Low weight perfect matchings

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Abstract

Answering a question posed by Caro, Hansberg, Lauri, and Zarb, we show that for every positive integer \( n \) and every function \( \sigma : E(K_{4n}) \to \{-1,1\} \) with \( \sigma (E(K_{4n})) = 0 \), there is a perfect matching \( M \) in \( K_{4n} \) with \( \sigma (M) = 0 \). Strengthening the consequence of a result of Caro and Yuster, we show that for every positive integer \( n \) and every function \( \sigma : E(K_{4n}) \to \{-1,1\} \) with \( |\sigma (E(K_{4n}))| < n^2 + 11n + 2 \), there is a perfect matching \( M \) in \( K_{4n} \) with \( |\sigma (M)| \leq 2 \). Both these results are best possible.

Mathematics Subject Classifications: 05C22, 05C70

1 Introduction

In [2] Caro, Hansberg, Lauri, and Zarb considered connected graphs \( G \) together with a function \( \sigma : E(G) \to \{-1,1\} \) labeling the edges of \( G \) with \(-1 \) or \(+1\), and they studied conditions that imply the existence of different types of spanning trees \( T \) with

\[
|\sigma (E(T))| = \left| \sum_{e \in E(T)} \sigma (e) \right| \leq 1,
\]

where, as usual, for a set \( E \) of edges, \( \sigma (E) \) is just the sum of \( \sigma (e) \) over all \( e \) in \( E \). As a variation of this problem, they ask whether, for every positive integer \( n \) and every labeling \( \sigma : E(K_{4n}) \to \{-1,1\} \) of the edges of the complete graph \( K_{4n} \) of order \( 4n \) with \( \sigma (E(K_{4n})) = 0 \), there is a perfect matching \( M \) in \( K_{4n} \) with \( \sigma (M) = 0 \). We answer their question in the affirmative.

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Theorem 1. For every positive integer \( n \) and every function \( \sigma : E(K_{4n}) \rightarrow \{-1, 1\} \) with \( \sigma(E(K_{4n})) = 0 \), there is a perfect matching \( M \) in \( K_{4n} \) with \( \sigma(M) = 0 \).

In order to put our result into some wider perspective, we briefly discuss the notion of local amoebas studied in [2, 3]: A graph \( G \) of order \( n \) is a local amoeba if, for every two isomorphic copies \( H \) and \( H' \) of \( G \) in \( K_n \), there is a sequence \( G_0, \ldots, G_k \) of isomorphic copies of \( G \) in \( K_n \) such that \( H = G_0, H' = G_k \), and each \( G_{i+1} \) arises from \( G_i \) by an edge-replacement, that is, \( E(G_{i+1}) = (E(G_i) \setminus \{e\}) \cup \{e'\} \) for some \( e \in E(G_i) \) and \( e' \in E(K_n) \setminus E(G_i) \). Note that a path \( P_n \) is an example of a local amoeba, and that local amoebas are defined by an exchange property that very closely resembles the well-known exchange property of the bases of a matroid: For every two bases \( B \) and \( B' \) of a matroid, and for every \( e' \in B' \setminus B \), there is some \( e \in B \setminus B' \) such that \( (B \setminus \{e\}) \cup \{e'\} \) is again a basis of the matroid. Now, perfect matchings of a complete graph are a very natural example of spanning subgraphs that are no local amoebas, and for which, consequently, the machinery developed in [2] fails. In fact, perfect matchings have the following slightly weaker exchange property: If \( M \) and \( M' \) are two distinct perfect matchings in \( K_n \), then, considering an \( M-M' \)-alternating cycle defined by their symmetric difference \( M \Delta M' \), it follows that there are two edges \( e_1, e_2 \) in \( M \), one edge \( e_1' \) in \( M' \), and one further edge \( e_2' \) which might not belong to \( M \cup M' \), such that \( M'' = (M \setminus \{e_1, e_2\}) \cup \{e_1', e_2'\} \) is a perfect matching in \( K_n \) for which \( |M' \Delta M''| \) is strictly smaller than \( |M' \Delta M| \), that is, one can transform \( M \) into \( M' \) by a sequence of exchange operations removing and adding two edges, and not just one, at every step.

