Low weight perfect matchings

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Abstract

Answering a question posed by Caro, Hansberg, Lauri, and Zarb, we show that for every positive integer n and every function $\sigma\colon E(K_{4n}) \to \{-1,1\}$ with $\sigma(E(K_{4n})) = 0$, there is a perfect matching M in $K_{4n}$with $\sigma(M) = 0$. Strengthening the consequence of a result of Caro and Yuster, we show that for every positive integer n and every function $\sigma\colon E(K_{4n}) \to \{-1,1\}$ with $|\sigma(E(K_{4n}))| < n^2+11n+2$, there is a perfect matching M in K_{4n} with $|\sigma(M)| \leqslant 2$. Both these results are best possible.

Mathematics Subject Classifications: 05C22, 05C70

1 Introduction

In [2] Caro, Hansberg, Lauri, and Zarb considered connected graphs G together with a function $\sigma\colon E(G) \to \{-1,1\}$ labeling the edges of G with -1 or $+1$, and they studied conditions that imply the existence of different types of spanning trees T with

$$|\sigma(E(T))| = \left|\sum_{e\in E(T)} \sigma(e)\right| \leqslant 1,$$

where, as usual, for a set E of edges, $\sigma(E)$ is just the sum of $\sigma(e)$ over all e in E. As a variation of this problem, they ask whether, for every positive integer n and every labeling $\sigma\colon E(K_{4n}) \to \{-1,1\}$ of the edges of the complete graph K_{4n} of order $4n$ with $\sigma(E(K_{4n})) = 0$, there is a perfect matching M in K_{4n} with $\sigma(M) = 0$. We answer their question in the affirmative.

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Theorem 1. *For every positive integer n and every function $\sigma\colon E(K_{4n}) \to \{-1,1\}$ with $\sigma(E(K_{4n})) = 0$, there is a perfect matching M in K_{4n} with $\sigma(M) = 0$.*

In order to put our result into some wider perspective, we briefly discuss the notion of local amoebas studied in [2, 3]: A graph G of order n is a *local amoeba* if, for every two isomorphic copies H and H' of G in K_n, there is a sequence $G_0, \ldots, G_k$ of isomorphic copies of G in K_n such that $H = G_0$, $H' = G_k$, and each G_{i+1} arises from G_i by an *edge-replacement*, that is, $E(G_{i+1}) = (E(G_i) \setminus \{e\}) \cup \{e'\}$ for some $e \in E(G_i)$ and $e' \in E(K_n) \setminus E(G_i)$. Note that a path P_n is an example of a local amoeba, and that local amoebas are defined by an exchange property that very closely resembles the well-known exchange property of the bases of a matroid: For every two bases B and B' of a matroid, and for every $e' \in B' \setminus B$, there is some $e \in B \setminus B'$ such that $(B \setminus \{e\}) \cup \{e'\}$ is again a basis of the matroid. Now, perfect matchings of a complete graph are a very natural example of spanning subgraphs that are no local amoebas, and for which, consequently, the machinery developed in [2] fails. In fact, perfect matchings have the following slightly weaker exchange property: If M and M' are two distinct perfect matchings in K_n, then, considering an M-M'-alternating cycle defined by their symmetric difference $M \Delta M'$, it follows that there are two edges e_1, e_2 in M, one edge e'_1 in M', and one further edge e'_2, which might not belong to $M \cup M'$, such that $M'' = (M \setminus \{e_1, e_2\}) \cup \{e'_1, e'_2\}$ is a perfect matching in K_n for which $|M' \Delta M''|$ is strictly smaller than $|M' \Delta M|$, that is, one can transform M into M' by a sequence of exchange operations removing and adding two edges, and not just one, at every step.

