

An update on reconfiguring 10-colorings of planar graphs

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Abstract

The reconfiguration graph $R_k(G)$ for the k -colorings of a graph G has as vertex set the set of all possible proper k -colorings of G and two colorings are adjacent if they differ in the color of exactly one vertex. A result of Bousquet and Perarnau (2016) regarding graphs of bounded degeneracy implies that if G is a planar graph with n vertices, then $R_{12}(G)$ has diameter at most $6n$. We improve on the number of colors, showing that $R_{10}(G)$ has diameter at most $8n$ for every planar graph G with n vertices.

Mathematics Subject Classifications: 005C15

1 Introduction and result

Let G be a graph, and let k be a non-negative integer. A (proper) k -coloring of G is a function $\varphi : V(G) \rightarrow \{1, \dots, k\}$ such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E(G)$. The reconfiguration graph $R_k(G)$ of the k -colorings of G has as vertex set the set of all k -colorings of G , with two colorings adjacent if they differ in the color of exactly one vertex. That is, two k -colorings φ_1 and φ_2 are joined by a path in $R_k(G)$ if and only if we can transform φ_1 into φ_2 by recoloring vertices one by one, always keeping the coloring proper,

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and the number of recolorings needed is equal to the distance between φ_1 and φ_2 in $R_k(G)$. Hence, it is natural to ask how the diameter of $R_k(G)$ depends on k and the number of vertices of G , subject to various conditions ensuring the k -colorability of G .

The study of the reconfiguration graph for colorings was begun by the statistical physics community in the context of Glauber dynamics for random colorings; see for example [14, 18]. It has also recently attracted the attention because of its connections to the existence of FPTAS for the number of colorings, but also for its own structural and computational merit. For example, typical questions include deciding whether two colorings belong to the same component of the reconfiguration graph, or that of determining the diameter of its components. For more details, we refer the reader to the surveys by van den Heuvel [17] and by Nishimura [15].

A graph is k -degenerate if every subgraph of the graph contains a vertex of degree at most k . Clearly, every k -degenerate graph G is $(k + 1)$ -colorable, but $R_{k+1}(G)$ may be disconnected (e.g. in the case $G = K_{k+1}$, but there are many more instances [2]). On the other hand, $R_{k+2}(G)$ is always connected [8]. Cereceda [7] conjectured the following.

Conjecture 1. If G is a k -degenerate graph on n vertices, then $R_{k+2}(G)$ has diameter $O(n^2)$.

This bound would be best possible [3]. Although the conjecture has resisted several efforts, there have been some partial results surrounding it [1, 6, 5, 9, 10, 11, 13]. The most important breakthrough is a theorem of Bousquet and Heinrich [5] where it was shown, among other results, that $R_{k+2}(G)$ has diameter $O(n^{k+1})$. In particular, the conjecture is still open even for $k = 2$.

Bousquet and Perarnau [6] gave the following bound in the situation when the number of colors is substantially larger than $k + 2$.

Theorem 2 (Bousquet and Perarnau [6, Theorem 1]). *If G is a k -degenerate graph on n vertices and $c \geq 2k + 2$, then $R_c(G)$ has diameter at most $(k + 1)n$.*

It was also shown by Bartier and Bousquet [4] that $R_{k+4}(G)$ has diameter $O(n)$ for every k -degenerate chordal graph G of bounded maximum degree. Another result in this direction was obtained by the second author [12] by showing that, for every graph G of maximum average degree at most $k + \epsilon$ ($0 \leq \epsilon < 1$), the reconfiguration graph $R_{k+2}(G)$ has diameter $O(n(\log n)^{k+1})$. In particular, for any fixed k , this diameter is $O(n^2)$, confirming Conjecture 1 for this class of graphs.

Planar graphs are 5-degenerate and have maximum average degree less than 6, and thus the aforementioned results imply that if G is a planar graph with n vertices, then $R_8(G)$ has diameter $O(n(\log n)^7)$ and $R_{12}(G)$ has diameter at most $6n$. This motivates the following question.

Problem 3. What is the minimum integer κ such that for every planar graph G with n vertices, $R_\kappa(G)$ has diameter $O(n)$?

The object of this paper is to show $\kappa \leq 10$, improving on the bound 12 following from Theorem 2.

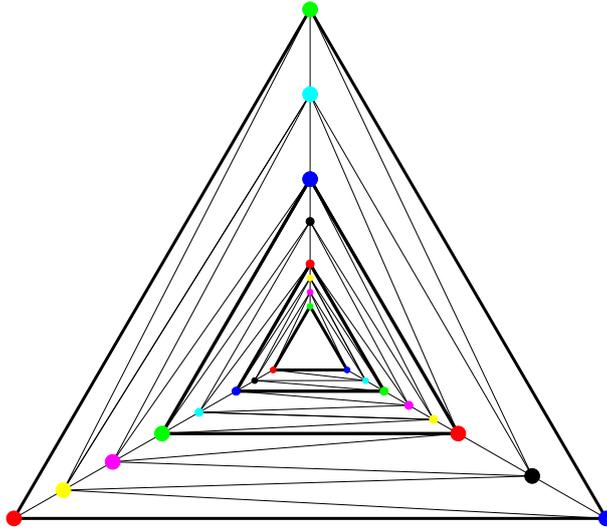


Figure 1: A graph obtained by cutting a toroidal drawing of K_7 and gluing.

Theorem 4. *Let G be a planar graph on n vertices. Then $R_{10}(G)$ has diameter at most $8n$.*

Consider the coloring of the icosahedron graph D where the opposite vertices get the same color. This gives a 6-coloring of D where the closed neighborhood of each vertex contains all 6 colors, and hence this 6-coloring forms an isolated vertex in $R_6(D)$. Consequently, $R_6(G)$ does not even need to be connected for planar graphs, implying $\kappa \geq 7$. However, not much is known about $R_7(G)$ for planar graphs G . The 5-degenerate graphs for which $R_7(G)$ has quadratic diameter constructed in [3] (paths with four apex vertices) are non-planar. A natural candidate for a planar graph G with $R_7(G)$ of quadratic diameter is as follows: Consider an embedding of K_7 on the torus. Cut this drawing along a non-contractible triangle (that is, a triangle not bounding a disk) and glue together many copies of the resulting cylinder (Figure 1 shows the graph obtained from gluing three copies). We obtain a planar graph with a 7-coloring such that the closed neighborhood of all but six vertices contains all 7 colors, so to recolor this graph, one has to “propagate” from the ends of the cylinder. However, this graph G is 3-degenerate and chordal, and thus $R_7(G)$ in fact has linear diameter by the aforementioned result of Bartier and Bousquet [4]. Hence, we cannot exclude the possibility that the answer to Problem 3 is $\kappa = 7$.

2 Outline of the proof

In this section, we lay out our strategy for proving Theorem 4. Let us start off by noting that Theorem 4 will follow as an immediate consequence to the following theorem.

Theorem 5. *Let G be a planar graph. Let α be a 10-coloring of G . Then there exists a sequence of recolorings from α to some 9-coloring of G that recolors every vertex either at most once, to a color distinct from 10, or exactly twice, first to the color 10 and then to a color distinct from 10.*

Theorem 4 follows by a standard argument.

Proof of Theorem 4. Let α and β be 10-colorings of G . To prove the theorem, it suffices to show that we can recolor α to β by at most $8n$ recolorings.

By Theorem 5, we can recolor α to some 9-coloring α_1 of G by at most $2n$ recolorings and β to some 9-coloring β_1 by at most $2n$ recolorings. By [16], there exists a partition of $V(G)$ into an independent set I and a 3-degenerate graph D . From α_1 and β_1 recolor the vertices in I to color 10 (the color that is not used in α_1 and β_1). Let α_2 and β_2 denote the restrictions of α_1 and β_1 to D . Applying Theorem 2, the distance between α_2 and β_2 in $R_9(D)$ is at most $4|V(D)|$, and thus we can recolor α_2 to β_2 by at most $4|V(D)|$ recolorings without using the color 10. This completes the proof. \square

The rest of this paper will be devoted to the proof of Theorem 5. In order to prove the theorem, we must first make a few definitions. A *scene* is a pair (G, α) , where G is a plane graph and α is a 10-coloring of G . We say that a sequence of recolorings from α to some coloring γ of G is *valid* if γ uses only colors $1, \dots, 9$ and every vertex v of G is recolored either at most once (to the color $\gamma(v)$) or exactly twice, first to the color 10 and then to the color $\gamma(v)$. We say that the scene (G, α) is *recolorable* if G admits a valid sequence of recolorings starting from α .

The scene (G, α) is said to be a *minimal counterexample* if (G, α) is not recolorable and all scenes (G', β) such that

- $|V(G')| < |V(G)|$, or
- $|V(G')| = |V(G)|$ and $|E(G')| > |E(G)|$, or
- $G' = G$ and $|\beta^{-1}(10)| > |\alpha^{-1}(10)|$

are recolorable.

Our aim will be to exclude the existence of a minimal counterexample, which will prove Theorem 5. We begin with an easy proposition.

