-2}\} \cup \{v_i : 1 \leq i \leq n - 2\}$, $G_n[S_2]$ is isomorphic to $P_n$.

Finally assume we delete an edge of the form $w_iw_j$ for some $i, j$ such that $i \neq j$, $i - j \not\equiv \pm 1 \mod n - 1$. Due to the symmetry, we may without loss of generality assume that the edge we deleted was $w_1w_j$ for some $j$ such that $3 \leq j \leq n - 2$.

Then if $3 < j < n - 2$, for $S_3 = \{w_1, w_{j-1}, w_{j}, w_{n-1}\} \cup \{v_i : 1 \leq i \leq j - 2\} \cup \{v_i : j \leq i \leq n - 3\}$, $G_n[S_3]$ is isomorphic to $P_n$, if $j = 3$, for $S'_3 = \{w_1, w_3\} \cup \{v_i : 3 \leq i \leq n - 1\}$, $G_n[S'_3]$ is isomorphic to $P_n$, and if $j = n - 2$, for $S''_3 = \{w_1, w_{n-2}\} \cup \{v_i : 1 \leq i \leq n - 2\}$, $G_n[S''_3]$ is isomorphic to $P_n$.

**Claim 4.** For each $n \geq 6$, adding any edge of $G_n^c$ to $G_n$ creates an induced copy of $P_n$.

**Proof.** The edge we add can be one of three types: $v_iv_j$, $v_iw_j$ or $w_iw_j$ for some $1 \leq i, j \leq n - 1$; we consider these cases separately.

First assume we add an edge of the form $w_iw_j$. Then we must have $i - j \equiv \pm 1 \mod n - 1$ and due to the symmetry, we may without loss of generality assume that the edge we added was $w_1w_{n-1}$.

Then for $T_1 = \{w_1, w_{n-1}\} \cup \{v_i : 1 \leq i \leq n - 2\}$, $G_n[T_1]$ is isomorphic to $P_n$.

Next assume we add an edge of the form $v_iv_j$ for some $i, j$ such that $i \neq j$, $i - j \not\equiv \pm 1 \mod n - 1$. Due to the symmetry, we may without loss of generality assume that the edge we added was $v_1v_j$ for some $j$ such that $3 \leq j \leq n - 2$.

Then if $3 < j < n - 2$, for $T_2 = \{w_{j-2}, w_{j-1}, w_{n-1}\} \cup \{v_i : 1 \leq i \leq j - 2\} \cup \{v_i : j \leq i \leq n - 2\}$, $G_n[T_2]$ is isomorphic to $P_n$, while if $j = 3$, for $T'_2 = \{w_2, w_{n-2}, w_{n-1}\} \cup \{v_i : 3 \leq i \leq n - 2\}$, $G_n[T'_2]$ is isomorphic to $P_n$.

Finally assume we add an edge of the form $v_iw_j$ for some $i \neq j$. Due to the symmetry, we may without loss of generality assume that the edge we added was $v_1w_j$ for some $j$ such that $2 \leq j \leq n - 1$.

Then if $2 \leq j < n - 3$, for $T_3 = \{w_{j-1}, w_{j}, w_{j+1}\} \cup \{v_i : 1 \leq i \leq j - 1\} \cup \{v_i : j + 1 \leq i \leq n - 2\}$, $G_n[T_3]$ is isomorphic to $P_n$, if $j = n - 2$, for $T'_3 = \{w_{n-3}, w_{n-2}, w_{n-1}\} \cup \{v_i : 1 \leq i \leq n - 3\}$, $G_n[T'_3]$ is isomorphic to $P_n$, and if $j = n - 1$, for $T''_3 = \{w_{n-2}, w_{n-1}\} \cup \{v_i : 1 \leq i \leq n - 2\}$, $G_n[T''_3]$ is isomorphic to $P_n$.

This now completes the proof that the construction indeed works.

Finally, let us note that for $n = 5$, our construction would give a graph that does contain an induced copy of $P_3$, leaving the question whether there exists a $P_3$-IS graph open.

**Note Added.** After the submission of this paper, it was pointed out to the author through personal correspondence that the case of $P_3$-IS graph was solved by Bonamy, Groenland, Johnston, Morrison and Scott in previously unpublished work. Their work can now be found in the following post [3].
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References