Under the hypothesis of Theorem 1, the existence of a perfect matching \( M \) in \( K_{4n} \) with \( |\sigma(M)| \leq 2 \) already follows from more general results due to Caro and Yuster, cf. Theorem 1.1 in [4]. More precisely, Caro and Yuster showed that the weaker hypothesis \( |\sigma(E(K_{4n}))| \leq 2(4n - 1) \) suffices for the existence of such a perfect matching \( M \) with \( |\sigma(M)| \leq 2 \). As observed in [2], for infinitely many positive integers \( n \), there are functions \( \sigma : E(K_{4n}) \rightarrow \{-1, 1\} \) with \( |\sigma(E(K_{4n}))| = 4\sqrt{n} - 2 \) such that \( \sigma(M) \neq 0 \) for every perfect matching \( M \) in \( K_{4n} \). Slightly modifying their construction, we obtain the following proposition, which implies that Theorem 1 is best possible for infinitely many values of \( n \).

Proposition 2. For infinitely many positive integers \( n \), there is a function \( \sigma : E(K_{4n}) \rightarrow \{-1, 1\} \) with \( |\sigma(E(K_{4n}))| = 2 \) such that \( \sigma(M) \neq 0 \) for every perfect matching \( M \) in \( K_{4n} \).

Considering the construction in the proof of Proposition 2 suggests that zero weight perfect matchings are excluded rather by parity reasons than by the value of the imbalance \( |\sigma(E(K_{4n}))| \) of \( \sigma \). We confirm this with our second main result showing that much weaker conditions on the imbalance imply the existence of low weight perfect matchings.

Theorem 3. For all positive integers \( n \) and \( k \) such that \( k \geq 2 \), and every function \( \sigma : E(K_{4n}) \rightarrow \{-1, 1\} \) with \( |\sigma(E(K_{4n}))| < n(n - 1) + k(6n - 1) + k^2 \), there is a perfect matching \( M \) in \( K_{4n} \) with \( |\sigma(M)| \leq 2k - 2 \).

For \( k = 2 \), Theorem 3 implies the following strengthening of the above-mentioned consequence of the result of Caro and Yuster.
Corollary 4. For every positive integer $n$ and every function $\sigma : E(K_{4n}) \to \{-1, 1\}$ with $|\sigma (E(K_{4n}))| < n^2 + 11n + 2$, there is a perfect matching $M$ in $K_{4n}$ with $|\sigma (M)| \leq 2$.

Both, Theorem 3 and, hence, also Corollary 4 are best possible. If, for instance, $\sigma : E(K_{4n}) \to \{-1, 1\}$ is such that the graph $(V(K_{4n}), \sigma^{-1}(1))$ consists of a clique of order $3n + 2$ and $n - 2$ isolated vertices, then $|\sigma (M)| \geq 4$ for every perfect matching $M$ in $K_{4n}$ while $|\sigma (E(K_{4n}))| = n^2 + 11n + 2$. In conjunction, Theorem 1, Proposition 2, and Corollary 4 imply an interesting behavior: If $n$ is a positive integer and $\sigma$ is a $\pm 1$-labeling of the edges of $K_{4n}$, then, in order to force the existence of a zero weight perfect matching, one needs to require zero imbalance, that is, $|\sigma (E(K_{4n}))| = 0$, while $|\sigma (M)| > 2$ for every perfect matching $M$ in $K_{4n}$ already forces a quadratic imbalance, that is, $|\sigma (E(K_{4n}))|$ is at least quadratic in $n$.

All proofs are given in the next section.

For a survey concerning related results, we refer the reader to [1] and the introduction of [2].

