Saturation number of Berge stars in random hypergraphs

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Abstract

Let G be a graph. We say an r-uniform hypergraph H is a Berge-G if there exists a bijection $\phi : E(G) \to E(H)$ such that $e \subseteq \phi(e)$ for each $e \in E(G)$. Given a family of r-uniform hypergraphs $\mathcal{F}$ and an r-uniform hypergraph H, a spanning sub-hypergraph H' of H is $\mathcal{F}$-saturated in H if H' is $\mathcal{F}$-free, but adding any edge in $E(H)\backslash E(H')$ to H' creates a copy of some $F \in \mathcal{F}$. The saturation number of $\mathcal{F}$ is the minimum number of edges in an $\mathcal{F}$-saturated spanning sub-hypergraph of H. In this paper, we asymptotically determine the saturation number of Berge stars in random r-uniform hypergraphs.

Mathematics Subject Classifications: 05C65, 05C35, 05C80

1 Introduction

Given a family of graphs $\mathcal{F}$, a graph G is *$\mathcal{F}$-saturated* if G does not contain any $F \in \mathcal{F}$ as a subgraph, but adding any missing edge to G creates a copy of some $F \in \mathcal{F}$. In other

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words, G is $\mathcal{F}$-saturated if and only if it is an edge-maximal $\mathcal{F}$-free graph. The maximum possible number of edges in a graph G that is $\mathcal{F}$-saturated is known as the Turán number of $\mathcal{F}$. The study of Turán numbers for various families of graphs is a cornerstone of extremal combinatorics.

On the other hand, the minimum number of edges in an $\mathcal{F}$-saturated graph with n vertices, denoted by $\mathrm{sat}(n, \mathcal{F})$, is called the *saturation number* of $\mathcal{F}$. Saturation numbers were first studied by Erdős, Hajnal and Moon [6] and since then have been researched extensively. In 1986, Kászonyi and Tuza [9] showed that saturation numbers are always linear. That is, $\mathrm{sat}(n, F) = O(n)$ for any graph F. In the same paper, they also determined the saturation number of star $K_{1,s}$. To be specific, they proved that

$$\mathrm{sat}(n, K_{1,s}) = \begin{cases} \binom{n-s}{2} + \binom{s}{2}, & \text{if } s+1 \leqslant n < 3s/2, \\ \left\lceil \dfrac{(s-1)n}{2} - \dfrac{s^2}{8} \right\rceil, & \text{if } n \geqslant 3s/2. \end{cases} \tag{1}$$

For more results on graph saturation, we refer the reader to the survey [7].

Graph saturation has been generalized in several natural ways, including studying other host graphs besides the complete graph, and the saturation number of hypergraphs. Recall that a hypergraph $H = (V(H), E(H))$ is a pair consisting of a vertex set $V(H)$, and a set $E(H)$ of subsets of $V(H)$, the edges of H. An *r-uniform hypergraph* or simply r-graph is a hypergraph such that all its edges have size r. Throughout this paper, we always assume that $r \geqslant 2$ is an integer.

To state our result precisely, we introduce some terminology and notation. Given a family of r-uniform hypergraphs $\mathcal{F}$ and an r-uniform hypergraph H, a spanning sub-hypergraph H' of H is *$\mathcal{F}$-saturated* in H if H' is $\mathcal{F}$-free, but adding any edge in $E(H)\backslash E(H')$ to H' creates a copy of some $F \in \mathcal{F}$. The minimum number of edges in an $\mathcal{F}$-saturated spanning sub-hypergraph of H is called the *saturation number* of $\mathcal{F}$, denoted by $\mathrm{sat}(H, \mathcal{F})$. Note that with this general notation, $\mathrm{sat}(n, \mathcal{F}) = \mathrm{sat}(K_n^{(r)}, \mathcal{F})$, where $K_n^{(r)}$ is the complete r-uniform hypergraph on n vertices.

