P_n-induced-saturated graphs exist for all $n \geqslant 6$

Vojtěch Dvořák*
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge, United Kingdom
vd273@cam.ac.uk

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Abstract

Let P_n be a path graph on n vertices. We say that a graph G is P_n-induced-saturated if G contains no induced copy of P_n, but deleting any edge of G as well as adding to G any edge of G^c creates such a copy. Martin and Smith (2012) showed that there is no P_4-induced-saturated graph. On the other hand, there trivially exist P_n-induced-saturated graphs for $n = 2, 3$. Axenovich and Csikós (2019) ask for which integers $n \geqslant 5$ do there exist P_n-induced-saturated graphs. Räty (2019) constructed such a graph for $n = 6$, and Cho, Choi and Park (2019) later constructed such graphs for all $n = 3k$ for $k \geqslant 2$. We show by a different construction that P_n-induced-saturated graphs exist for all $n \geqslant 6$, leaving only the case $n = 5$ open.

Mathematics Subject Classifications: 05C75

1 Introduction

Given graphs G, H, we say G is H*-saturated* if G contains no subgraph isomorphic to H, but adding any edge from G^c to G creates a subgraph isomorphic to H. Related problems have been extensively studied (see for instance a survey of Faudree, Faudree and Schmitt [5]).

In 2012, Martin and Smith [6] introduced the notion of induced saturation on trigraphs. As a special case of this more general framework, there arises the notion of induced-saturated graphs, first studied in its own right by Behrens et al. [2] and later also by Axenovich and Csikós [1]. Given graphs G, H, we say G is H*-induced-saturated* if G contains no induced subgraph isomorphic to H, but deleting any edge of G creates an induced subgraph isomorphic to H, and adding any new edge to G from G^c also creates an

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induced subgraph isomorphic to H. Throughout the rest of the note, we will abbreviate a H-induced-saturated graph as a H-IS graph.

While for any graph H, there exist H-saturated graphs, the same is not true for H-IS graphs. Indeed, for instance for a path on 4 vertices P_4, Martin and Smith [6] showed that there exists no P_4-IS graph.

On the other hand, it is easy to see that there do exist P_2-IS and P_3-IS graphs. This leads to a question, asked by Axenovich and Csikós [1], for what integers $n \geqslant 5$ do there exist P_n-IS graphs. Räty [7] was the first to make a progress on this question, showing by an algebraic construction that there exists a P_6-IS graph. Cho, Choi and Park [4] later showed that in fact for any $k \geqslant 2$, there exists a P_{3k}-IS graph. We use a different construction to settle the question completely, with the exception of the case $n = 5$.

Theorem 1. *For each $n \geqslant 6$, there is a P_n-induced-saturated graph.*

In Section 2, we describe our construction of a P_n-IS graph G_n for each $n \geqslant 6$. Then in Section 3, we check that the graph G_n is actually P_n-IS.

2 Construction

We will construct, for each $n \geqslant 6$, a P_n-IS graph G_n. Our construction has been inspired by the observation of Cho, Choi and Park [4] that the Petersen graph is P_6-IS. We let

$$V(G_n) = \{v_1, \dots, v_{n-1}, w_1, \dots, w_{n-1}\}$$

Further, the edge set $E(G_n)$ of G_n is defined as follows. For $1 \leqslant i, j \leqslant n-1$, we have $v_i w_j \in E(G_n)$ if and only if $i = j$. For $1 \leqslant i, j \leqslant n-1$, we have $v_i v_j \in E(G_n)$ if and only if $i - j \equiv \pm 1 \mod n-1$. And finally for $1 \leqslant i, j \leqslant n-1$, we have $w_i w_j \in E(G_n)$ if and only if $i \neq j$ and $i - j \not\equiv \pm 1 \mod n-1$.

Note that the graph G_6 is isomorphic to the Petersen graph. Labelled graph G_7 is illustrated in the Figure 1 below, and (unlabelled) graphs G_6, G_7, G_8 are illustrated in the Figure 2 below.

In the rest of the paper, we will prove that for each $n \geqslant 6$, G_n is P_n-IS, by checking the three properties that we need by the definition of an induced saturation.

3 Proof that the construction works

Claim 2. *For each $n \geqslant 6$, G_n contains no induced copy of P_n.*

Proof. For $n = 6$, the result is easy to check by hand. So throughout rest of the proof assume that $n \geqslant 7$. Also assume for contradiction that we have an induced copy of P_n in G_n.