2 Proofs

We start with the proof of our first main result.

Proof of Theorem 1. We suppose, for a contradiction, that $\sigma : E(K_{4n}) \to \{-1, 1\}$ is such that $\sigma (E(K_{4n})) = 0$ but that $\sigma (M) \neq 0$ for every perfect matching $M$ in $K_{4n}$. First, we consider the case $n = 1$. The edge set of $K_4$ is the union of three edge-disjoint perfect matchings $M_1$, $M_2$, and $M_3$. Since $\sigma (M_i) \neq 0$ for every $i$, we obtain $\sigma (M_i) \in \{-2, 2\}$, which implies the contradiction $\sigma (E(K_{4n})) = \sigma (M_1) + \sigma (M_2) + \sigma (M_3) \neq 0$. Hence, we may assume that $n \geq 2$. We call an edge $e$ a plus-edge if $\sigma (e) = 1$, and a minus-edge if $\sigma (e) = -1$. Since $\sigma (E(K_{4n})) = 0$,

there are exactly $\frac{1}{2} \binom{4n}{2} = 4n^2 - n$ plus-edges and minus-edges in $K_{4n}$, respectively. (1)

For a matching $M$, we denote by $M^+$ and $M^-$ the sets of plus-edges and minus-edges in $M$, respectively. We choose a perfect matching $M$ in $K_{4n}$ such that $|\sigma (M)|$ is as small as possible. Possibly replacing $\sigma$ with $-\sigma$, we may assume that $\sigma (M) > 0$. Since $M$ contains $2n$ edges, $\sigma (M)$ is even, which implies $\sigma (M) \geq 2$.

We start with some easy observations.

Claim 5. For every two edges $e$ in $M^+$ and $f$ in $M^-$, there are no two disjoint minus-edges between $e$ and $f$. In particular, there are at most two minus-edges between $e$ and $f$.

Proof of Claim 5. If there are two disjoint minus-edges $e'$ and $f'$ between two edges $e \in M^+$ and $f \in M^-$, then the perfect matching $N = (M \setminus \{e, f\}) \cup \{e', f'\}$ satisfies $0 \leq \sigma (N) = \sigma (M) - 2 < \sigma (M)$, contradicting the choice of $M$.  

Claim 6. There are two edges $e$ and $f$ in $M^+$ such that there exists a minus-edge between $e$ and $f$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 27(4) (2020), #P4.49 3
Proof of Claim 6. Suppose, for a contradiction, that there is no minus-edge between vertices in \( V(M^+) \); that is, more formally, the subgraph of \( K_{4n} \) induced by the set \( V(M^+) \) of vertices that are incident with a plus-edge from \( M \) contains no minus-edge of \( K_{4n} \). For \( m^+ = |M^+| \), we have \( m^+ \geq n + 1 \). By Claim 5, at least half the \( 2m^+(4n - 2m^+) \) edges between \( V(M^+) \) and \( V(M^-) \) are plus-edges, and, hence, the total number of plus-edges is at least

\[
\binom{2m^+}{2} + \frac{1}{2} \cdot 2m^+(4n - 2m^+) = (4n - 1)m^+ \geq (4n - 1)(n + 1) > 4n^2 - n,
\]

contradicting (1).

\( \square \)

Claim 7. For every two edges \( u_1u_2 \) and \( v_1v_2 \) in \( M^+ \), if \( u_1v_1 \) is a minus-edge, then \( u_2v_2 \) is also a minus-edge. Furthermore, \( \sigma(M) = 2 \).