Let G be a graph. We say an r-uniform hypergraph H is a Berge-G if there exists a bijection $\phi : E(G) \to E(H)$ such that $e \subseteq \phi(e)$ for each edge $e \in E(G)$. Recently, extremal problems for Berge hypergraphs have attracted the attention of a lot of researchers, see, e.g., [3, 2, 4, 5, 8, 13]. In 2018, Austhof and English [2] studied the saturation number of Berge stars. They proved that

$$\mathrm{sat}(n, \textit{Berge-}K_{1,s}) = \min_{a \in [n],\, \binom{a-1}{r-1} \leqslant s-2} \left\lceil \frac{(s-1)(n-a)}{r} \right\rceil + \binom{a}{r}$$

for large n, which generalizes equation (1) to uniform hypergraphs. In 2019, English et al. [5] proved that $\mathrm{sat}(n, \textit{Berge-}F) = O(n)$ for any graph F and uniformities $3 \leqslant r \leqslant 5$.

In recent years, some classic extremal problems were extended to random analogues. The random r-uniform hypergraph $H^r(n, p)$ is the probability space of all r-uniform hypergraphs with vertex set $[n] := \{1, 2, \ldots, n\}$, and each edge is chosen with probability

p independently of all the other edges. In particular, for $r = 2$ this model reduces to the well known Erdős-Rényi random graph $G(n,p)$. In 2017, Korándi and Sudakov [10] initiated the study of graph saturation in random graphs. More precisely, they proved that with high probability

$$\mathrm{sat}(G(n,p), K_m) = (1+o(1))n \cdot \log_{1/(1-p)} n$$

for each fixed $p \in (0,1)$. Let us recall that an event holds with high probability (w.h.p. for short) in $G(n,p)$ if its probability goes to 1 as n tends to infinity. In 2018, Mohammadian and Tayfeh-Rezaie [11] asymptotically determined the saturation number of stars in random graphs. It is proved that w.h.p.

$$\mathrm{sat}(G(n,p), K_{1,s}) = \frac{(s-1)n}{2} - (1+o(1))(s-1) \cdot \log_{1/(1-p)} n \tag{2}$$

for every fixed $p \in (0,1)$, which supplements the early work of Zito [14].

The main goal of this paper is to extend (2) to random hypergraphs. To be specific, we asymptotically determine the saturation number of Berge stars in random r-uniform hypergraphs.

Using a similar but more complicated technique than that of [11], we can prove the main result of this paper.

Theorem 1. *Let $p \in (0,1)$ be a fixed number and $s \geqslant 2$. Then w.h.p.*

$$\mathrm{sat}(H^r(n,p), \textit{Berge-}K_{1,s}) = \frac{s-1}{r}\left(n - (1+o(1))\big(r! \cdot \log_{1/(1-p)} n\big)^{1/(r-1)}\right).$$

Note that the proof of Theorem 1 is a combination of Theorem 4 and Theorem 9. Taking $r = 2$, we obtain the result of Mohammadian and Tayfeh-Rezaie described in Equation (2), so Theorem 1 generalizes that result. Let us note that most of our results are about n tending to infinity, so we tacitly assume that n is large enough throughout this paper.

2 Lower bound on the saturation number of Berge stars

We start this section with some notation. Let H be a hypergraph and S be a subset of $V(H)$, the sub-hypergraph of H induced by S is the hypergraph $H[S]$ consisting of all edges of H that are contained in S. A set $X \subseteq V(H)$ is an *independent set* of H if X contains no edges of H. We also denote by $\alpha_k(H)$ the maximum cardinality of a subset S of $V(H)$ such that the maximum degree of $H[S]$ is at most k.

Lemma 2 ([11])**.** *Let X be a binomial random variable with parameters n and $p \in (0,1)$. Then*

$$\mathbf{P}(X \leqslant s) \leqslant \binom{n}{s}(1-p)^{n-s}$$

for any $s \in \{0,1,\ldots,n\}$.