Lemma 6. *If (G, α) is a minimal counterexample, then G is a triangulation and the color 10 appears in the closed neighborhood of every vertex of G under α .*

Proof. Suppose that G is not a triangulation; then for some face f of G , there exist distinct non-adjacent vertices u and v incident with f . If $\alpha(u) \neq \alpha(v)$, we insert the edge uv . If $\alpha(u) = \alpha(v)$ we identify u and v into a new vertex u' . The resulting graph G' is plane and, by minimality, (G', α) is recolorable (we consider α to be a coloring of G' by defining $\alpha(u') = \alpha(u) = \alpha(v)$). As any valid sequence of recolorings in G' easily translates into a valid sequence of recolorings in G , this shows that G must be a triangulation.

Suppose that the color 10 does not appear on some vertex v of G or any of its neighbors. Recolor v to the color 10 and let α' denote the resulting coloring. By minimality, (G, α') is recolorable. It follows, by definition, that (G, α) is recolorable. \square

We now analyze the structure of a minimal counterexample (G, α) by showing that G cannot contain a number of induced subgraphs whose vertices are of prescribed degrees (here and in Section 3). Afterwards, we will show that no such minimal counterexample exists using the discharging method (see Section 4).

Let H be an induced subgraph of G . By the minimality of (G, α) , there exists a valid sequence of recolorings in $G - V(H)$ from the restriction of α to $G - V(H)$ to some coloring γ of $G - V(H)$. Let us define a list assignment L^H for H by setting

$$L^H(v) = \{1, \dots, 9\} \setminus \left(\bigcup_{u \in N_G(v) \setminus V(H)} \{\alpha(u), \gamma(u)\} \right)$$

for each $v \in V(H)$. We say that L^H is an *assignment of available colors* to H in (G, α) ; let us remark that there may be several different assignments of available colors, corresponding to different colorings of $G - V(H)$.

We have the following proposition. A sequence of recolorings of H is said to be a *once-only recoloring* if every vertex of H is recolored at most once. The induced subgraph H of G is said to be *reducible* in (G, α) if for every assignment of available colors L^H to H , there exists a once-only recoloring of H from the restriction of α to some L^H -coloring of H .

Lemma 7. *In a minimal counterexample (G, α) , no induced subgraph of G is reducible.*

Proof. Let H be an induced subgraph of G . By minimality, $G - V(H)$ has a valid sequence of recolorings σ to some coloring γ . Let L^H be the corresponding assignment of available colors to H . Suppose for a contradiction H is reducible. Then there exists a once-only recoloring σ' of H from the restriction of α to some L^H -coloring γ_H of H . But σ' followed by σ is a valid sequence of recolorings in G . Indeed, recoloring of a vertex $v \in V(H)$ according to σ' does not conflict with the colors of its neighbors in $G - V(H)$, since if $u \in V(G) \setminus V(H)$ and $uv \in E(G)$, then $\alpha(u) \notin L^H(v)$. Afterwards, recolorings of $u \in V(G) \setminus V(H)$ do not conflict with the color of its neighbors $v \in V(H)$, since u can only be recolored to 10 or $\gamma(u)$ and neither of these colors belongs to $L^H(v)$. This is a contradiction. \square

It is often convenient to focus just on the sizes of the lists. For a function $s : X \rightarrow \mathbb{N}$ with $V(H) \subseteq X$, we say that a list assignment L for H is an *s-list assignment* if $|L(v)| \geq s(v)$ for every $v \in V(H)$. Let

$$s_G^H(v) = 9 - 2(\deg_G v - \deg_H v)$$

and

$$s_{G,\alpha}^H(v) = 9 - 2(\deg_G v - \deg_H v) + |(N_G(v) \cap \alpha^{-1}(10)) \setminus V(H)|$$

for $v \in V(H)$.

Remark 8. Notice, by definition, that any assignment of available colors to H in (G, α) is an $s_{G, \alpha}^H$ -list assignment, and thus also an s_G^H -list assignment.

A *motif* M consists of a graph H_M , a 10-coloring α_M of H_M , and an assignment L_M of subsets of $\{1, \dots, 9\}$ to vertices of H_M . For an induced subgraph F of H_M , a motif M' is an F -restriction of M if $H_{M'} = F$, $\alpha_{M'}$ is the restriction of α_M to F , and $L_{M'}(v) \subseteq L_M(v)$ for $v \in V(F)$. The motif M is *oo-recolorable (to γ)* if there exists a once-only recoloring of H_M from α_M to an L_M -coloring γ of H_M . For a scene (G, α) and an induced subgraph H of G , we say a motif M is *induced by H* if $H_M = H$ and α_M is the restriction of α to H , and L_H is an $s_{G, \alpha}^H$ -list assignment. We use the following easy consequence of Lemma 7 and Remark 8 to constrain minimal counterexamples.

Lemma 9. *Let (G, α) be a minimal counterexample. If H is an induced subgraph of G , then there exist a motif M induced by H in (G, α) that is not oo-recolorable.*

Proof. Let α_H be the restriction of α to H . By Lemma 7, H is not reducible, and thus for some assignment L^H of available colors to H in (G, α) , there does not exist any once-only recoloring from α_H to an L^H -coloring of H . Let M be the motif with $H_M = H$, $\alpha_M = \alpha_H$, and $L_M = L^H$. Then M is not oo-recolorable, and since L^H is an $s_{G, \alpha}^H$ -list assignment by Remark 8, the motif M is induced by H . \square

In the next section, we show a number of motifs that are oo-recolorable, and thus they cannot be induced in a minimal counterexample. Before we do that, let us point out the aspects of our argument that we consider to be novel: Our original plan was to restrict ourselves to once-only recolorings; this enables us to apply the method of reducible configurations which has not been previously used in the area, since we only need to forbid two colors (the initial and the final color) per neighbor outside of the configuration. A bit of a breakthrough for us then was the seemingly counterintuitive notion of valid sequences of recolorings, where we introduce new vertices of color 10 in order to eventually eliminate the color 10. This enables us to assume that color 10 appears in the closed neighborhood of every vertex, which is extremely useful in proving the reducibility of configurations.

3 Structure of minimal counterexample

In this section, we show in a series of lemmas that if (G, α) is a minimal counterexample, then G has minimum degree at least five and does not contain any of the graphs in Figure 2 as induced subgraphs with the prescribed degrees of vertices. Let us start with a trivial observation.

Observation 10. *Suppose M is a motif. If $|V(H_M)| = 1$ and $|L_M(v)| \geq 1$ for the unique vertex $v \in V(H_M)$, then M is oo-recolorable.*

Corollary 11. *If (G, α) is a minimal counterexample, then G has minimum degree at least five.*

Proof. Consider a vertex $v \in V(G)$. By Lemma 9, there exists a motif M induced by v that is not oo-colorable, and thus $|L_M(v)| = 0$ by Observation 10. But $|L_M(v)| \geq s_G^v(v) = 9 - 2 \deg_G v$, implying $\deg_G v \geq 5$. \square

In order to facilitate the proofs that the graphs in Figure 2 are reducible, we first require a number of auxiliary lemmas. Consider a motif M . For brevity, let $V(M) = V(H_M)$, and for $v \in V(M)$, let $N_M(v) = N_{H_M}(v)$ and $\deg_M v = \deg_{H_M} v$. Let us also define $\deg'_M(v) = \deg_M v - |\alpha^{-1}(10) \cap N_M(v)|$ as the number of neighbors of v in M whose color is not 10. For a vertex $v \in V(M)$, let $M - v$ denote the $(H_M - v)$ restriction of M with L_{M-v} equal to the restriction of L_M to $H_M - v$.

Lemma 12. *Let M be a motif and let v be a vertex of M . If $|L_M(v)| > \deg_M v + \deg'_M v$ and $M - v$ is oo-recolorable, then M is oo-recolorable.*

Proof. By assumptions, $M - v$ is oo-recolorable to some coloring γ , via a sequence σ of recolorings. Since $|L_M(v)| > \deg_M v + \deg'_M v$ and $10 \notin L_M(v)$, there exists a color $c \in L_M(v) \setminus \bigcup_{u \in N_M(v)} \{\alpha(u), \gamma(u)\}$. Hence, we can first recolor v to c and then perform the recolorings according to σ , showing that M is oo-recolorable. \square

Similarly, we obtain the following observation.

Lemma 13. *Let M be a motif and let v be a vertex of M . If $\alpha_M(v) = 10$ and $|L_M(v)| > \deg_M v$ and $M - v$ is oo-recolorable, then M is oo-recolorable.*

Proof. By assumptions, $M - v$ is oo-recolorable to some coloring γ , via a sequence σ of recolorings. We can first perform the recolorings σ in M , as they do not conflict with the color 10 of v . Finally, we can recolor v to a color in $L_M(v) \setminus \bigcup_{u \in N_M(v)} \{\gamma(u)\}$, which exists since $|L(v)| > \deg_M v$. This shows M is oo-recolorable. \square

For a motif M , a vertex $v \in V(H_M)$, and a color c , let $M - (v \rightarrow c)$ denote the $(H_M - v)$ -restriction of M with $L_{M-(v \rightarrow c)}(u)$ equal to $L_M(u) \setminus c$ for $u \in N_M(v)$ and to $L_M(u)$ for all other vertices.