Proof of Claim 7. If \( u_1u_2 \) and \( v_1v_2 \) are two edges in \( M^+ \) such that \( u_1v_1 \) is a minus-edge and \( u_2v_2 \) is a plus-edge, then the perfect matching \( N = (M \setminus \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\} \) satisfies \( 0 \leq \sigma(N) = \sigma(M) - 2 < \sigma(M) \), contradicting the choice of \( M \). This implies the first part of the statement. Now, suppose, for a contradiction, that \( \sigma(M) > 2 \). Since \( \sigma(M) \) is even, we have \( \sigma(M) \geq 4 \). By Claim 6, there are two edges \( u_1u_2 \) and \( v_1v_2 \) in \( M^+ \) such that \( u_1v_1 \) and \( u_2v_2 \) are both minus-edges. Now, the perfect matching \( N \) as above satisfies \( 0 \leq \sigma(N) = \sigma(M) - 4 < \sigma(M) \), contradicting the choice of \( M \). This completes the proof of the claim.

Since \( \sigma(M) = 2 \), the matching \( M \) contains exactly \( n + 1 \) plus-edges and \( n - 1 \) minus-edges. We call a perfect matching \( N \) in \( K_{4n} \) good if \( \sigma(N) = \sigma(M) \). If \( N \) is a good matching, then an edge \( e^+ \) in \( N^+ \) is called special if there is no minus-edge between vertices in \( V(N^+ \setminus \{e^+\}) \), that is, all minus-edges between vertices in \( V(N^+) \) are adjacent with \( e^+ \).

We distinguish the following two cases.

Case 1. Every good matching contains a special edge.

Let \( e^+ \) be a special edge in \( M \).

Claim 8. For every edge \( e \) in \( M^+ \setminus \{e^+\} \), there exist only minus-edges between \( e^+ \) and \( e \).

Proof of Claim 8. Suppose, for a contradiction, that there is a plus-edge between \( e^+ \) and some edge \( e \) in \( M^+ \setminus \{e^+\} \). By Claim 7, there are at least two plus-edges between \( e^+ \) and \( e \). Since \( e^+ \) is special, it follows that there are at most \( 4n - 2 \) minus-edges between vertices in \( V(M^+) \). Therefore, by Claim 5, the total number of plus-edges is at least

\[
\binom{2n + 2}{2} - (4n - 2) + \frac{1}{2} \cdot (2n + 2)(2n - 2) = 4n^2 - n + 1,
\]

contradicting (1).

\( \square \)
Claim 9. There is no plus-edge between vertices in $V(M^-)$. Furthermore, there is an edge $e^-$ in $M^-$ such that there are exactly three plus edges between $e^-$ and $e^+$, and, for every two edges $e$ in $M^+$ and $e'$ in $M^-$ with $(e, e') \neq (e^+, e^-)$, there are exactly two plus-edges between $e$ and $e'$.

Proof of Claim 9. By Claim 8, there are exactly $4n$ minus-edges between vertices in $V(M^+)$, and, hence, exactly $\binom{2n+2}{2} - 4n$ plus-edges between vertices in $V(M^+)$. By Claim 5, there are at least $\frac{1}{2} \cdot (2n + 2)(2n - 2)$ plus-edges between $V(M^+)$ and $V(M^-)$. Since
\[
\left(\frac{2n + 2}{2}\right) - 4n + \frac{1}{2} \cdot (2n + 2)(2n - 2) = 4n^2 - n - 1,
\]
observation (1) implies the existence of exactly one further plus-edge not yet accounted for.

Suppose, for a contradiction, that $V(M^-)$ contains a plus-edge $f_1$, that is, there are two edges $e_1$ and $e_2$ in $M^-$ such that $f_1$ lies between $e_1$ and $e_2$. Let $f_2$ be the edge between $e_1$ and $e_2$ that is disjoint from $f_1$. Since, by (1), $f_1$ is the only plus-edge between vertices in $V(M^-)$, it follows that $f_2$ is a minus-edge. Let $e_3$ be in $M^+ \setminus \{e^+\}$, and let $f_3$ and $f_4$ be disjoint minus-edges between $e_3$ and $e^+$. Now, the perfect matching $N = (M \setminus \{e^+, e_1, e_2, e_3\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of $M$.