Lemma 3. *Let $p \in (0,1)$ be a fixed number and $k \geqslant 1$. Then w.h.p.*

$$\alpha_k(H^r(n,p)) \leqslant (1+o(1))\big(r! \cdot \log_{1/(1-p)} n\big)^{1/(r-1)}.$$

Proof. For convenience, we denote $q := 1/(1-p)$ and

$$i := \Big\lceil \big(r! \cdot \log_q n + k(r-1)! \cdot \log_q \log_q n\big)^{1/(r-1)} \Big\rceil + r - 1. \tag{3}$$

Let X_i be the number of induced sub-hypergraphs in $H^r(n,p)$ on i vertices with at most $\lfloor ik/r \rfloor$ edges. For any $A \subseteq V(H^r(n,p))$ with $|A| = i$, let Y_A be the number of edges in $H^r(n,p)[A]$.

Our first goal is to estimate the expectation $\mathbf{E}(X_i)$. By Theorem 2 we have

$$\mathbf{P}(Y_A \leqslant \lfloor ik/r \rfloor) \leqslant \binom{\binom{i}{r}}{\lfloor ik/r \rfloor} (1-p)^{\binom{i}{r} - \lfloor ik/r \rfloor}.$$

Noting that the function

$$f(x) := \left(\frac{e\binom{i}{r}}{x}\right)^x$$

is non-decreasing in $x \in (0, \binom{i}{r})$, we have

$$\begin{aligned}
\mathbf{P}(Y_A \leqslant \lfloor ik/r \rfloor) &\leqslant \left(\frac{e\binom{i}{r}}{\lfloor ik/r \rfloor}\right)^{\lfloor ik/r \rfloor} (1-p)^{\binom{i}{r} - \lfloor ik/r \rfloor} \\
&\leqslant \left(\frac{re\binom{i}{r}}{ik}\right)^{ik/r} (1-p)^{\binom{i}{r} - ik/r}.
\end{aligned}$$

In view of the above inequality, we can give an estimation of $\mathbf{E}(X_i)$ as follows:

$$\begin{aligned}
\mathbf{E}(X_i) &= \sum_{A \subseteq V(H^r(n,p)),\, |A| = i} \mathbf{P}(Y_A \leqslant \lfloor ik/r \rfloor) \\
&\leqslant \binom{n}{i} \left(\frac{re\binom{i}{r}}{ik}\right)^{ik/r} (1-p)^{\binom{i}{r} - ik/r} \\
&\leqslant \left(\frac{ne}{i}\right)^i \left(\frac{re\binom{i}{r}}{ik}\right)^{ik/r} (1-p)^{\binom{i}{r} - ik/r} \\
&\leqslant \left(\left(\frac{e^{r+1+r/k}}{(1-p)r^{r-1}k}\right)^k i^{(r-1)k-r} n^r (1-p)^{\binom{i-1}{r-1}}\right)^{i/r}.
\end{aligned}$$

To finish the proof, we shall prove that $i^{(r-1)k-r}n^r(1-p)^{\binom{i-1}{r-1}} = o(1)$, and therefore $\mathbf{E}(X_i) = o(1)$. Indeed, by some algebra we have

$$\begin{aligned} i^{(r-1)k-r}n^r(1-p)^{\binom{i-1}{r-1}} &\leqslant \big(2(r!\cdot\log_q n)^{1/(r-1)}\big)^{(r-1)k-r}\, n^r(1-p)^{\binom{i-1}{r-1}} \\ &\leqslant (2^{r-1}r!)^k(\log_q n)^{k-1}n^r(1-p)^{(i-r+1)^{r-1}/(r-1)!} \\ &\leqslant \frac{(2^{r-1}r!)^k}{\log_q n}, \end{aligned}$$

the last inequality follows from the fact that $(1-p)^{(i-r+1)^{r-1}/(r-1)!} \leqslant n^{-r}(\log_q n)^{-k}$ by (3). Therefore, $\mathbf{E}(X_i) = o(1)$.

Finally, by the Markov inequality, $\mathbf{P}(X_i > 0) \leqslant \mathbf{E}(X_i) = o(1)$. Hence, w.h.p. $X_i = 0$, which yields that for any $X \subseteq V(H^r(n,p))$ with $|X| = i$, the number of edges of $H^r(n,p)[X]$ is at least $\lfloor ik/r \rfloor + 1$. Hence, the maximum degree Δ of $H^r(n,p)[X]$ satisfies $i\Delta \geqslant r(\lfloor ik/r \rfloor + 1)$. Consequently,

$$\Delta \geqslant \frac{r(\lfloor ik/r \rfloor + 1)}{i} > k.$$

Therefore, we have

$$\alpha_k(H^r(n,p)) < i = (1+o(1))\big(r!\cdot\log_{1/(1-p)} n\big)^{1/(r-1)},$$

completing the proof of Theorem 3. □

In view of Theorem 3, we can obtain a lower bound of the saturation number of Berge stars.

