A counterexample to a conjecture on Schur positivity of chromatic symmetric functions of trees

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Abstract

We show that no tree on twenty vertices with maximum degree ten has Schur positive chromatic symmetric function, thereby providing a counterexample to a conjecture of Dahlberg, She and van Willigenburg.

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Among the many nice results on chromatic symmetric functions in the paper [1] of Dahlberg, She, and van Willigenburg is Theorem 39 therein, which says that no bipartite graph on $n$ vertices with a vertex of degree more than $\left\lceil \frac{n}{2} \right\rceil$ has Schur positive chromatic symmetric function. In particular, Theorem 39 applies to trees. A near-converse to Theorem 39 for trees is posed in [1, Conjecture 42], which says that for every $n \geq 2$, there is a tree $T$ on $n$ vertices, one of which has degree $\left\lfloor \frac{n}{2} \right\rfloor$, such that the chromatic symmetric function of $T$ is Schur positive. The authors of [1] confirmed this conjecture for $n \leq 19$, using computer calculations. The conjecture turns out to be false for $n = 20$, as we show here. We use SageMath [2] calculations after a preparatory proposition that reduces the number of trees that we must examine.

We give the requisite definitions and reiterate more formally. Given a (finite, loopless, simple) graph $G = (V,E)$, a proper coloring of $G$ is a function $\kappa$ from $V$ to the set $\mathbb{P}$ of positive integers such that $\kappa(v) \neq \kappa(w)$ whenever $\{v,w\} \in E$. We fix an infinite set $x := \{x_i : i \in \mathbb{P}\}$ of pairwise commuting variables, and write $K(G)$ for the set of all proper colorings of $G$. To each proper coloring $\kappa$ one associates a monomial

$$x^\kappa := \prod_{v \in V} x_{\kappa(v)}.$$
The chromatic symmetric function $X_G$ of $G$ is the sum of all such monomials,

$$X_G(x) := \sum_{\pi \in K(G)} x^\pi.$$  

Chromatic symmetric functions were introduced by Stanley in [5] and have drawn considerable attention. Various results and conjectures, including the above-mentioned theorem and conjecture from [1], relate the structure of $G$ to the expansion of $X_G$ in terms of one or more familiar bases for the algebra $\Lambda$ of symmetric functions. Recall that if $B$ is a basis for $\Lambda$ and $f$ a basis for $\Lambda$, we call $f$ B-positive if, when we expand $f = \sum_{b \in B} \alpha_b b$, each $\alpha_b$ is non-negative. The Schur basis for $\Lambda$ is a fundamental object in symmetric function theory. See for example [3, Chapter 7] for basic properties of Schur functions and other rudimentary facts about symmetric functions that will be used herein without reference.

We prove the following result, thereby disproving Conjecture 42 of [1].

**Theorem 1.** If $T$ is a tree on twenty vertices, one of which has degree ten, then $X_T(x)$ is not Schur positive.

A stable partition of $G$ is a set partition $\pi : V = \bigcup_{j=1}^{k} \pi_j$ with each $\pi_j$ an independent set in $G$. We assume without loss of generality that $|\pi_j| \geq |\pi_{j+1}|$ for each $j \in [n-1]$. Setting $\lambda_j = |\pi_j|$ for each $j$, we get that $\lambda := (\lambda_1, \ldots, \lambda_k)$ is a partition of the integer $|V|$. We call $\lambda$ the type of $\pi$. Given another partition $\mu = (\mu_1, \ldots, \mu_\ell)$ of $|V|$, we write $\mu \preceq \lambda$ if $\lambda$ dominates $\mu$, that is, if $\sum_{j=1}^{m} \mu_j \leq \sum_{j=1}^{m} \lambda_j$ for all $m \in [k]$. Our proof of Theorem 1 rests on the following basic result, due to Stanley. This result follows quickly from the fact that if $\mu \preceq \lambda$, then when the Schur function $s_\lambda$ is expanded in the monomial basis, the coefficient of $m_\mu$ is positive.