First we claim that, since $n \geqslant 7$, among any five mutually disjoint vertices of the form w_i, w_j, w_k, w_l, w_m for some $1 \leqslant i < j < k < l < m \leqslant n-1$, some three form a triangle K_3 in G_n. To see that, note that G_n necessarily contains at least one of the

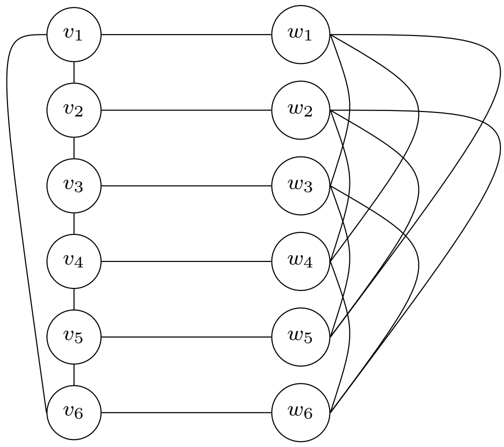

Figure 1: Labelled graph G_7

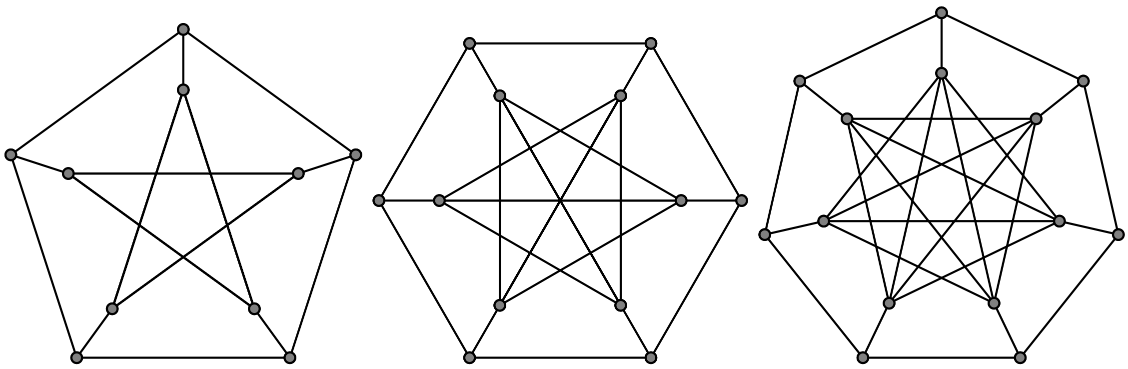

Figure 2: Graphs G_6, G_7, G_8

edges $w_iw_j, w_jw_k, w_kw_l, w_lw_m, w_mw_i$ and due to the symmetry, we may without loss of generality assume that G_n contains an edge w_iw_j. But then $w_iw_jw_l$ forms a triangle.

Write W for $\{w_1, \ldots, w_{n-1}\} \subset V(G_n)$. Since P_n is acyclic, we must have at most four vertices from W in our induced copy of P_n. We also must have at least one vertex from W in our induced copy of P_n, since $|V(G_n) \setminus W| = n-1 < n = |V(P_n)|$. We will consider four cases depending on the number of vertices of W in our induced copy of P_n.

If we have one vertex from W in our induced copy of P_n, then we know our induced copy contains all of the vertices $v_1, \ldots, v_{n-1}$, but these form a cycle, which gives a contradiction.

If we have two vertices from W in our induced copy of P_n, we may without loss of generality assume that our induced copy contains all of the vertices $v_1, \ldots, v_{n-2}$, but not the vertex v_{n-1}. Since P_n contains no vertex of degree more than two, we know our copy of P_n can not contain any of the vertices $w_2, \ldots, w_{n-3}$. But looking at all three two-element subsets of the set $\{w_{n-2}, w_{n-1}, w_1\}$, we see that adding none of these subsets to the set $\{v_1, v_2, \ldots, v_{n-2}\}$ will create an induced copy of P_n.

Next assume we have three vertices w_i, w_j, w_k from W in our induced copy of P_n. Since P_n is acyclic, we know we must have at least one of the relations $i - j \equiv \pm 1$

mod $n-1, i-k \equiv \pm 1 \mod n-1, j-k \equiv \pm 1 \mod n-1$ to hold, else w_i, w_j, w_k would form a triangle. Consider two subcases depending on if one or two of the relations above hold (since $n > 4$, we know all three can not hold simulateneously).