Theorem 4. *Let $p \in (0,1)$ be a fixed number and $s \geqslant 2$. Then w.h.p.*

$$\mathrm{sat}(H^r(n,p), \textit{Berge-}K_{1,s}) \geqslant \frac{s-1}{r}\left(n - (1+o(1))\big(r!\cdot\log_{1/(1-p)} n\big)^{1/(r-1)}\right).$$

Proof. Let H' be a Berge-$K_{1,s}$-saturated spanning sub-hypergraph of $H^r(n,p)$, and let $A \subseteq V(H')$ be the set of vertices with degree at most $s-2$ in H'. Then each vertex in $V(H^r(n,p))\backslash A$ is of degree at least $s-1$ in H'. Therefore,

$$r|E(H')| \geqslant \sum_{v\in V(H^r(n,p))\backslash A} \deg_{H'}(v) \geqslant (s-1)(n-|A|).$$

Clearly, $H'[A] = H^r(n,p)[A]$. Hence, $|A| \leqslant \alpha_{s-2}(H^r(n,p))$. It follows from Theorem 3 that

$$\begin{aligned} |E(H')| &\geqslant \frac{(s-1)(n-|A|)}{r} \\ &\geqslant \frac{s-1}{r}\left(n - (1+o(1))\big(r!\cdot\log_{1/(1-p)} n\big)^{1/(r-1)}\right), \end{aligned}$$

completing the proof of Theorem 4. □

3 Upper bound on the saturation number of Berge stars

In this section, we will give an upper bound on the saturation number of Berge stars in random hypergraphs. Before continuing, we need the following lemma.

Lemma 5 ([12]). *Let H be a fixed r-uniform hypergraph on n vertices with maximum degree Δ. There exists a constant $C > 0$ such that for $p = C(\ln n/n)^{1/\Delta}$, w.h.p. the random r-uniform hypergraph $H^r(n,p)$ contains a copy of H.*

Lemma 6. *Let $r \geqslant 2$ and $a = \left\lfloor \left(r! \cdot (\log_{1/(1-p)} n - 3 \cdot \log_{1/(1-p)} \log_{1/(1-p)} n)\right)^{1/(r-1)} \right\rfloor - 2$, and*

$$A_i = \frac{\binom{a}{i}\binom{n-a}{a-i}}{\binom{n}{a}(1-p)^{\binom{i}{r}}},\ i \geqslant r.$$

Then $\lim_{n\to\infty} \sum_{i=r}^{a} A_i = 0$.

Proof. For any $r \leqslant i \leqslant a$, our main goal is to show that $A_i \leqslant A_r$. To this end, note that $A_{r+1} \leqslant A_r$, hence it suffices to show that $A_i \leqslant A_r$ for $i \geqslant r+2$.

By simple algebra, we get

$$\begin{aligned}\frac{A_{j+1}}{A_j} &= \frac{(a-j)^2}{(j+1)(n-2a+j+1)} \cdot (1-p)^{\binom{j}{r}-\binom{j+1}{r}} \\ &\leqslant \left(1 + O\left(\frac{(\log_{1/(1-p)} n)^{1/(r-1)}}{n}\right)\right) \frac{a^2(1-p)^{\binom{j}{r}-\binom{j+1}{r}}}{(j+1)n}.\end{aligned}$$

Therefore, we obtain

$$\frac{A_i}{A_r} = \prod_{j=r}^{i-1} \frac{A_{j+1}}{A_j} \leqslant (1+o(1)) \left(\frac{a^2}{n}\right)^{i-r} \frac{r!(1-p)^{1-\binom{i}{r}}}{i!}.$$

Noting that $e > (1+1/j)^j$ for each $j \in [i]$, we have $i! > ((i+1)/e)^i > (i/e)^i$. Therefore,

$$\frac{A_i}{A_r} \leqslant r!(1+o(1)) \left(\frac{a^2}{n}\right)^{i-r} \left(\frac{e}{i}\right)^i (1-p)^{1-\binom{i}{r}}. \tag{4}$$

To simplify inequality (4), we need the following claim.