Lemma 14. *Let M be a motif, let v be a vertex of M , and consider any color $c \in L_M(v) \setminus \bigcup_{u \in N_M(v)} \{\alpha(u)\}$. If the motif $M - (v \rightarrow c)$ is oo-recolorable, then M is oo-recolorable.*

Proof. By assumptions, $M - (v \rightarrow c)$ is oo-recolorable via a sequence σ of recolorings. We can first recolor v to c (since no neighbor of v has color c) and then perform the recolorings σ in M . For a neighbor u of v , the recoloring of u according to σ does not conflict with the color c , since $c \notin L_{M-(v \rightarrow c)}(u)$. This shows M is oo-recolorable. \square

In particular, repeatedly applying Lemma 14 until a motif with single vertex is obtained and using Observation 10, we have the following consequence.

Corollary 15. *Let M be a motif. If $|L_M(v)| > \deg_M v$ for every $v \in V(M)$, then M is oo-recolorable.*

For a motif M , a vertex $v \in V(M)$, and a color $c \in L_M(v)$, let $M \ominus (v \rightarrow c)$ denote the $(H_M - v)$ -restriction of M with $L_{M \ominus (v \rightarrow c)}(u)$ equal to $L_M(u) \setminus (\{\alpha_M(v), c\})$ for $u \in N_M(v)$ and to $L_M(u)$ for all other vertices. In case that $|L_M(v)| = 1$, we write $M \ominus v$ for brevity, since the color c is uniquely determined in this case. Let us also remark that in case that $\alpha_M(v) = 10$, we have $M \ominus (v \rightarrow c) = M - (v \rightarrow c)$.

Lemma 16. *Let M be a motif, let v be a vertex of M , and consider any color $c \in L_M(v)$. If the motif $M \ominus (v \rightarrow c)$ is oo-recolorable, then M is oo-recolorable.*

Proof. By assumptions, $M \ominus (v \rightarrow c)$ is oo-recolorable via a sequence σ of recolorings. This sequence of recolorings can also be performed in M , since no neighbor of v can be assigned the color $\alpha_M(v)$. Finally, we can recolor v to c , since no neighbor may end up with the color c . This shows M is oo-recolorable. \square

We will generally repeatedly use the preceding claims to simplify the motif obtained by Lemma 9, often to one contradicting Corollary 15. For brevity, let us introduce a notation for this kind of arguments. Suppose $m \geq |V(M)|$ is a positive integer and $V(M) = \{v_i : i \in I\}$ for some $I \subseteq \{1, \dots, m\}$. A vector (s_1, \dots, s_m) describes M if s_i is an integer smaller or equal to $|L(v_i)|$ for $i \in I$ and $s_i = \bullet$ for $i \in \{1, \dots, m\} \setminus I$. Furthermore, a segment of this vector can be enclosed in square brackets; this indicates that there exists an index i in this segment such that $\alpha_M(v_i) = 10$. By $M \sim (s_1, \dots, \underline{s_i}, \dots, s_m) \xrightarrow{L^n} (s'_1, \dots, s'_m) \sim M'$, we mean the following: The motif M is described by the vector (s_1, \dots, s_m) , and applying Lemma n with $v = v_i$, we obtain a motif M' described by (s'_1, \dots, s'_m) , such that if M is not oo-colorable, then M' also is not oo-colorable. In case Lemma 14 or Lemma 16 with more than one color choice is applied, we also specify the color c over the arrow. In case the resulting motif M' is not further discussed (e.g., a contradiction with Corollary 15 is obtained), the $\sim M'$ part is omitted. We can also chain several such statements in the natural way. In all the arguments, we without loss of generality assume that $|L(v_i)| = s_i$, implicitly removing extra colors from the lists if needed.

Recall that by Lemma 6, the color 10 appears in the closed neighborhood of every vertex of a minimal counterexample. The proofs of Lemmas 17–23 below mainly rely on Observation 10 and Corollary 15 (which, in turn, relies on Lemma 14).

Lemma 17. *Let (G, α) be a minimal counterexample and let v_1 and v_2 be adjacent vertices of G . If $\deg v_1 = \deg v_2 = 5$, then either $\alpha(v_1) = 10$ or $\alpha(v_2) = 10$.*

Proof. By Lemma 9, there exist a motif M induced by $H = G[\{v_1, v_2\}]$ in (G, α) that is not oo-recolorable. If neither u nor v has color 10, then since the color 10 appears in the closed neighborhood of every vertex, we have $s_{G, \alpha}^H(u) \geq 2$ and $s_{G, \alpha}^H(v) \geq 2$. However, this contradicts Corollary 15. \square

We also need the following three easy observations.

Lemma 18. *Let M be a motif such that H_M is an edge with vertices v_1 and v_2 . If M is described by $(2, 1)$, then M is oo-recolorable unless $\alpha_M^{-1}(10) \cap V(M) = \emptyset$, $L_M(v_1) = \{\alpha_M(v_1), \alpha_M(v_2)\}$ and $L_M(v_2) = \{\alpha_M(v_1)\}$.*

Proof. Suppose that M is not oo-recolorable. If there exists a color $c_2 \in L_M(v_2) \setminus \{\alpha_M(v_1)\}$, then $M \sim (2, \underline{1}) \xrightarrow{L14, c_2} (1, \bullet)$, contradicting Observation 10. It follows that $L_M(v_2) = \{\alpha_M(v_1)\}$. Hence, if there exists a color $c_1 \in L_M(v_1) \setminus \{\alpha_M(v_1), \alpha_M(v_2)\}$, then $M \sim (2, 1) \xrightarrow{L14, c_1} (\bullet, 1)$, contradicting Observation 10. Therefore, we have $L_M(v_1) = \{\alpha_M(v_1), \alpha_M(v_2)\}$, and in particular $\alpha_M^{-1}(10) \cap V(M) = \emptyset$. \square

Lemma 19. *Let M be a motif such that H_M is a path $v_1v_2v_3$. If M is described by $(2, 2, 2)$ and $\alpha^{-1}(10) \cap V(H_M) \neq \emptyset$, then M is oo-recolorable.*

Proof. Suppose for a contradiction M is not oo-recolorable and that $\alpha^{-1}(10) \cap V(H_M) \neq \emptyset$. If $\alpha(v_1) = 10$, then $M \sim ([\underline{2}], 2, 2) \xrightarrow{L13} (\bullet, 2, 2)$, contradicting Corollary 15. It follows by symmetry that $\alpha(v_2) = 10$; but then $M \sim (\underline{2}, [\underline{2}], \underline{2}) \xrightarrow{L12} (\bullet, [\underline{2}], \bullet)$, contradicting Observation 10. \square

Lemma 20. *Let M be a motif such that H_M is a path $v_1v_2v_3$. If M is described by $(1, 4, 1)$, then M is oo-recolorable.*

Proof. Suppose M is not oo-recolorable. If there exists a color $c_1 \in L_M(v_1) \setminus \{\alpha_M(v_2)\}$, then $M \sim (\underline{1}, 4, 1) \xrightarrow{L14, c_1} (\bullet, \underline{3}, 1) \xrightarrow{L12} (\bullet, \bullet, 1)$, contradicting Observation 10. So we can assume by symmetry that $L_M(v_1) = L_M(v_3) = \{\alpha_M(v_2)\}$. But then for $c_2 \in L_M(v_2) \setminus \{\alpha_M(v_1), \alpha_M(v_2), \alpha_M(v_3)\}$, we have $M \sim (1, \underline{4}, 1) \xrightarrow{L14, c_2} (1, \bullet, 1)$, contradicting Corollary 15. \square

We now make two observations about triangles in a minimal counterexample.

Lemma 21. *Let (G, α) be a minimal counterexample. If G contains a triangle T with vertices v_1, v_2 , and v_3 such that v_1 has degree five and v_2 and v_3 have degree at most six, then $\alpha^{-1}(10) \cap V(T) \neq \emptyset$.*

Proof. By Lemma 9, there exists a motif M induced by T in (G, α) that is not oo-recolorable. Suppose for a contradiction that no vertex of T has color 10. Since the color 10 appears in the closed neighborhood of every vertex, we have $s_{G, \alpha}^T(v_1) \geq 4$ and $s_{G, \alpha}^T(v_2), s_{G, \alpha}^T(v_3) \geq 2$. If there existed a color $c \in L_M(v_2) \setminus \{\alpha(v_1), \alpha(v_3)\}$, we would have $M \sim (4, \underline{2}, 2) \xrightarrow{L14, c} (\underline{3}, \bullet, 1) \xrightarrow{L12} (\bullet, \bullet, 1)$, contradicting Observation 10. Therefore, $L_M(v_2) = \{\alpha(v_1), \alpha(v_3)\}$, and by symmetry, $L_M(v_3) = \{\alpha(v_1), \alpha(v_2)\}$. Then, letting c' be a color in $L_M(v_1) \setminus \{\alpha(v_1), \alpha(v_2), \alpha(v_3)\}$, we have $M \sim (\underline{4}, 2, 2) \xrightarrow{L14, c'} (\bullet, 2, 2)$, contradicting Corollary 15. \square