Under the hypothesis of Theorem 1, the existence of a perfect matching M in K_{4n} with $|\sigma(M)| \leqslant 2$ already follows from more general results due to Caro and Yuster, cf. Theorem 1.1 in [4]. More precisely, Caro and Yuster showed that the weaker hypothesis $|\sigma(E(K_{4n}))| \leqslant 2(4n-1)$ suffices for the existence of such a perfect matching M with $|\sigma(M)| \leqslant 2$. As observed in [2], for infinitely many positive integers n, there are functions $\sigma\colon E(K_{4n}) \to \{-1,1\}$ with $\sigma(E(K_{4n})) = 4\sqrt{n} - 2$ such that $\sigma(M) \neq 0$ for every perfect matching M in K_{4n}. Slightly modifying their construction, we obtain the following proposition, which implies that Theorem 1 is best possible for infinitely many values of n.

Proposition 2. *For infinitely many positive integers n, there is a function $\sigma\colon E(K_{4n}) \to \{-1,1\}$ with $\sigma(E(K_{4n})) = 2$ such that $\sigma(M) \neq 0$ for every perfect matching M in K_{4n}.*

Considering the construction in the proof of Proposition 2 suggests that zero weight perfect matchings are excluded rather by parity reasons than by the value of the imbalance $|\sigma(E(K_{4n}))|$ of σ. We confirm this with our second main result showing that much weaker conditions on the imbalance imply the existence of low weight perfect matchings.

Theorem 3. *For all positive integers n and k such that $k \geqslant 2$, and every function $\sigma\colon E(K_{4n}) \to \{-1,1\}$ with $|\sigma(E(K_{4n}))| < n(n-1) + k(6n-1) + k^2$, there is a perfect matching M in K_{4n} with $|\sigma(M)| \leqslant 2k-2$.*

For $k = 2$, Theorem 3 implies the following strengthening of the above-mentioned consequence of the result of Caro and Yuster.

Corollary 4. *For every positive integer n and every function* $\sigma\colon E(K_{4n}) \to \{-1,1\}$ *with* $|\sigma(E(K_{4n}))| < n^2 + 11n + 2$, *there is a perfect matching* M *in* K_{4n} *with* $|\sigma(M)| \leqslant 2$.

Both, Theorem 3 and, hence, also Corollary 4 are best possible. If, for instance, $\sigma\colon E(K_{4n}) \to \{-1,1\}$ is such that the graph $(V(K_{4n}), \sigma^{-1}(1))$ consists of a clique of order $3n+2$ and $n-2$ isolated vertices, then $|\sigma(M)| \geqslant 4$ for every perfect matching M in K_{4n} while $|\sigma(E(K_{4n}))| = n^2 + 11n + 2$. In conjunction, Theorem 1, Proposition 2, and Corollary 4 imply an interesting behavior: If n is a positive integer and σ is a ±1-labeling of the edges of K_{4n}, then, in order to force the existence of a zero weight perfect matching, one needs to require zero imbalance, that is, $|\sigma(E(K_{4n}))| = 0$, while $|\sigma(M)| > 2$ for every perfect matching M in K_{4n} already forces a quadratic imbalance, that is, $|\sigma(E(K_{4n}))|$ is at least quadratic in n.

All proofs are given in the next section.

For a survey concerning related results, we refer the reader to [1] and the introduction of [2].

2 Proofs

We start with the proof of our first main result.

Proof of Theorem 1. We suppose, for a contradiction, that $\sigma\colon E(K_{4n}) \to \{-1,1\}$ is such that $\sigma(E(K_{4n})) = 0$ but that $\sigma(M) \neq 0$ for every perfect matching M in K_{4n}. First, we consider the case $n = 1$. The edge set of K_4 is the union of three edge-disjoint perfect matchings M_1, M_2, and M_3. Since $\sigma(M_i) \neq 0$ for every i, we obtain $\sigma(M_i) \in \{-2,2\}$, which implies the contradiction $\sigma(E(K_{4n})) = \sigma(M_1) + \sigma(M_2) + \sigma(M_3) \neq 0$. Hence, we may assume that $n \geqslant 2$. We call an edge e a *plus-edge* if $\sigma(e) = 1$, and a *minus-edge* if $\sigma(e) = -1$. Since $\sigma(E(K_{4n})) = 0$,

$$\text{there are exactly } \frac{1}{2}\binom{4n}{2} = 4n^2 - n \text{ plus-edges and minus-edges in } K_{4n}\text{, respectively.} \quad (1)$$

For a matching M, we denote by M^+ and M^- the sets of plus-edges and minus-edges in M, respectively. We choose a perfect matching M in K_{4n} such that $|\sigma(M)|$ is as small as possible. Possibly replacing σ with $-\sigma$, we may assume that $\sigma(M) > 0$. Since M contains $2n$ edges, $\sigma(M)$ is even, which implies $\sigma(M) \geqslant 2$.