Claim 7. *$r!\binom{i}{r} \leqslant (i-r)(i+2)^{r-1}$, where $i \geqslant r+2$.*

Proof of Theorem 7. Assume that $i \geqslant r+2$, then we have

$$\begin{aligned} r!\binom{i}{r} &= \frac{i!}{(i-r+2)!} \cdot (i-r+2)(i-r+1) \\ &\leqslant (i+2)^{r-2}(i-r+2)(i-r+1) \\ &\leqslant (i-r)(i+2)^{r-1}. \end{aligned}$$

The last inequality follows from a simple fact that

$$(i-r)(i+2)-(i-r+2)(i-r+1)=(r-1)(i-r)-2\geqslant 0.$$

The proof of the claim is completed.

Finally, in view of (4) and Theorem 7 we deduce that

$$\begin{aligned}\frac{A_i}{A_r}&\leqslant r!(1-p)(1+o(1))\left(\frac{a^2e}{ni}(1-p)^{-\binom{i}{r}/(i-r)}\right)^{i-r}\\&\leqslant r!(1-p)(1+o(1))\left(\frac{a^2e}{ni}(1-p)^{-(i+2)^{r-1}/r!}\right)^{i-r}.\end{aligned}$$

Since $i\leqslant a$, we have

$$(1-p)^{-(i+2)^{r-1}/r!}\leqslant\frac{n}{(\log_{1/(1-p)}n)^3}.$$

Combining these two inequalities, we see

$$\frac{A_i}{A_r}\leqslant r!(1-p)(1+o(1))\left(\frac{a^2e}{i(\log_{1/(1-p)}n)^3}\right)^{i-r}\leqslant 1.$$

It follows that

$$\sum_{i=r}^{a}A_i\leqslant aA_r=\frac{a\binom{a}{r}\binom{n-a}{a-r}}{(1-p)\binom{n}{a}}=o(1),$$

completing the proof of Theorem 6. □

An r-uniform hypergraph is *nearly-d-regular* if every vertex has degree either d or $d-1$, and less than r vertices have degree $d-1$. A hypergraph H is called *linear* if every pair of edges intersects in at most one vertex. The following theorem guarantees the existence of nearly-regular linear hypergraphs.

Theorem 8 ([2])**.** *Let $d\geqslant 1$ and $r\geqslant 2$. Then for all sufficiently large n, there exists a nearly-d-regular r-uniform linear hypergraph on n vertices.*

Armed with Theorem 5, Theorem 6 and Theorem 8, we are ready to propose our main result in this section.

Theorem 9. *Let $p\in(0,1)$ be a fixed number. Then w.h.p.*

$$\mathrm{sat}(H^r(n,p),\textit{Berge-}K_{1,s})\leqslant\frac{s-1}{r}\left(n-(1+o(1))\big(r!\cdot\log_{1/(1-p)}n\big)^{1/(r-1)}\right).$$

Proof. For short, we denote $H:=H^r(n,p)$ and let

$$a=\Big\lfloor\big(r!\cdot(\log_{1/(1-p)}n-3\cdot\log_{1/(1-p)}\log_{1/(1-p)}n)\big)^{1/(r-1)}\Big\rfloor-2.$$

Fix a nearly-$(s-1)$-regular r-uniform linear hypergraph R on $n-a$ vertices. For any $A \subseteq V(H)$ with $|A| = a$, let

$$X_A = \begin{cases} 1, & \text{if } A \text{ is an independent set in } H \text{ and } R \subseteq H[V(H)\backslash A], \\ 0, & \text{otherwise,} \end{cases}$$

and set $X := \sum_{A \subseteq V(H),\, |A|=a} X_A$.