Lemma 22. *Let M be a motif such that H_M is a triangle with vertices v_1, v_2 , and v_3 . If M is described by $(4, 3, 1)$, then M is oo-recolorable, and if M is described by $(3, 3, 1)$ or $(3, 3, 2)$, then M is oo-recolorable unless $\alpha_M^{-1}(10) = \emptyset$ and $L_M(v_1) = L_M(v_2) = \{\alpha_M(v_1), \alpha_M(v_2), \alpha_M(v_3)\}$ and $L_M(v_3) \subseteq \{\alpha_M(v_1), \alpha_M(v_2)\}$.*

Proof. Suppose first that M is described by $(3, 3, 1)$ or $(3, 3, 2)$, and that M is not oo-recolorable. If there exists $c_3 \in L_M(v_3) \setminus \{\alpha_M(v_1), \alpha_M(v_2)\}$, then $M \sim (3, 3, \underline{1}) \xrightarrow{L14, c_3} (2, 2, \bullet)$, contradicting Corollary 15. Hence, we have $L_M(v_3) \subseteq \{\alpha_M(v_1), \alpha_M(v_2)\}$, and by symmetry we can assume $\alpha_M(v_1) \in L_M(v_3)$. If there exists a color $c_1 \in L_M(v_1) \setminus \{\alpha_M(v_1), \alpha_M(v_2), \alpha_M(v_3)\}$, then we can first recolor v_1 by c_1 , then v_3 by $\alpha_M(v_1)$ and finally v_2 by a color in $L_M(v_2) \setminus \{\alpha_M(v_1), c_1\}$, showing that M is oo-recolorable, a contradiction. Therefore $L_M(v_1) = \{\alpha_M(v_1), \alpha_M(v_2), \alpha_M(v_3)\}$, and in particular $\alpha_M^{-1}(10) = \emptyset$. If $L_M(v_2) \neq L_M(v_1)$, then there would exist $c_2 \in L_M(v_2) \setminus \{\alpha_M(v_1), \alpha_M(v_2), \alpha_M(v_3)\}$, and $M \sim (3, \underline{3}, 1) \xrightarrow{L14, c_2} (3, \bullet, 1) \xrightarrow{L12} (\bullet, \bullet, 1)$, contradicting Observation 10. This gives the characterization of non-oo-recolorable motifs described by $(3, 3, 1)$ or $(3, 3, 2)$.

Suppose now M is described by $(4, 3, 1)$; then we can delete a color from $L_M(v_1)$ to obtain a motif M' described by $(3, 3, 1)$, but with $L_{M'}(v_1) \neq L_{M'}(v_2)$. The motif M' is oo-recolorable by the previous paragraph, and thus M is oo-recolorable as well. \square

We also require the following observation on diamonds in a minimal counterexample.

Lemma 23. *Let (G, α) be a minimal counterexample. Let v_1, \dots, v_4 be distinct vertices of G such that the subgraph F of G induced by $\{v_1, v_2, v_3, v_4\}$ contains all possible edges except for v_2v_4 . If $\deg v_1 \leq 7$, $\deg v_2 \leq 5$ and $\deg v_3, \deg v_4 \leq 6$, then $\alpha^{-1}(10) \cap V(F) \neq \emptyset$.*

Proof. By Lemma 9, there exists a motif M induced by F in (G, α) that is not oo-recolorable. Suppose for a contradiction no vertex of F has color 10. Since the color 10 appears in the closed neighborhood of every vertex, M is described by $(2, 4, 4, 2)$. If there exists a color $c_4 \in L_M(v_4) \setminus \{\alpha(v_1), \alpha(v_3)\}$, then $M \sim (2, 4, 4, \underline{2}) \xrightarrow{L14, c_4} (1, 4, 3, \bullet)$, contradicting Lemma 22. Therefore $L_M(v_4) = \{\alpha(v_1), \alpha(v_3)\}$. If there exists a color $c_1 \in L_M(v_1) \setminus \{\alpha(v_2), \alpha(v_3), \alpha(v_4)\}$, then $M \sim (\underline{2}, 4, 4, 2) \xrightarrow{L14, c_1} (\bullet, \underline{3}, 3, 1) \xrightarrow{L12} (\bullet, \bullet, \underline{3}, 1) \xrightarrow{L12} (\bullet, \bullet, \bullet, 1)$, contradicting Observation 10. Hence, $L_M(v_1) \subseteq \{\alpha(v_2), \alpha(v_3), \alpha(v_4)\}$. If there exists a color $c_3 \in L_M(v_3) \setminus \{\alpha(v_1), \dots, \alpha(v_4)\}$, then $M \sim (2, 4, \underline{4}, 2) \xrightarrow{L14, c_3} (2, \underline{3}, \bullet, 2) \xrightarrow{L12} (2, \bullet, \bullet, 2)$, contradicting Corollary 15. Therefore, $L_M(v_3) = \{\alpha(v_1), \dots, \alpha(v_4)\}$, and in particular $\alpha(v_2) \neq \alpha(v_4)$. Choose a color $c_2 \in L_M(v_2) \setminus \{\alpha(v_1), \alpha(v_2), \alpha(v_3)\}$.

- If $\alpha(v_2) \in L_M(v_1)$, we first recolor v_2 to c_2 , then v_1 to $\alpha(v_2)$, and finally v_4 to $\alpha(v_1)$.
- Otherwise, $L_M(v_1) = \{\alpha(v_3), \alpha(v_4)\}$. We first recolor v_2 to c_2 , then v_3 to $\alpha(v_2)$, then v_1 to $\alpha(v_3)$, and finally v_4 to $\alpha(v_1)$. \square

We are now ready to demonstrate that the graphs in Figure 2 are reducible.

Lemma 24. *If (G, α) is a minimal counterexample, then G contains none of the induced subgraphs with prescribed vertex degrees depicted in Figure 2.*

Proof. Suppose for a contradiction C is one of the graphs depicted in Figure 2 and contained in G as an induced subgraph with the prescribed degrees of vertices. By Lemma 9, there exist a motif M induced by C in (G, α) that is not oo-recolorable. We prove that each of the cases are reducible separately, starting with C_1 and working our way towards C_{16} . We fix the labeling of vertices as indicated in Figure 2.

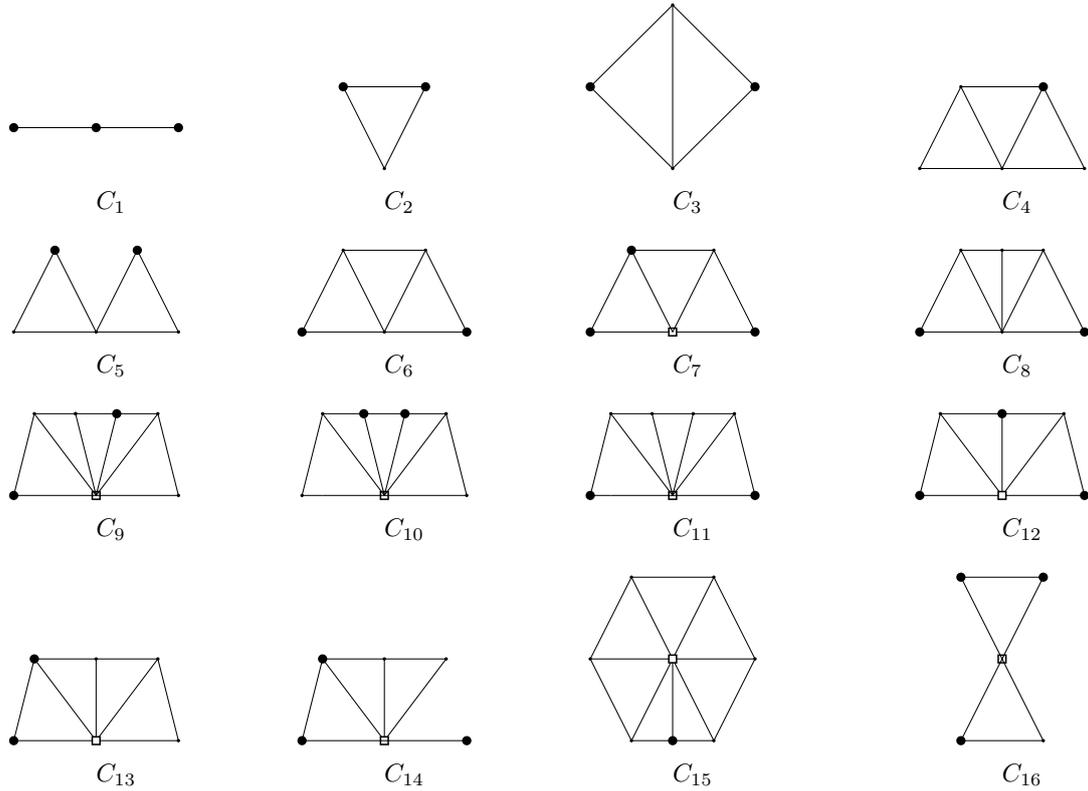


Figure 2: Reducible induced subgraphs, where \square denotes a vertex of degree at most seven, \bullet denotes a vertex of degree five and \cdot denotes a vertex of degree at most six.