We start with some easy observations.

Claim 5. *For every two edges e in* M^+ *and f in* M^-, *there are no two disjoint minus-edges between e and f. In particular, there are at most two minus-edges between e and f.*

Proof of Claim 5. If there are two disjoint minus-edges e' and f' between two edges $e \in M^+$ and $f \in M^-$, then the perfect matching $N = (M \setminus \{e,f\}) \cup \{e', f'\}$ satisfies $0 \leqslant \sigma(N) = \sigma(M) - 2 < \sigma(M)$, contradicting the choice of M. □

Claim 6. *There are two edges e and f in* M^+ *such that there exists a minus-edge between e and f.*

Proof of Claim 6. Suppose, for a contradiction, that there is no minus-edge between vertices in $V(M^+)$; that is, more formally, the subgraph of K_{4n} induced by the set $V(M^+)$ of vertices that are incident with a plus-edge from M contains no minus-edge of K_{4n}. For $m^+ = |M^+|$, we have $m^+ \geqslant n+1$. By Claim 5, at least half the $2m^+(4n-2m^+)$ edges between $V(M^+)$ and $V(M^-)$ are plus-edges, and, hence, the total number of plus-edges is at least

$$\binom{2m^+}{2} + \frac{1}{2} \cdot 2m^+(4n-2m^+) = (4n-1)m^+ \geqslant (4n-1)(n+1) > 4n^2 - n,$$

contradicting (1). □

Claim 7. *For every two edges u_1u_2 and v_1v_2 in M^+, if u_1v_1 is a minus-edge, then u_2v_2 is also a minus-edge. Furthermore, $\sigma(M) = 2$.*

Proof of Claim 7. If u_1u_2 and v_1v_2 are two edges in M^+ such that u_1v_1 is a minus-edge and u_2v_2 is a plus-edge, then the perfect matching $N = (M \setminus \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ satisfies $0 \leqslant \sigma(N) = \sigma(M) - 2 < \sigma(M)$, contradicting the choice of M. This implies the first part of the statement. Now, suppose, for a contradiction, that $\sigma(M) > 2$. Since $\sigma(M)$ is even, we have $\sigma(M) \geqslant 4$. By Claim 6, there are two edges u_1u_2 and v_1v_2 in M^+ such that u_1v_1 and u_2v_2 are both minus-edges. Now, the perfect matching N as above satisfies $0 \leqslant \sigma(N) = \sigma(M) - 4 < \sigma(M)$, contradicting the choice of M. This completes the proof of the claim. □

Since $\sigma(M) = 2$, the matching M contains exactly $n+1$ plus-edges and $n-1$ minus-edges. We call a perfect matching N in K_{4n} *good* if $\sigma(N) = \sigma(M)$. If N is a good matching, then an edge e^+ in N^+ is called *special* if there is no minus-edge between vertices in $V(N^+ \setminus \{e^+\})$, that is, all minus-edges between vertices in $V(N^+)$ are adjacent with e^+.

We distinguish the following two cases.

Case 1. *Every good matching contains a special edge.*

Let e^+ be a special edge in M.

Claim 8. *For every edge e in $M^+ \setminus \{e^+\}$, there exist only minus-edges between e^+ and e.*

Proof of Claim 8. Suppose, for a contradiction, that there is a plus-edge between e^+ and some edge e in $M^+ \setminus \{e^+\}$. By Claim 7, there are at least two plus-edges between e^+ and e. Since e^+ is special, it follows that there are at most $4n-2$ minus-edges between vertices in $V(M^+)$. Therefore, by Claim 5, the total number of plus-edges is at least

$$\binom{2n+2}{2} - (4n-2) + \frac{1}{2} \cdot (2n+2)(2n-2) = 4n^2 - n + 1,$$

contradicting (1). □

Claim 9. *There is no plus-edge between vertices in $V(M^-)$. Furthermore, there is an edge e^- in M^- such that there are exactly three plus edges between e^+ and e^-, and, for every two edges e in M^+ and e' in M^- with $(e,e') \neq (e^+,e^-)$, there are exactly two plus-edges between e and e'.*