It follows that there is no plus-edge between vertices in $V(M^-)$ and that there are exactly $\frac{1}{2} \cdot (2n + 2)(2n - 2) + 1$ plus-edges between $V(M^+)$ and $V(M^-)$. By Claim 5, this implies that there is an edge $\hat{e}$ in $M^+$ and an edge $e^-$ in $M^-$ such that there are exactly three plus edges between $\hat{e}$ and $e^-$, and, for every two edges $e$ in $M^+$ and $e'$ in $M^-$ with $(e, e') \neq (\hat{e}, e^-)$, there are exactly two plus-edges between $e$ and $e'$. In order to complete the proof of the claim, it remains to show that $\hat{e} = e^+$. Suppose, for a contradiction, that $\hat{e} \neq e^+$. Since $|M^+| = n + 1 \geq 3$, there is an edge $e_1$ in $M^+ \setminus \{e^+, \hat{e}\}$. Let $f_1$ and $f_2$ be two disjoint minus-edges between $e^+$ and $e_1$, and let $f_3$ and $f_4$ be two disjoint plus-edges between $\hat{e}$ and $e^-$. The perfect matching $N = (M \setminus \{e^+, \hat{e}, e^-, e_1\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of $M$. Hence, $\hat{e} = e^+$, which completes the proof of the claim.

Let $f^+$ and $f^-$ be two disjoint edges between $e^+$ and $e^-$ such that $f^+$ is a plus-edge and $f^-$ is a minus-edge. The perfect matching $N = (M \setminus \{e^+, e^-\}) \cup \{f^+, f^\}$ satisfies $\sigma(N) = \sigma(M)$, that is, $N$ is good. By the assumption in Case 1, $N$ contains a special edge. Since there are exactly three plus-edges between $f^+$ in $N^+$ and $f^-$ in $N^-$, Claim 9 implies that $f^+$ is a special edge in $N$. If $u^-$ is such that $\{u^-\} = e^- \cap f^+$, then Claim 8 implies that there are only minus-edges between $u^-$ and $V(M^+ \setminus \{e^+\}) = V(N^+ \setminus \{f^+\})$. Let $f_1$ be the plus-edge between $e^+$ and $e^-$ that is not incident with $u^-$. Let $e \in M^+ \setminus \{e^+\}$. Let $f_2$ and $f_3$ be two disjoint edges between $e^+ \cup e^-$ and $e$ that are disjoint from $f_1$, in particular, $f_2$ and $f_3$ are both minus-edges. Now, the perfect matching $M' = (M \setminus \{e^+, e^-, e\}) \cup \{f_1, f_2, f_3\}$ satisfies $\sigma(M') = \sigma(M) - 2 = 0$, contradicting the choice of $M$, which completes the proof in this case.
Case 2. Some good matching contains no special edge.

In this case, we may assume that the good matching $M$ chosen at the beginning of the proof contains no special edge.

First, suppose, for a contradiction, that there are two edges $e_1$ in $M^+$ and $e_2$ in $M^−$ such that there are two disjoint plus-edges $f_1$ and $f_2$ between $e_1$ and $e_2$. Since $e_1$ is not a special edge in $M$, Claim 7 implies the existence of two edges $e_3$ and $e_4$ both in $M^+ \{e_1\}$ such that there are two disjoint minus-edges $f_3$ and $f_4$ between $e_3$ and $e_4$. Now, the perfect matching $N = (M \{e_1, e_2, e_3, e_4\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of $M$. Hence, there are no two such edges as $e_1$ and $e_2$. Together with Claim 5 this implies that there are exactly $\frac{1}{2} \cdot (2n + 2)(2n - 2) = 2(n + 1)(n - 1)$ plus-edges between $V(M^+)$ and $V(M^−)$. Consequently, there are exactly $\frac{1}{2} \cdot (2n + 2)(2n - 2) = 2(n + 1)(n - 1)$ minus-edges between $V(M^+)$ and $V(M^−)$.