(C1) By Lemma 17, either $\alpha(v_1) = \alpha(v_3) = 10$, or $\alpha(v_2) = 10$. In the former case, $M \sim ([1], \underline{3}, [1]) \xrightarrow{L12} (1, \bullet, 1)$, contradicting Corollary 15. In the latter case, $M \sim (1, [\underline{3}], 1) \xrightarrow{L13} (1, \bullet, 1)$, again contradicting Corollary 15.

(C2) By Lemma 17 and symmetry, we can assume $\alpha(v_3) = 10$. But then

$$M \sim (1, 3, [\underline{3}]) \xrightarrow{L13} (1, \underline{3}, \bullet) \xrightarrow{L12} (1, \bullet, \bullet),$$

contradicting Observation 10.

(C3) If $\alpha(v_2) = \alpha(v_4) = 10$, then $M \sim (3, [\underline{3}], 3, [\underline{3}]) \xrightarrow{L13} (3, \bullet, 3, \bullet)$, contradicting Corollary 15. Hence, by Lemma 21 and symmetry, we can assume $\alpha(v_3) = 10$. Now, there exists a color $c \in L_M(v_1) \setminus \{\alpha(v_2), \alpha(v_4)\}$. However, then $M \sim (\underline{3}, 3, [3], 3) \xrightarrow{L14,c} (\bullet, 2, [2], 2)$, contradicting Lemma 19.

(C4) If $\alpha(v_1) = 10$, then $M \sim ([\underline{5}], 1, 3, 5, 1) \xrightarrow{L13} (\bullet, 1, 3, \underline{5}, 1) \xrightarrow{L12} (\bullet, 1, \underline{3}, \bullet, 1) \xrightarrow{L12} (\bullet, 1, \bullet, \bullet, 1)$, contradicting Corollary 15. If $\alpha(v_3) = \alpha(v_5) = 10$, then

$$M \sim (5, 1, [3], \underline{5}, [1]) \xrightarrow{L12} (\underline{5}, 1, [3], \bullet, [1]) \xrightarrow{L12} (\bullet, 1, [\underline{3}], \bullet, 1) \xrightarrow{L13} (\bullet, 1, \bullet, \bullet, 1),$$

contradicting Corollary 15. Hence, Lemma 21 implies $\alpha(v_4) = 10$, and thus $M \sim (5, 1, 3, \underline{5}, 1) \xrightarrow{L13} (5, 1, 3, \bullet, 1) \sim M'$. Let $\{c_5\} = L_{M'}(v_5)$. We have $M' \sim (5, 1, 3, \bullet, \underline{1}) \xrightarrow{L16} (3, 1, 3, \bullet, \bullet)$, and thus Lemma 22 implies that $L_M(v_1)$ is the disjoint union of $L_M(v_3) = \{\alpha(v_1), \alpha(v_2), \alpha(v_3)\}$ and $\{\alpha(v_5), c_5\}$. In particular, $c_5 \neq \alpha(v_1)$, and thus $M' \sim (5, 1, 3, \bullet, \underline{1}) \xrightarrow{L14, c_5} (4, 1, 3, \bullet, \bullet)$, contradicting Lemma 22

(C5) If $\alpha(v_1) = 10$, then $M \sim (\underline{5}, 1, 3, 3, 1) \xrightarrow{L13} (\bullet, 1, \underline{3}, \underline{3}, 1) \xrightarrow{L12} (\bullet, 1, \bullet, \bullet, 1)$, contradicting Corollary 15. If $\alpha(v_3) = \alpha(v_4) = 10$, then $M \sim (5, 1, \underline{3}, \underline{3}, 1) \xrightarrow{L13} (\underline{5}, 1, \bullet, \bullet, 1) \xrightarrow{L12} (\bullet, 1, \bullet, \bullet, 1)$, again contradicting Corollary 15. If $\alpha(v_3) = \alpha(v_5) = 10$, then $M \sim (5, 1, \underline{3}, 3, [1]) \xrightarrow{L13} (5, \underline{1}, \bullet, 3, [1]) \xrightarrow{L16} (3, \bullet, \bullet, 3, [1])$, contradicting Lemma 22. Hence, by Lemma 21 and symmetry, we can assume $\alpha(v_2) = \alpha(v_5) = 10$. But then, for any $c_3 \in L_M(v_3) \setminus (L_M(v_2) \cup \{\alpha(v_1)\})$ and $c_4 \in L_M(v_4) \setminus (L_M(v_3) \cup \{\alpha(v_1)\})$, $M \sim (5, [1], \underline{3}, 3, [1]) \xrightarrow{L14, c_3} (4, [1], \bullet, \underline{3}, [1]) \xrightarrow{L14, c_4} (\underline{3}, [1], \bullet, \bullet, [1]) \xrightarrow{L12} (\bullet, [1], \bullet, \bullet, [1])$, contradicting Corollary 15.

(C6) If $\alpha(v_1) = 10$, then $M \sim (\underline{5}, 3, 3, 3, 3) \xrightarrow{L13} (\bullet, 3, 3, 3, 3)$, contradicting Corollary 15. Note that at most one of the adjacent vertices v_3 and v_4 can have color 10. Hence, by Lemma 21 and symmetry, we can assume $\alpha(v_2) = 10$. But then, for any $c_3 \in L_M(v_3) \setminus \{\alpha(v_1), \alpha(v_4)\}$, $M \sim (5, \underline{3}, 3, 3, 3) \xrightarrow{L13} (5, \bullet, \underline{3}, 3, 3) \xrightarrow{L14, c_3} (4, \bullet, \bullet, 2, 3)$, contradicting Lemma 22.

(C7) If $\alpha(v_2) = 10$, then $M \sim (3, \underline{3}, 5, 3, 3) \xrightarrow{L13} (3, \bullet, \underline{5}, 3, 3) \xrightarrow{L12} (3, \bullet, \bullet, 3, 3)$, which contradicts Corollary 15. Hence, by Lemma 17 we have $\alpha(v_3) = 10$. It follows that $M \sim (3, 3, \underline{5}, 3, 3) \xrightarrow{L13} (3, \underline{3}, \bullet, 3, 3) \xrightarrow{L12} (3, \bullet, \bullet, 3, 3)$, which again contradicts Corollary 15.

(C8) If $\alpha(v_1) = 10$, then

$$M \sim (\underline{7}, 3, 3, 3, 3, 3) \xrightarrow{L13} (\bullet, 3, 3, 3, 3, 3),$$

which contradicts Corollary 15. If $\alpha(v_2) = \alpha(v_6) = 10$, then

$$M \sim (7, \underline{3}, 3, 3, 3, \underline{3}) \xrightarrow{L13} (\underline{7}, \bullet, 3, 3, 3, \bullet) \xrightarrow{L12} (\bullet, \bullet, 3, 3, 3, \bullet),$$

again contradicting Corollary 15. If $\alpha(v_3) = \alpha(v_5) = 10$, choose $c_1 \in L_M(v_1) \setminus \{\alpha(v_2), \alpha(v_4), \alpha(v_6)\}$; we have

$$M \sim (\underline{7}, 3, \underline{3}, 3, \underline{3}, 3) \xrightarrow{L14, c_1} (\bullet, \underline{2}, [2], 2, [2], \underline{2}) \xrightarrow{L12} (\bullet, \bullet, [2], 2, [2], \bullet),$$

contradicting Lemma 19.

Hence, by Lemma 21 and symmetry, we can assume $\alpha(v_2) = \alpha(v_5) = 10$. For $c_3 \in L_M(v_3) \setminus \{\alpha(v_1), \alpha(v_4)\}$, we have $M \sim (7, \underline{3}, 3, 3, \underline{3}, 3) \xrightarrow{L13} (7, \bullet, \underline{3}, 3, \underline{3}, 3) \xrightarrow{L14, c_3} (\underline{6}, \bullet, \bullet, 2, [3], 3) \xrightarrow{L12} (\bullet, \bullet, \bullet, 2, [3], 3)$, contradicting Corollary 15.

(C9) If $\alpha(v_1) = 10$, then we have $M \sim ([\underline{7}], 3, 3, 3, 5, 3, 1) \xrightarrow{L13} (\bullet, 3, 3, 3, \underline{5}, 3, 1) \xrightarrow{L12} (\bullet, 3, 3, 3, \bullet, \underline{3}, 1) \xrightarrow{L12} (\bullet, 3, 3, 3, \bullet, \bullet, 1)$, which contradicts Corollary 15. Therefore by Lemma 23 we can assume that at least one of v_2, v_3, v_4 has color 10 and at least one of v_5, v_6, v_7 has color 10. Choose a color $c_6 \in L_M(v_6) \setminus \{\alpha(v_6), \alpha(v_7)\}$, let $L_M(v_7) = \{c_7\}$, and choose a color $c_1 \in L_M(v_1) \setminus (\{c_6, c_7\} \cup \bigcup_{i=2}^7 \{\alpha(v_i)\})$. Then $M \sim ([\underline{7}], [3, 3, 3], [5, 3, 1]) \xrightarrow{L14, c_1} (\bullet, [2, 2, 2], [4, 2, 1]) \sim M'$, where $L_{M'}(v_6) \neq \{\alpha(v_6), \alpha(v_7)\}$.