If two of the relations above hold, we may without loss of generality assume that we have precisely the vertices w_1, w_2, w_3 from W in our induced copy of P_n. Then note that we must have v_2 in our copy of P_n too, else the degree of w_2 in this copy would be zero. Also, w_2 has degree one in our copy of P_n, hence it forms one of the endpoints of P_n. Since P_n is connected, our copy of it must also contain one of the vertices v_1 or v_3, due to the symmetry we may without loss of generality assume it contains v_1. But then it can not contain v_3, else it would contain a cycle $v_1v_2v_3w_3w_1$, hence w_3 also has degree one in our copy of P_n and it forms another of the endpoints of P_n. But then we conclude $n \leqslant 5$, since distance of the endpoints of P_n in our copy of it is at most four as we have a path $w_3w_1v_1v_2w_2$ connecting them, giving us a desired contradiction.

If just one of the relations above holds, we may without loss of generality assume that we have precisely the vertices w_1, w_2, w_j for some j such that $4 \leqslant j \leqslant n-2$ from W in our induced copy of P_n. We can not have v_j in our copy, else w_j would have degree three in the copy. As P_n is connected and $n > 3$, we must have either v_1 or v_2 in our copy, and we can not have both, as then it would contain a cycle $v_1v_2w_2w_jw_1$. If we have v_1 but not v_2 in our copy of P_n, we can easily see that as P_n is connected and $j \geqslant 4$, it can contain none of the vertices $v_2, v_3, \ldots, v_j$, and hence contains at most $n-1$ vertices, giving us a contradiction. If we have v_2 but not v_1 in our copy, we conclude analogously by noting our copy of P_n contains none of the vertices $v_1, v_{n-1}, \ldots, v_j$ and $j \leqslant n-2$.

Finally assume we have four vertices w_i, w_j, w_k, w_l from W in our induced copy of P_n. If one of these four vertices is connected to all of the others, we must have a triangle in our copy of P_n (since at least one of the three pairs of the other three vertices is connected too) and reach a contradiction. So due to this observation and the symmetry, it is enough to consider configurations w_1, w_2, w_l, w_{l+1} where $3 \leqslant l \leqslant n-2$.

First consider the case $l = 3$ (the case $l = n-2$ is analogous). In that case, we can not have v_1 or v_4 included in our copy of P_n, since that would mean degree of w_1 or w_4 respectively in the copy would be at least three. But then as P_n is connected, none of the vertices $v_4, v_5, \ldots, v_{n-2}, v_{n-1}, v_1$ can be in the copy, so our path P_n has at most six vertices and hence $n \leqslant 6$, which is a contradiction.

Finally consider the case $3 < l < n-2$. In this case $w_1w_lw_2w_{l+1}$ is a cycle, contradicting that P_n is acyclic. □

Claim 3. *For each $n \geqslant 6$, deleting any edge of G_n creates an induced copy of P_n.*

Proof. The edge we delete can be one of three types: v_iv_j, v_iw_i or w_iw_j for some $1 \leqslant i, j \leqslant n-1$; we consider these cases separately.

First assume we delete an edge of the form v_iv_j. Then we must have $i - j \equiv \pm 1 \mod n-1$ and due to the symmetry, we may without loss of generality assume that the edge we deleted was v_1v_{n-1}.

Then for $S_1 = \{w_1\} \cup \{v_i : 1 \leqslant i \leqslant n-1\}$, $G_n[S_1]$ is isomorphic to P_n.

Next assume we delete an edge of the form v_iw_i. Then due to the symmetry, we may without loss of generality assume that the edge we deleted was v_1w_1.

Then for $S_2 = \{w_1, w_{n-2}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\}$, $G_n[S_2]$ is isomorphic to P_n.

Finally assume we delete an edge of the form w_iw_j for some i, j such that $i \neq j$, $i - j \not\equiv \pm 1 \mod n-1$. Due to the symmetry, we may without loss of generality assume that the edge we deleted was w_1w_j for some j such that $3 \leqslant j \leqslant n-2$.