**Lemma 2** (Proposition 1.5 of [4]). If $X_G(x)$ is Schur positive and $G$ admits a stable partition of type $\lambda$, then $G$ admits a stable partition of type $\mu$ whenever $\mu \preceq \lambda$.

**Corollary 3.** Assume that $T = (V, E)$ is a tree on $2n$ vertices and $v \in V$ has degree $n$ in $T$. If $X_T(x)$ is Schur positive, then every $x \in V$ that is neither $v$ nor a neighbor of $v$ is a leaf in $T$.

**Proof.** As $T$ is connected and bipartite, $T$ has a unique bipartition $\pi : V = \pi_1 \cup \pi_2$. If $X_T(x)$ is Schur positive, then $\pi$ has type $(n, n)$ by Lemma 2. We assume without loss of generality that $v \in \pi_1$. Then the neighborhood $N_T(v)$ is contained in $\pi_2$ and so $\pi_2 = N_T(v)$. Were the claim of the corollary false, some $z \in V$ would be at distance three from $v$ in $T$ and therefore lie in $\pi_2$, which is impossible.

For each partition $\nu = (\nu_1, \ldots, \nu_t)$ of $n - 1$, let $T(\nu)$ be a tree on $2n$ vertices in which one vertex $v$ has exactly $n$ neighbors $v_1, \ldots, v_n$, and for $1 \leq i \leq t$, $v_i$ has exactly $\nu_i$ neighbors other than $v$ (each of which is necessarily a leaf). The next result follows immediately from Corollary 3.

**Corollary 4.** If $T$ is a tree on $2n$ vertices, one of which has degree $n$, and $X_T(x)$ is Schur positive, then there is some partition $\nu$ of $n - 1$ such that $T$ is isomorphic with $T(\nu)$.
Theorem 1 follows from the next result, which we prove by inspection using SageMath calculations.

**Proposition 5.** If $\nu$ is a partition of the integer nine, then $X_{T(\nu)}$ is not Schur positive.

Our computations reveal in particular that if $n = 10$ and $\nu_1 \geq 6$, then the coefficient of $s_{(9,9,2)}$ in the Schur expansion of $X_{T(\nu)}(x)$ is negative; and if $n = 10$ and $\nu_1 \leq 5$, then the coefficient of $s_{(3,3,2,2,2,2,2)}$ in the Schur expansion of $X_{T(\nu)}(x)$ is negative. This Schur expansion has can have as few as four negative coefficients (when $\nu$ is one of $(6,2,1),(6,1,1,1)$ or $(5,4)$) and as many as thirty (when $\nu$ is one of $(2,2,2,2,1),(2,2,1,1,1)$ or $(1,1,1,1,1,1,1,1,1)$). Our programs, along with the complete Schur expansion of $X_{T(\nu)}(x)$ for each partition $\nu$ of nine, can be found at [https://github.com/emmanuellasa/Schur_Decomposition_20](https://github.com/emmanuellasa/Schur_Decomposition_20).

We close with some comments. In addition to Schur positivity, it is of interest to study $e$-positivity of chromatic symmetric functions, that is, positivity with respect to the basis of elementary symmetric functions. (See in particular [5, Section 5] and [6, Section 5].) Dahlberg, She and van Willigenburg posit in [1, Conjecture 41] that the chromatic symmetric function of a tree with a vertex a degree at least four cannot be $e$-positive (that is, a non-negative linear combination of elementary symmetric functions). In the preprint [7], K. Zheng proves a similar but slightly weaker claim: if a tree $T$ has a vertex of degree at least six, then $X_T(x)$ is not $e$-positive. Together, [1, Conjectures 41 and 42] suggest that trees behave differently with respect to Schur positivity than they do with respect to $e$-positivity. (This is in contrast to a conjecture of Stanley and Stembridge found in [5, 6].) Given Zheng’s result and ours, it is natural to ask whether there exists some constant $k$ such that every tree with a vertex of degree at least $k$ cannot have Schur positive symmetric function.

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**References**