If $\alpha(v_5) = 10$, then we can continue with

$$M' \sim (\bullet, [2, 2, 2], [\underline{4}], 2, 1) \xrightarrow{L13} (\bullet, [2, 2, 2], \bullet, \underline{2}, 1) \xrightarrow{L18} (\bullet, [2, 2, 2], \bullet, \bullet, \bullet),$$

which contradicts Lemma 19. If $\alpha(v_6) = 10$, then $M' \sim (\bullet, [2, 2, 2], 4, [2], \underline{1}) \xrightarrow{L14} (\bullet, [2, 2, 2], 4, [\underline{1}], \bullet) \xrightarrow{L16} (\bullet, [2, 2, 2], \underline{3}, \bullet, \bullet) \xrightarrow{L12} (\bullet, [2, 2, 2], \bullet, \bullet, \bullet)$, which again contradicts Lemma 19. Finally, suppose $\alpha(v_7) = 10$. Then

$$M' \sim (\bullet, [2, 2, 2], 4, 2, [\underline{1}]) \xrightarrow{L16} (\bullet, [2, 2, 2], 4, 1, \bullet) \sim M^*.$$

If $\alpha(v_4) = 10$, then

$$M^* \sim (\bullet, 2, 2, [2], \underline{4}, 1, \bullet) \xrightarrow{L12} (\bullet, 2, 2, [\underline{2}], \bullet, 1, \bullet) \xrightarrow{L19} (\bullet, \bullet, \bullet, \bullet, \bullet, 1, \bullet),$$

which contradicts Observation 10. If $\alpha(v_3) = 10$, then $M^* \sim (\bullet, \underline{2}, [2], 2, 4, 1, \bullet) \xrightarrow{L12} (\bullet, \bullet, [\underline{2}], 2, 4, 1) \xrightarrow{L13} (\bullet, \bullet, \bullet, 2, 4, 1)$, which contradicts Lemma 20. If $\alpha(v_2) = 10$, then $M^* \sim (\bullet, [\underline{2}], 2, 2, 4, 1, \bullet) \xrightarrow{L13} (\bullet, \bullet, \underline{2}, 2, 4, 1) \xrightarrow{L14} (\bullet, \bullet, \bullet, 1, 4, 1)$, which again contradicts Lemma 20.

(C10) By Lemma 17 and symmetry, we can assume v_5 has color 10. By Lemma 23, it follows that v_2 or v_3 has color 10. We have $M \sim (7, [1, 3], 5, [\underline{5}], 3, 1) \xrightarrow{L13} (7, [1, 3], \underline{5}, \bullet, 3, 1) \xrightarrow{L12} (7, [1, 3], \bullet, \bullet, 3, 1) \sim M'$. Let $L_{M'}(v_i) = \{c_i\}$ for $i \in \{2, 7\}$ and choose $c_6 \in L_{M'}(v_6) \setminus \{\alpha(v_6), \alpha(v_7)\}$. Then there exists a color $c_1 \in L_{M'}(v_1) \setminus \{c_2, c_6, c_7, \alpha(v_2), \alpha(v_3), \alpha(v_6), \alpha(v_7)\}$, and

$$M' \sim ([\underline{7}], [1, 3], \bullet, \bullet, 3, 1) \xrightarrow{L14, c_1} (\bullet, [1, 2], \bullet, \bullet, 2, 1) \sim M^*,$$

where $L_{M^*}(v_6) \neq \{\alpha(v_6), \alpha(v_7)\}$. This contradicts Lemma 18.

(C11) If $\alpha(v_1) = 10$, then $M \sim ([\underline{7}], 3, 3, 3, 3, 3, 3) \xrightarrow{L13} (\bullet, 3, 3, 3, 3, 3, 3)$, contradicting Corollary 15. Lemma 23 thus implies $\alpha^{-1}(10) \cap \{v_2, v_3, v_4\} \neq \emptyset$ and $\alpha^{-1}(10) \cap \{v_5, v_6, v_7\} \neq \emptyset$. Now,

- for each $i \in \{2, 7\}$, if $\alpha(v_i) = 10$, apply Lemma 13 to v_i , and

- for all $i \in \{2, \dots, 7\}$, if $\alpha(v_i) \neq 10$ and $\deg'_M(v_i) \leq 2$, then choose a color $c_i \in L_M(v_i) \setminus \alpha(N_M(v_i))$ and apply Lemma 14.

Let M' denote the resulting motif. Note that one of the two cases always applies for $i = 2$ and $i = 7$, and thus $v_2, v_7 \notin V(M')$.

Suppose that for some $i \in \{3, 4\}$, we have $v_i \in V(M')$ and $\alpha(v_i) \neq 10$. By the construction of M' , we have $\deg'_M(v_i) = 3$, and thus $\alpha(v_{i-1}) \neq 10 \neq \alpha(v_{i+1})$. Since $\alpha^{-1}(10) \cap \{v_2, v_3, v_4\} \neq \emptyset$, it follows that $i = 4$ and $\alpha(v_2) = 10$. Hence, the number of vertices $v_i \in V(M')$ such that $i \in \{3, 4\}$ and $\alpha(v_i) \neq 10$ is at most $|\alpha^{-1}(10) \cap \{v_2\}|$. Symmetrically, the number of vertices $v_i \in V(M')$ such that $i \in \{5, 6\}$ and $\alpha(v_i) \neq 10$ is at most $|\alpha^{-1}(10) \cap \{v_7\}|$.

Therefore, $\deg'_M(v_1) \leq |\alpha^{-1}(10) \cap \{v_2, v_7\}|$. On the other hand, we have $|L_M(v_1)| = 7 = |V(M)|$. Note that during the construction of M' , the colors are removed from the list of v_1 only in the second case; hence,

$$\begin{aligned} |L_{M'}(v_1)| &\geq |L_M(v_1)| - (|V(M)| - |V(M')|) + |\alpha^{-1}(10) \cap \{v_2, v_7\}| \\ &= |V(M')| + |\alpha^{-1}(10) \cap \{v_2, v_7\}| > \deg_{M'}(v_1) + \deg'_M(v_1). \end{aligned}$$

Therefore, M' is oo-recolorable by Lemma 12 applied to v_1 and by Corollary 15. This is a contradiction.

(C12) If $\alpha(v_1) = 10$, let $c_1 \in L_M(v_1)$. We have

$$M \sim ([\underline{5}], 3, 3, 5, 3, 3) \xrightarrow{L16, c_1} (\bullet, \underline{2}, 2, 4, 2, \underline{2}) \xrightarrow{L14} (\bullet, \bullet, 1, 4, 1, \bullet),$$

contradicting Lemma 20. If $\alpha(v_4) = 10$, we have

$$M \sim (5, 3, 3, [\underline{5}], 3, 3) \xrightarrow{L13} (\underline{5}, 3, 3, \bullet, 3, 3) \xrightarrow{L14} (\bullet, 2, 2, \bullet, 2, 2),$$

which contradicts Corollary 15. If $\alpha(v_3) = 10$, then $M \sim (\underline{5}, 3, [3], 5, 3, 3) \xrightarrow{L14} (\bullet, 2, [2], \underline{4}, 2, 2) \xrightarrow{L12} (\bullet, 2, [2], \bullet, 2, 2)$, contradicting Corollary 15. The case $\alpha(v_5) = 10$ is symmetric. Therefore, Lemma 23 implies $\alpha(v_2) = \alpha(v_6) = 10$, and thus $M \sim (5, [3], 3, 5, 3, [3]) \xrightarrow{L13} (5, \bullet, 3, 5, 3, \bullet)$, contradicting Corollary 15.

(C13) By Lemma 17, either $\alpha(v_2) = 10$ or $\alpha(v_3) = 10$, and thus either

$$M \sim (5, [3], 5, 3, 3, 1) \xrightarrow{L13} \sim (5, \bullet, \underline{5}, 3, 3, 1) \xrightarrow{L12} (5, \bullet, \bullet, 3, 3, 1) \sim M',$$

or

$$M \sim (5, 3, [\underline{5}], 3, 3, 1) \xrightarrow{L13} \sim (5, \underline{3}, \bullet, 3, 3, 1) \xrightarrow{L12} (5, \bullet, \bullet, 3, 3, 1) \sim M'.$$

Let $\{c_6\} = L_{M'}(v_6)$; we have $M' \sim (5, \bullet, \bullet, 3, 3, \underline{1}) \xrightarrow{L16, c_6} (3, \bullet, \bullet, 3, 1, \bullet) \sim M^*$, and by Lemma 22, we have $L_{M^*}(v_1) = L_{M'}(v_4) = \{\alpha(v_1), \alpha(v_4), \alpha(v_5)\}$. Consequently, $L_{M'}(v_1) = \{\alpha(v_1), \alpha(v_4), \alpha(v_5), \alpha(v_6), c_6\}$, and in particular $c_6 \notin \{\alpha(v_1), \alpha(v_5)\}$. Therefore $M' \sim (5, \bullet, \bullet, 3, 3, \underline{1}) \xrightarrow{L14, c_6} (4, \bullet, \bullet, 3, 2, \bullet)$, contradicting Lemma 22.