Proof of Claim 9. By Claim 8, there are exactly $4n$ minus-edges between vertices in $V(M^+)$, and, hence, exactly $\binom{2n+2}{2} - 4n$ plus-edges between vertices in $V(M^+)$. By Claim 5, there are at least $\frac{1}{2}\cdot(2n+2)(2n-2)$ plus-edges between $V(M^+)$ and $V(M^-)$. Since

$$\left(\binom{2n+2}{2} - 4n\right) + \frac{1}{2}\cdot(2n+2)(2n-2) = 4n^2 - n - 1,$$

observation (1) implies the existence of exactly one further plus-edge not yet accounted for.

Suppose, for a contradiction, that $V(M^-)$ contains a plus-edge f_1, that is, there are two edges e_1 and e_2 in M^- such that f_1 lies between e_1 and e_2. Let f_2 be the edge between e_1 and e_2 that is disjoint from f_1. Since, by (1), f_1 is the only plus-edge between vertices in $V(M^-)$, it follows that f_2 is a minus-edge. Let e_3 be in $M^+ \setminus \{e^+\}$, and let f_3 and f_4 be disjoint minus-edges between e_3 and e^+. Now, the perfect matching $N = (M \setminus \{e^+, e_1, e_2, e_3\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of M.

It follows that there is no plus-edge between vertices in $V(M^-)$ and that there are exactly $\frac{1}{2}\cdot(2n+2)(2n-2)+1$ plus-edges between $V(M^+)$ and $V(M^-)$. By Claim 5, this implies that there is an edge $\hat{e}$ in M^+ and an edge e^- in M^- such that there are exactly three plus edges between $\hat{e}$ and e^-, and, for every two edges e in M^+ and e' in M^- with $(e,e') \neq (\hat{e},e^-)$, there are exactly two plus-edges between e and e'. In order to complete the proof of the claim, it remains to show that $\hat{e} = e^+$. Suppose, for a contradiction, that $\hat{e} \neq e^+$. Since $|M^+| = n+1 \geqslant 3$, there is an edge e_1 in $M^+ \setminus \{e^+, \hat{e}\}$. Let f_1 and f_2 be two disjoint minus-edges between e^+ and e_1, and let f_3 and f_4 be two disjoint plus-edges between $\hat{e}$ and e^-. The perfect matching $N = (M \setminus \{e^+, \hat{e}, e^-, e_1\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of M. Hence, $\hat{e} = e^+$, which completes the proof of the claim. □

Let f^+ and f^- be two disjoint edges between e^+ and e^- such that f^+ is a plus-edge and f^- is a minus-edge. The perfect matching $N = (M \setminus \{e^+, e^-\}) \cup \{f^+, f^-\}$ satisfies $\sigma(N) = \sigma(M)$, that is, N is good. By the assumption in Case 1, N contains a special edge. Since there are exactly three plus-edges between f^+ in N^+ and f^- in N^-, Claim 9 implies that f^+ is a special edge in N. If u^- is such that $\{u^-\} = e^- \cap f^+$, then Claim 8 implies that there are only minus-edges between u^- and $V(M^+ \setminus \{e^+\}) = V(N^+ \setminus \{f^+\})$. Let f_1 be the plus-edge between e^+ and e^- that is not incident with u^-. Let $e \in M^+ \setminus \{e^+\}$. Let f_2 and f_3 be two disjoint edges between $e^+ \cup e^-$ and e that are disjoint from f_1, in particular, f_2 and f_3 are both minus-edges. Now, the perfect matching $M' = (M \setminus \{e^+, e^-, e\}) \cup \{f_1, f_2, f_3\}$ satisfies $\sigma(M') = \sigma(M) - 2 = 0$, contradicting the choice of M, which completes the proof in this case.