Our first goal is to show that $\mathbf{P}(X = 0) = o(1)$ with high probability. To this end, note that for any $0 < \varepsilon < 1/5$ we have

$$\mathbf{E}(X_A) \geqslant (1-p)^{\binom{a}{r}}(1-\varepsilon) \tag{5}$$

from Theorem 5. By linearity of expectation we deduce that

$$\mathbf{E}(X) \geqslant \binom{n}{a}(1-p)^{\binom{a}{r}}(1-\varepsilon). \tag{6}$$

Moreover, for subsets S, $T \subseteq V(H)$ of size a with $|S \cap T| = i$, we find that

$$\mathbf{E}(X_S X_T) \leqslant (1-p)^{2\binom{a}{r}-\binom{i}{r}}, \tag{7}$$

where $\binom{i}{r} = 0$ if $i \leqslant r-1$.

By the Chebyshev's inequality (See [1, Theorem 4.3.1]), we have

$$\mathbf{P}(X=0) \leqslant \frac{\mathbf{Var}(X)}{\mathbf{E}(X)^2} = \sum_{\substack{S,\,T \subseteq V(H),\\ |S|=|T|=a}} \frac{\mathbf{E}(X_S X_T) - \mathbf{E}(X_S)\cdot\mathbf{E}(X_T)}{\mathbf{E}(X)^2}.$$

It follows from (5) – (7) and Theorem 6 that

$$\begin{aligned}
\mathbf{P}(X=0) &\leqslant \sum_{i=0}^{a} \sum_{\substack{S,\,T \subseteq V(H),\\ |S|=|T|=a,\, |S\cap T|=i}} \frac{\mathbf{E}(X_S X_T) - \mathbf{E}(X_S)\cdot\mathbf{E}(X_T)}{\mathbf{E}(X)^2} \\
&\leqslant \binom{n}{a} \sum_{i=0}^{a} \frac{\binom{a}{i}\binom{n-a}{a-i}\cdot\left((1-p)^{2\binom{a}{r}-\binom{i}{r}} - (1-p)^{2\binom{a}{r}}(1-\varepsilon)^2\right)}{\binom{n}{a}^2 (1-p)^{2\binom{a}{r}}(1-\varepsilon)^2} \\
&= \sum_{i=0}^{a} \frac{\binom{a}{i}\binom{n-a}{a-i}}{\binom{n}{a}(1-p)^{\binom{i}{r}}} \cdot \frac{1-(1-p)^{\binom{i}{r}}(1-\varepsilon)^2}{(1-\varepsilon)^2} \\
&\leqslant 3\varepsilon \sum_{i=0}^{r-1} \frac{\binom{a}{i}\binom{n-a}{a-i}}{\binom{n}{a}} + 2\sum_{i=r}^{a} \frac{\binom{a}{i}\binom{n-a}{a-i}}{\binom{n}{a}(1-p)^{\binom{i}{r}}} \\
&= o(1).
\end{aligned}$$

Hence, $\mathbf{P}(X=0) = o(1)$. This shows that w.h.p. there is $A \subseteq V(H)$ with $|A| = a$ such that A is an independent set in H and $H[V(H)\backslash A]$ has a copy of R as a sub-hypergraph.

Let H' be a Berge-$K_{1,s}$-saturated spanning sub-hypergraph of H such that

$$H'[V(H)\backslash A] = R.$$

Denote by α and β the numbers of vertices of degree $s-1$ and $s-2$ in $H'[V(H)\backslash A]$ respectively, where $\alpha+\beta = n-a$ and $\beta \leqslant r-1$. Clearly, the number of edges in H' which have nonempty intersection with A and $V(H)\backslash A$ is at most β. Therefore,

$$\begin{aligned} r|E(H')| &\leqslant \alpha(s-1)+\beta(s-2)+r\beta \\ &= (s-1)(\alpha+\beta)+(r-1)\beta \\ &\leqslant (s-1)(n-a)+(r-1)^2, \end{aligned}$$

which yields that

$$|E(H')| \leqslant \frac{s-1}{r}\left(n-(1+o(1))\big(r!\cdot\log_{1/(1-p)} n\big)^{1/(r-1)}\right),$$

completing the proof. □

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