(C14) By Lemma 17, either $\alpha(v_2) = 10$ or $\alpha(v_3) = 10$, and thus either

$$M \sim (5, [3]_1, 5, 3, 1, 1) \xrightarrow{L13} \sim (5, \bullet, \underline{5}, 3, 1, 1) \xrightarrow{L12} (5, \bullet, \bullet, 3, 1, 1) \sim M',$$

or

$$M \sim (5, 3, [5], 3, 1, 1) \xrightarrow{L13} \sim (5, \underline{3}, \bullet, 3, 1, 1) \xrightarrow{L12} (5, \bullet, \bullet, 3, 1, 1) \sim M'.$$

Let $\{c_6\} = L_{M'}(v_6)$; we have $M' \sim (5, \bullet, \bullet, 3, 1, \underline{1}) \xrightarrow{L16, c_6} (3, \bullet, \bullet, 3, 1, \bullet) \sim M^*$, and by Lemma 22, we have $L_{M^*}(v_1) = L_{M'}(v_4) = \{\alpha(v_1), \alpha(v_4), \alpha(v_5)\}$. Consequently, $L_{M'}(v_1) = \{\alpha(v_1), \alpha(v_4), \alpha(v_5), \alpha(v_6), c_6\}$, and in particular $c_6 \neq \alpha(v_1)$. Therefore $M' \sim (5, \bullet, \bullet, 3, 1, \underline{1}) \xrightarrow{L14, c_6} (4, \bullet, \bullet, 3, 1, \bullet)$, contradicting Lemma 22.

(C15) In this case M is described by $(9, 3, 3, 3, 3, 3, 3, 3)$. Repeatedly apply Lemma 14 to the vertices v_2, \dots, v_8 as long as there exists $i \in \{2, \dots, 8\}$ such that the list of v_i contains a color not appearing on its neighbors; let M' denote the resulting motif. Note that $|L_{M'}(v_1)| > |V(M')|$ and that $|L_{M'}(v_i)| = \deg_{M'} v_i$ and $L_{M'}(v_i) \subseteq \alpha(V(M'))$ for $i \in \{2, \dots, 8\}$ such that $v_i \in V(M')$. Hence, there exists a color $c_1 \in |L_{M'}(v_1)| \setminus \alpha(V(M'))$, and this color does not appear in the lists of vertices of $\{v_2, \dots, v_8\} \cap V(M')$. Applying Lemma 14, $M' - (v_1 \rightarrow c_1)$ contradicts Corollary 15.

(C16) By Lemma 17 and symmetry, we can assume that $\alpha(v_5) = 10$. If $\alpha^{-1}(10) \cap \{v_2, v_3\} \neq \emptyset$, then $M \sim (3, [3, 1], 3, [3]) \xrightarrow{L13} (3, [3, 1], \underline{3}, \bullet) \xrightarrow{L12} (3, [3, 1], \bullet, \bullet)$, which contradicts Lemma 22. Therefore, the color 10 does not appear in the closed neighborhood of v_2 in C_{16} . Since the color 10 appears in the closed neighborhood of every vertex in G , we have $s_{G, \alpha}^C(v_2) \geq 4$, and thus $M \sim (3, 4, 1, 3, [3]) \xrightarrow{L13} (3, 4, 1, \underline{3}, \bullet) \xrightarrow{L12} (3, 4, 1, \bullet, \bullet)$, which contradicts Lemma 22. \square

4 Discharging phase

Consider a plane triangulation G , a vertex $v \in V(G)$ of degree $k \geq 3$, and its neighbors v_1, \dots, v_k in the clockwise order around G . We say that the subgraph of G consisting of the cycle $v_1 \dots v_k$, the vertex v , and the edges vv_i for $i = 1, \dots, k$ is a *wheel*, v is its *center* and v_1, \dots, v_k its *rim*. Note that a wheel is not necessarily an induced subgraph of G . Let T be the triangle bounding the outer face of G . Let C be a graph and $d : V(C) \rightarrow \mathbb{N}$ a function assigning a *prescribed degree* to each vertex of C . We say that C with the prescribed degrees d *appears* in G if there exists a wheel W in G and an injective function $f : V(C) \rightarrow V(W)$ such that

- for distinct $x, y \in V(C)$, xy is an edge of C if and only if $f(x)f(y)$ is an edge of W ,
- for all $x \in V(C)$, $\deg_G f(x) \leq d(x)$, and
- $f(V(C)) \cap V(T) = \emptyset$.

Hence, C is an induced subgraph of W , but the subgraph C of G is not necessarily induced (since W may not be an induced subgraph of G). Let us remark that the last technical condition from the definition of appearance will be later used to deal with this issue.

Lemma 25. *Suppose G is a plane triangulation such that every vertex not incident with the outer face of G has degree at least five. If $|V(G)| \geq 4$, then one of the graphs with prescribed degrees depicted in Figure 2 appears in G .*

Proof. Suppose for a contradiction none of these graphs appears in G . We assign the initial charge $\text{ch}_0(v) = 10 \cdot \deg v - 60$ to each vertex v of G . Since G is a triangulation, we have $|E(G)| = 3|V(G)| - 6$ by Euler's formula, and thus

$$\sum_{v \in V(G)} \text{ch}_0(v) = 20|E(G)| - 60|V(G)| = -120. \quad (1)$$

A vertex is *big* if it either has degree at least 7 or it is incident with the outer face of G , *medium* if it has degree six and is not incident with the outer face of G , and *small* if it has degree five and is not incident with the outer face of G .

Our general aim will be to show that, provided G does not contain any of the graphs depicted in Figure 2, we can redistribute the charge of vertices so that the total of their final charges violates (1), and this contradiction will clearly complete the proof.

Let us now describe the redistribution rules. Let us remark that for a rule sending some amount of charge from a vertex v to another vertex u , we also specify faces incident with v through which the charge *leaves* v , and an edge e incident with u along which the charge *arrives* to u . Additionally, we specify a face incident with e through which the charge *passes*. This is for accounting purposes—e.g., in order to bound the amount of charge leaving a vertex, we will bound the amount of charge which leaves the vertex through each incident face.

- (R1) A big vertex v sends 2 units of charge to each adjacent small vertex u along the edge vu ; of this charge, one unit leaves v and passes through one of the faces incident with the edge uv , while the other unit leaving v passes through the other face incident with uv .
- (R2) Suppose vux is a face of G , v is big, u is small and x is medium or small. Then v sends 1 unit of charge to u ; the charge leaves v and passes through the face vux to arrive to u along the edge xu .
- (R3) Suppose v_1, \dots, v_m for some $m \in \{3, \dots, 6\}$ are consecutive neighbors of a medium vertex x in the clockwise or the counterclockwise order, v_1 is small, v_2, \dots, v_{m-1} are medium and v_m is big. Then v_m sends 1 unit of charge to v_1 ; the charge leaves v_m through the face $xv_{m-1}v_m$ and passes through the face xv_1v_2 to arrive to v_1 along the edge v_2v_1 .

Note that (R2) applies in addition to the two units of charge sent by v to u by (R1), but the charge arrives to u along a different edge. In case x is small, the charge is also being

sent from v to x by (R2) with the roles of u and x exchanged. Furthermore, note that (R3) may possibly send charge from v_m to v_1 twice around the same vertex x , once in the clockwise direction, once in the counterclockwise one (when x is the center of a wheel whose rim contains v_1 and v_m and every other vertex of the rim is medium); similarly, note that (R3) may also possibly send charge to v_1 twice through the face xv_1v_2 (when xv_1v_2 is adjacent to another face $v_m xv_2$ and $v_1v_m \notin E(G)$ – thus, v_m would send a unit of charge clockwise around x and another unit of charge anticlockwise around v_2 , where v_2 here plays the role of x).

We now analyze the final charge $\text{ch}(v)$ of each vertex v of G after the redistribution of the charge. Clearly, for a medium vertex v , we have $\text{ch}(v) = \text{ch}_0(v) = 0$.

Consider now a small vertex z . We claim that for each edge $e = wz$ incident with z and each face $f = wzx$ incident with e , a unit of charge passes through f to arrive to z along e , and thus $\text{ch}(z) = \text{ch}_0(z) + 10 \times 1 = 0$. Indeed, if w is big, then this is the case by (R1). If w is not big and x is big, then a unit of charge passing through f arrives to z along e from x by (R2). If neither w nor x is big, then since C_2 does not appear in G , both of them are medium. Since C_4 does not appear in G , x has a neighbor y distinct from z that is not medium. Let $v_1 = z, v_2 = w, v_3, \dots, v_m$ be the neighbors of x in order, where v_3, \dots, v_{m-1} are medium and v_m is not medium. Since C_3, C_6, C_8 and C_2 do not appear in G , the vertex v_m is not small, and thus v_m is big. Consequently, a unit of charge passing through f arrives to z along e from v_m by (R3).

Suppose now v is a vertex of degree $d \geq 7$ not incident with the outer face of G . For a face $f = vxy$, let $t_v(f)$ denote the total amount of charge that leaves v through f ; in the following, we omit the subscript v as long as the vertex v is fixed. If both x and y are small, then $t(f) = 4$ since two units leave through f by (R1), one arriving along the edge vx and the other along vy , and two by (R2), arriving along the edge xy in both directions. If x is small and y is medium or vice versa, then $t(f) = 2$ since one unit leaves through f by (R1) and one by (R2). If both x and y are medium, then $t(f) \leq 2$, since at most two units leave through f by (R3). If x is small and y is big or vice versa, then $t(f) = 1$, since only one unit leaves through f by (R1). Otherwise, $t(f) = 0$.