Case 2. *Some good matching contains no special edge.*

In this case, we may assume that the good matching M chosen at the beginning of the proof contains no special edge.

First, suppose, for a contradiction, that there are two edges e_1 in M^+ and e_2 in M^- such that there are two disjoint plus-edges f_1 and f_2 between e_1 and e_2. Since e_1 is not a special edge in M, Claim 7 implies the existence of two edges e_3 and e_4 both in $M^+ \setminus \{e_1\}$ such that there are two disjoint minus-edges f_3 and f_4 between e_3 and e_4. Now, the perfect matching $N = (M \setminus \{e_1, e_2, e_3, e_4\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of M. Hence, there are no two such edges as e_1 and e_2. Together with Claim 5 this implies that there are exactly $\frac{1}{2} \cdot (2n+2)(2n-2) = 2(n+1)(n-1)$ plus-edges between $V(M^+)$ and $V(M^-)$. Consequently, there are exactly $\frac{1}{2} \cdot (2n+2)(2n-2) = 2(n+1)(n-1)$ minus-edges between $V(M^+)$ and $V(M^-)$.

Next, suppose, for a contradiction, that the number of plus-edges between vertices in $V(M^-)$ is odd. In this case, there are two edges e_1 and e_2 in M^- and two disjoint edges f_1 and f_2 between e_1 and e_2 such that f_1 is a plus-edge and f_2 is a minus-edge. By Claim 6 and Claim 7, there are two edges e_3 and e_4 in M^+ and two disjoint minus-edges f_3 and f_4 between e_3 and e_4. Again, the perfect matching $N = (M \setminus \{e_1, e_2, e_3, e_4\}) \cup \{f_1, f_2, f_3, f_4\}$ satisfies $\sigma(N) = \sigma(M) - 2 = 0$, contradicting the choice of M. Hence, the number of plus-edges between vertices in $V(M^-)$ is even, say equal to $2k$ for some integer k. By (1), the number of minus-edges between vertices in $V(M^+)$ is

$$(4n^2 - n) - 2(n+1)(n-1) - \left(\binom{2n-2}{2} - 2k\right) = 4n + 2k - 1,$$

which is odd. Nevertheless, by Claim 7, the number of minus-edges between vertices in $V(M^+)$ is even, which is a contradiction, and completes the proof. □

The following is based on a construction given at the end of [2].

Proof of Proposition 2. Let n be such that there exists a positive even integer k with $4n = k^2 + 4$. Let (A, B) be a partition of the vertex set of K_{4n} with $|A| = \frac{1}{2}(k^2 + k) + 2$ and $|B| = \frac{1}{2}(k^2 - k) + 2$. Now, we define a function $\sigma\colon E(K_{4n}) \to \{-1, 1\}$ such that all edges between A and B receive the value 1 and all remaining edges receive the value -1. Note that

$$\begin{aligned} |\sigma^{-1}(1)| = |A| \cdot |B| &= \left(\frac{k^2+k}{2} + 2\right)\left(\frac{k^2-k}{2} + 2\right) = \frac{(k^2+4)(k^2+3)+4}{4} \\ &= \frac{1}{2}\binom{4n}{2} + 1 \end{aligned}$$

and thus,

$$\sigma\left(E(K_{4n})\right) = |\sigma^{-1}(1)| - |\sigma^{-1}(-1)| = \left(\frac{1}{2}\binom{4n}{2} + 1\right) - \left(\frac{1}{2}\binom{4n}{2} - 1\right) = 2.$$

Now, suppose, for a contradiction, that M is a perfect matching in K_{4n} with $\sigma(M) = 0$. Clearly, M contains n plus-edges between A and B. Hence, the number of vertices in A that are not covered by a plus-edge in M is

$$|A| - n = \left(\frac{k^2+k}{2} + 2\right) - \frac{k^2+4}{4} = \frac{k}{2}\left(\frac{k}{2}+1\right) + 1.$$

Since this is an odd number, by construction, not all these vertices can be covered by minus-edges in M, which is a contradiction, and completes the proof. □

For the proof of our second main result, Theorem 3, we need the following extremal result about matchings due to Erdős and Gallai [5].