Then if $3 < j < n-2$, for $S_3 = \{w_1, w_{j-1}, w_j, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant j-2\} \cup \{v_i : j \leqslant i \leqslant n-3\}$, $G_n[S_3]$ is isomorphic to P_n, if $j = 3$, for $S_3' = \{w_1, w_3\} \cup \{v_1\} \cup \{v_i : 3 \leqslant i \leqslant n-1\}$, $G_n[S_3']$ is isomorphic to P_n, and if $j = n-2$, for $S_3'' = \{w_1, w_{n-2}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\}$, $G_n[S_3'']$ is isomorphic to P_n. □

Claim 4. *For each $n \geqslant 6$, adding any edge of G_n^c to G_n creates an induced copy of P_n.*

Proof. The edge we add can be one of three types: v_iv_j, v_iw_j or w_iw_j for some $1 \leqslant i, j \leqslant n-1$; we consider these cases separately.

First assume we add an edge of the form w_iw_j. Then we must have $i - j \equiv \pm 1 \mod n-1$ and due to the symmetry, we may without loss of generality assume that the edge we added was w_1w_{n-1}.

Then for $T_1 = \{w_1, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\}$, $G_n[T_1]$ is isomorphic to P_n.

Next assume we add an edge of the form v_iv_j for some i, j such that $i \neq j$, $i - j \not\equiv \pm 1 \mod n-1$. Due to the symmetry, we may without loss of generality assume that the edge we added was v_1v_j for some j such that $3 \leqslant j \leqslant n-2$.

Then if $3 < j \leqslant n-2$, for $T_2 = \{w_{j-2}, w_{j-1}, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant j-2\} \cup \{v_i : j \leqslant i \leqslant n-2\}$, $G_n[T_2]$ is isomorphic to P_n, while if $j = 3$, for $T_2' = \{w_2, w_{n-2}, w_{n-1}\} \cup \{v_1\} \cup \{v_i : 3 \leqslant i \leqslant n-2\}$, $G_n[T_2']$ is isomorphic to P_n.

Finally assume we add an edge of the form v_iw_j for some $i \neq j$. Due to the symmetry, we may without loss of generality assume that the edge we added was v_1w_j for some j such that $2 \leqslant j \leqslant n-1$.

Then if $2 \leqslant j \leqslant n-3$, for $T_3 = \{w_{j-1}, w_j, w_{j+1}\} \cup \{v_i : 1 \leqslant i \leqslant j-1\} \cup \{v_i : j+1 \leqslant i \leqslant n-2\}$, $G_n[T_3]$ is isomorphic to P_n, if $j = n-2$, for $T_3' = \{w_{n-3}, w_{n-2}, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant n-3\}$, $G_n[T_3']$ is isomorphic to P_n and if $j = n-1$, for $T_3'' = \{w_{n-2}, w_{n-1}\} \cup \{v_i : 1 \leqslant i \leqslant n-2\}$, $G_n[T_3'']$ is isomorphic to P_n. □

This now completes the proof that the construction indeed works.

Finally, let us note that for $n = 5$, our construction would give a graph that does contain an induced copy of P_5, leaving the question whether there exists a P_5-IS graph open.

Note Added. After the submission of this paper, it was pointed out to the author through personal correspondence that the case of P_5-IS graph was solved by Bonamy, Groenland, Johnston, Morrison and Scott in previously unpublished work. Their work can now be found in the following post [3].

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References

[1] Maria Axenovich, Mónika Csikós. *Induced saturation of graphs.* Discrete Mathematics **342**(4) (2018), 1195–1212.

[2] Sarah Behrens, Catherine Erbes, Michael Santana, Derreck Yager, Elyse Yeager. *Graphs with induced-saturation number zero.* Electronic Journal of Combinatorics **23** (2016), no. 1, #P1.54.

[3] Marthe Bonamy, Carla Groenland, Tom Johnston, Natasha Morrison, Alex Scott. *Induced Saturation for P_5.* https://tomjohnston.co.uk/blog/2020-05-22-induced-saturation-for-paths.html.

[4] Eun-Kyung Cho, Ilkyoo Choi, Boram Park. *On induced saturation for paths.* European Journal of Combinatorics **91** (2021), 103204.

[5] Jill R. Faudree, Ralph J. Faudree, John R. Schmitt. *A Survey of Minimum Saturated Graphs.* Electronic Journal of Combinatorics **DS19** (2011) http://eudml.org/doc/227276.

[6] Ryan R. Martin, Jason J. Smith. *Induced Saturation Number.* Discrete Mathematics **312**(21) (2012), 3096–3106.

[7] Eero Räty. *Induced saturation of P_6.* Discrete Mathematics **343**(1) (2020), 111641, 3 pp.