Furthermore, consider the faces f_1 and f_2 following f in the clockwise order around v . Since C_1 does not appear in G , if $t(f) = 4$, then $t(f_1) \leq 2$ and $t(f_2) \leq 2$. Consequently, there are at most $\lfloor d/3 \rfloor$ faces f incident with v such that $t(f) = 4$. If $d \geq 8$, this implies

$$\text{ch}(v) \geq \text{ch}_0(v) - 2d - 2\lfloor d/3 \rfloor = 8d - 2\lfloor d/3 \rfloor - 60 \geq 0.$$

Hence, we can assume $d = 7$, and thus $\text{ch}_0(v) = 10$. Let v_1, \dots, v_7 be the neighbors of v in the clockwise order, and for $i = 1, \dots, 7$, let f_i be the face $vv_i v_{i+1}$ (where $v_8 = v_1$). Let $s = \sum_{i=1}^7 t(f_i)$ be the total amount of charge sent by v . We argue that $s \leq 10$, and thus $\text{ch}(v) = \text{ch}_0(v) - s \geq 0$. To do so, we discuss several cases.

- v is adjacent to two consecutive small vertices in the cycle consisting of neighbors of v . Thus v is incident with a face f such that $t(f) = 4$. By symmetry, we can assume $t(f_1) = 4$, and thus v_1 and v_2 are small. Since C_1 does not appear in G , v_3 and v_7 are not small.

If v_5 is small, then since C_{16} does not appear in G , both v_4 and v_6 are big and hence $t(f_4) = t(f_5) = 1$, $t(f_3) = t(f_6) = 0$, and $t(f_2), t(f_7) \leq 2$, implying $s \leq 10$. Hence, we can assume v_5 is not small.

Suppose v_6 and v_7 are both medium. Since C_{13} does not appear in G , v_5 is big, and thus $t(f_7) + t(f_6) + t(f_5) \leq 2 + 2 + 0 = 4$. Since C_{14} and C_{10} do not appear in G , v_4 is not small and v_3 and v_4 are not both medium, respectively, implying $t(f_3) = 0$ and $t(f_4) = 0$. Consequently, $s \leq 4 + 2 + 0 + 0 + 4 = 10$. Hence, assume v_6 and v_7 are not both medium, and symmetrically, that v_3 and v_4 are not both medium.

If v_4 is small, then since C_7 and C_{16} do not appear in G , v_3 and v_5 are big and $t(f_2) + t(f_3) + t(f_4) = 1 + 1 + 1 = 3$. Otherwise, since v_3 and v_4 are not both medium, we have $t(f_3) = 0$ and $t(f_2) + t(f_4) \leq 3$. Hence $t(f_2) + t(f_3) + t(f_4) \leq 3$, and symmetrically $t(f_7) + t(f_6) + t(f_5) \leq 3$. It follows that $s \leq 4 + 3 + 3 = 10$.

- *small vertices are not consecutive in the cycle consisting of neighbors of v .* Consequently, $t(f) \leq 2$ for each face incident with v and v is adjacent to at most three small vertices.

Before we proceed, let us make a useful observation:

(\star) For any $b \in \{1, \dots, 5\}$, if none of v_b, v_{b+1} and v_{b+2} is small, then $t(f_b) + t(f_{b+1}) \leq 3$.

This is clearly the case unless v_b, v_{b+1} , and v_{b+2} are all medium and $t(f_b) = t(f_{b+1}) = 2$. Then, let $v_b, v, v_{b+2}, z_3, z_2, z_1$ be the neighbors of v_{b+1} in order. Since $t(f_b) = t(f_{b+1}) = 2$, charge leaves v through f_b and f_{b+1} twice by (R3), and thus either both z_1 and z_3 are small, or none of z_1, z_2 , and z_3 is big and at least one of them is small. But then either C_5 or C_4 appears in G , which is a contradiction.

Let us now continue with the case analysis.

- *v is adjacent to three small vertices.* By symmetry we can assume v_1, v_3 , and v_5 are small. Since C_{12} does not appear in G , we can by symmetry assume v_2 is big, and hence $t(f_1) = t(f_2) = 1$. If v_4 is big, then $t(f_3) = t(f_4) = 1$ implying $s \leq 4 \times 1 + 3 \times 2 = 10$. Thus, since C_1 does not appear in G , we can assume v_4 is medium. Since C_9 does not appear in G , v_6 and v_7 cannot both be medium, and thus $t(f_6) = 0$. Consequently, $s \leq 1 + 1 + 2 + 2 + 2 + 0 + 2 = 10$.
- *v is adjacent to two small vertices, at distance two in the cycle on neighbors of v .* By symmetry we can assume v_1 and v_3 are small. If v_5 is big, then $t(f_4) = t(f_5) = 0$ and $s \leq 5 \times 2 = 10$. Hence, we can assume v_5 is medium, and by symmetry v_6 is medium. Since C_{11} does not appear in G , v_4 and v_7 are not both medium; by symmetry, we can assume v_7 is big, and thus $t(f_6) = 0$ and $t(f_7) = 1$. Furthermore, $t(f_4) + t(f_5) \leq 3$ by (\star), and thus $s \leq 2 + 2 + 2 + 3 + 0 + 1 = 10$.
- *v is adjacent to two small vertices, at distance three in the cycle on neighbors of v .* By symmetry we can assume v_1 and v_4 are small. If v_6 is big or both v_5

and v_7 are big, then $t(f_5) = t(f_6) = 0$ and $s \leq 5 \times 2 = 10$; hence, we can by symmetry assume v_5 and v_6 are medium. Since C_9 does not appear in G , v_2 and v_3 are not both medium, and thus $t(f_1) + t(f_2) + t(f_3) \leq 2 + 0 + 1 = 3$. Furthermore, $t(f_5) + t(f_6) \leq 3$ by (\star) , implying $s \leq 3 + 2 + 3 + 2 = 10$.

- v is adjacent to at most one small vertex. By symmetry we can assume no neighbor of v other than possibly v_1 is small. If v_i is big for some $i \in \{1, 3, 4, 5, 6\}$, then $t(f_{i-1}) = t(f_i) = 0$ (where $f_0 = f_7$) and $s \leq 5 \times 2 = 10$. Hence, we can assume v_i is medium for $i \in \{3, 4, 5, 6\}$ and v_1 is medium or small. Since C_{15} does not appear in G , v_2 and v_7 are not both medium; by symmetry, we can assume v_2 is big, and thus $t(f_1) + t(f_2) \leq 1$. By (\star) , $t(f_3) + t(f_4) \leq 3$ and $t(f_5) + t(f_6) \leq 3$, and thus $s \leq 1 + 2 \times 3 + 2 < 10$.

We conclude that every vertex not incident with the outer face of G has non-negative final charge.

Finally, let us consider a vertex v incident with the outer face of G . Since $|V(G)| \geq 4$ and G is a triangulation, we have $\deg(v) \geq 3$. Furthermore, the outer face f of G is incident only with big vertices by definition, and thus $t(f) = 0$. Since $t(f') \leq 4$ for every face $f' \neq f$ incident with v , we have $\text{ch}(v) \geq \text{ch}_0(v) - (\deg v - 1) \times 4 = 6 \deg v - 56 \geq -38$. Therefore, (1) together with the fact that no charge is created or lost in the redistribution process gives

$$-120 = \sum_{v \in V(G)} \text{ch}_0(v) = \sum_{v \in V(G)} \text{ch}(v) \geq 3 \times (-38) = -114,$$

which is a contradiction. □

Corollary 26. *If G is a plane triangulation of minimum degree at least five, then one of the graphs depicted in Figure 2 is an induced subgraph of G with prescribed vertex degrees.*

Proof. If G contains a separating triangle, then let T be a separating triangle in G such that the open disk in the plane bounded by T is minimal; otherwise, let T be the triangle bounding the outer face of G . Let G' be the induced subgraph of G drawn in the closed disk bounded by T . By Lemma 25, one of the graphs C with prescribed degrees depicted in Figure 2 appears in G , via a map $f : V(C) \rightarrow V(W)$ for a wheel W in G' . By the choice of G' , observe that G' does not contain any separating triangle, and thus W is an induced subgraph of G' , and thus also of G . Since C is an induced subgraph of W , it follows that C is an induced subgraph of G . Furthermore, $V(C) \cap V(T) = \emptyset$ by the last condition from the definition of appearance, and thus the vertices of $f(V(C))$ have the same degree in G' and in G . □

The proof of the main result is now straightforward.

Proof of Theorem 5. Suppose for a contradiction that there exists a non-recolorable scene (G, α) . Choose such a scene with the smallest number of vertices, among those with the largest number of edges, and among those with the largest number of vertices of color 10.

Then (G, α) is a minimal counterexample, and thus G is a triangulation by Lemma 6, has minimum degree at least five by Corollary 11, and does not contain any of the induced subgraphs with prescribed vertex degrees depicted in Figure 2. However, this contradicts Corollary 26. \square

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