Theorem 10 (Erdős and Gallai [5]). *If G is a graph of order $4n$ with matching number $n-k$ for some positive integer k, then the number of edges of G is at most $\binom{4n}{2} - \binom{3n+k}{2}$, with equality if and only if G is the complement of the disjoint union of a complete graph of order $3n+k$ and $n-k$ isolated vertices.*

We proceed to the proof of our second main result.

Proof of Theorem 3. Let $\sigma\colon E(K_{4n}) \to \{-1, 1\}$ be such that $\sigma\left(E(K_{4n})\right) \geqslant 0$ and such that $|\sigma(M)| \geqslant 2k \geqslant 4$ for every perfect matching M in K_{4n}. As observed above, $|\sigma(M)|$ is even for every perfect matching M in K_{4n}. Therefore, it remains to show that $\sigma\left(E(K_{4n})\right)$ is at least the term stated in the theorem.

We distinguish the following two cases.

Case 1. $\sigma(M) \leqslant 0$ *for some perfect matching M in K_{4n}.*

We choose a perfect matching M in K_{4n} with $\sigma(M) \leqslant 0$ such that $\sigma(M)$ is as large as possible. Since $\sigma(M) \leqslant -2k \leqslant -4$, we obtain $|M^-| \geqslant n + k$ and $|M^+| \leqslant n - k$, where we use the notation and terminology as in the proof of Theorem 1. The following two observations correspond to Claim 5 and Claim 6 within the proof of Theorem 1.

If there is an edge e in M^+ and an edge f in M^- such that there are two disjoint plus-edges e' and f' between e and f, then the perfect matching $N = (M \setminus \{e, f\}) \cup \{e', f'\}$ satisfies $\sigma(M) < \sigma(N) = \sigma(M) + 2 < 0$, contradicting the choice of M. Hence, between every edge in M^+ and every edge in M^-, there are at least two minus-edges, which implies that at least half the edges between $V(M^+)$ and $V(M^-)$ are minus-edges.

If there is no plus-edge between vertices in $V(M^-)$, then there are strictly more minus-edges than plus-edges, contradicting $\sigma\left(E(K_{4n})\right) \geqslant 0$. Hence, there are two edges e and f in M^- such that there exists a plus-edge e' between e and f. Let f' be the edge between e and f that is disjoint from e'. The perfect matching $N = (M \setminus \{e, f\}) \cup \{e', f'\}$ satisfies $\sigma(M) < \sigma(N) \leqslant \sigma(M) + 4 \leqslant -2k + 4 \leqslant 0$, contradicting the choice of M. This contradiction completes the proof in this case.

Case 2. $\sigma(M) > 0$ *for every perfect matching M in K_{4n}.*

Note that $\sigma(M) \geqslant 2k$ for every perfect matching M in K_{4n}. Let ν be the matching number of the graph $G = (V(K_{4n}), \sigma^{-1}(-1))$.

Suppose, for a contradiction, that $\nu > n-k$. If M is a maximum matching in G, then $V(G) \setminus V(M^-)$ is an independent set in G. This implies that there is a matching M^+ in $(V(K_{4n}), \sigma^{-1}(1))$ covering all vertices in $V(G) \setminus V(M^-)$. Note that $|M^+| = 2n - \nu < n+k$, and, hence, $M^- \cup M^+$ is a perfect matching in K_{4n} with $\sigma(M^- \cup M^+) < -(n-k)+(n+k) = 2k$, which is a contradiction. Hence, we have $\nu \leqslant n-k$.

Let $\nu = n - k'$ for some integer $k' \geqslant k$. By Theorem 10, we obtain

$$\begin{aligned} \sigma\left(E(K_{4n})\right) &= \binom{4n}{2} - 2m(G) \\ &\geqslant \binom{4n}{2} - 2\left(\binom{4n}{2} - \binom{3n+k'}{2}\right) \\ &= 2\binom{3n+k'}{2} - \binom{4n}{2} \\ &\geqslant 2\binom{3n+k}{2} - \binom{4n}{2} \\ &= n(n-1) + k(6n-1) + k^2, \end{aligned}$$

which completes the proof. □

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