Next, suppose, for a contradiction, that the number of plus-edges between vertices in $V(M^−)$ is odd. In this case, there are two edges $e_1$ and $e_2$ in $M^−$ and two disjoint edges $f_1$ and $f_2$ between $e_1$ and $e_2$ such that $f_1$ is a plus-edge and $f_2$ is a minus-edge. By Claim 6 and Claim 7, there are two edges $e_3$ and $e_4$ in $M^+$ and two disjoint minus-edges $f_3$ and $f_4$ between $e_3$ and $e_4$. Again, the perfect matching $N = (M \{e_1, e_2, e_3, e_4\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of $M$. Hence, the number of plus-edges between vertices in $V(M^−)$ is even, say equal to $2k$ for some integer $k$. By (1), the number of minus-edges between vertices in $V(M^+)$ is

$$(4n^2 - n) - 2(n + 1)(n - 1) - \binom{2n - 2}{2} - 2k = 4n + 2k - 1,$$

which is odd. Nevertheless, by Claim 7, the number of minus-edges between vertices in $V(M^+)$ is even, which is a contradiction, and completes the proof. \hfill \square

The following is based on a construction given at the end of [2].

Proof of Proposition 2. Let $n$ be such that there exists a positive even integer $k$ with $4n = k^2 + 4$. Let $(A, B)$ be a partition of the vertex set of $K_{4n}$ with $|A| = \frac{1}{2}(k^2 + k) + 2$ and $|B| = \frac{1}{2}(k^2 - k) + 2$. Now, we define a function $\sigma : E(K_{4n}) \to \{−1, 1\}$ such that all edges between $A$ and $B$ receive the value 1 and all remaining edges receive the value $−1$. Note that

$$|\sigma^−(1)| = |A| \cdot |B| = \left(\frac{k^2 + k}{2} + 2\right) \left(\frac{k^2 - k}{2} + 2\right) = \frac{(k^2 + 4)(k^2 + 3) + 4}{4}$$

$$= \frac{1}{2} \left(\frac{4n}{2}\right) + 1$$

and thus,

$$\sigma(E(K_{4n})) = |\sigma^−(1)| - |\sigma^−(-1)| = \left(\frac{1}{2} \left(\frac{4n}{2}\right) + 1\right) - \left(\frac{1}{2} \left(\frac{4n}{2}\right) - 1\right) = 2.$$
Now, suppose, for a contradiction, that \( M \) is a perfect matching in \( K_{4n} \) with \( \sigma(M) = 0 \). Clearly, \( M \) contains \( n \) plus-edges between \( A \) and \( B \). Hence, the number of vertices in \( A \) that are not covered by a plus-edge in \( M \) is
\[
|A| - n = \left( \frac{k^2 + k}{2} + 2 \right) - \frac{k^2 + 4}{4} = \frac{k}{2} \left( \frac{k}{2} + 1 \right) + 1.
\]
Since this is an odd number, by construction, not all these vertices can be covered by minus-edges in \( M \), which is a contradiction, and completes the proof.

For the proof of our second main result, Theorem 3, we need the following extremal result about matchings due to Erdős and Gallai [5].

**Theorem 10** (Erdős and Gallai [5]). If \( G \) is a graph of order \( 4n \) with matching number \( n - k \) for some positive integer \( k \), then the number of edges of \( G \) is at most \( \binom{n}{2} - \binom{3n+k}{2} \), with equality if and only if \( G \) is the complement of the disjoint union of a complete graph of order \( 3n \) and \( n - k \) isolated vertices.

We proceed to the proof of our second main result.

**Proof of Theorem 3.** Let \( \sigma : E(K_{4n}) \rightarrow \{-1,1\} \) be such that \( \sigma(E(K_{4n})) \geq 0 \) and such that \( |\sigma(M)| \geq 2k \geq 4 \) for every perfect matching \( M \) in \( K_{4n} \). As observed above, \( |\sigma(M)| \) is even for every perfect matching \( M \) in \( K_{4n} \). Therefore, it remains to show that \( \sigma(E(K_{4n})) \) is at least the term stated in the theorem.

We distinguish the following two cases.

**Case 1.** \( \sigma(M) \leq 0 \) for some perfect matching \( M \) in \( K_{4n} \).

We choose a perfect matching \( M \) in \( K_{4n} \) with \( \sigma(M) \leq 0 \) such that \( \sigma(M) \) is as large as possible. Since \( \sigma(M) \leq -2k \leq -4 \), we obtain \( |M^-| \geq n + k \) and \( |M^+| \leq n - k \), where we use the notation and terminology as in the proof of Theorem 1. The following two observations correspond to Claim 5 and Claim 6 within the proof of Theorem 1.

If there is an edge \( e \) in \( M^+ \) and an edge \( f \) in \( M^- \) such that there are two disjoint plus-edges \( e' \) and \( f' \) between \( e \) and \( f \), then the perfect matching \( N = (M \setminus \{e, f\}) \cup \{e', f'\} \) satisfies \( \sigma(M) > \sigma(N) = \sigma(M) + 2 < 0 \), contradicting the choice of \( M \). Hence, between every edge in \( M^+ \) and every edge in \( M^- \), there are at least two minus-edges, which implies that at least half the edges between \( V(M^+) \) and \( V(M^-) \) are minus-edges.

If there is no plus-edge between vertices in \( V(M^-) \), then there are strictly more minus-edges than plus-edges, contradicting \( \sigma(E(K_{4n})) \geq 0 \). Hence, there are two edges \( e \) and \( f \) in \( M^- \) such that there exists a plus-edge \( e' \) between \( e \) and \( f \). Let \( f' \) be the edge between \( e \) and \( f \) that is disjoint from \( e' \). The perfect matching \( N = (M \setminus \{e, f\}) \cup \{e', f'\} \) satisfies \( \sigma(M) < \sigma(N) \leq \sigma(M) + 4 \leq -2k + 4 \leq 0 \), contradicting the choice of \( M \). This contradiction completes the proof in this case.

**Case 2.** \( \sigma(M) > 0 \) for every perfect matching \( M \) in \( K_{4n} \).

Note that \( \sigma(M) \geq 2k \) for every perfect matching \( M \) in \( K_{4n} \). Let \( \nu \) be the matching number of the graph \( G = (V(K_{4n}), \sigma^{-1}(-1)) \).
Suppose, for a contradiction, that \( \nu > n - k \). If \( M \) is a maximum matching in \( G \), then \( V(G) \setminus V(M^-) \) is an independent set in \( G \). This implies that there is a matching \( M^+ \) in \((V(K_{4n}), \sigma^{-1}(1))\) covering all vertices in \( V(G) \setminus V(M^-) \). Note that \( |M^+| = 2n - \nu < n + k \), and, hence, \( M^- \cup M^+ \) is a perfect matching in \( K_{4n} \) with \( \sigma(M^- \cup M^+) < -(n - k) + (n + k) = 2k \), which is a contradiction. Hence, we have \( \nu \leq n - k \).

Let \( \nu = n - k' \) for some integer \( k' \geq k \). By Theorem 10, we obtain

\[
\sigma(E(K_{4n})) = \left( \binom{4n}{2} \right) - 2m(G) \\
\geq \left( \binom{4n}{2} \right) - 2 \left( \frac{\binom{4n}{2} - \binom{3n + k'}{2}}{2} \right) \\
= 2 \left( \frac{3n + k'}{2} \right) - \left( \frac{4n}{2} \right) \\
\geq 2 \left( \frac{3n + k}{2} \right) - \left( \frac{4n}{2} \right) \\
= n(n - 1) + k(6n - 1) + k^2,
\]

which completes the proof.

\[\square\]

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