Two-distance-primitive graphs

Wei Jin∗ †
School of Mathematics and Statistics
Central South University
Changsha, Hunan, 410075, P.R.China
School of Statistics
Jiangxi University of Finance and Economics
Nanchang, Jiangxi, 330013, P.R.China
jinweipei820163.com

Ci Xuan Wu‡
School of Statistics and Mathematics
Yunnan University of Finance and Economics
Kunming, 650221, P.R.China
wucixuan@gmail.com

Jin Xin Zhou
Department of Mathematics
Beijing Jiaotong University
Beijing, 100044, P.R.China
jxzhou@bjtu.edu.cn

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Abstract

A 2-distance-primitive graph is a vertex-transitive graph whose vertex stabilizer
is primitive on both the first step and the second step neighborhoods. Let Γ be
such a graph. This paper shows that either Γ is a cyclic graph, or Γ is a complete
bipartite graph, or Γ has girth at most 4 and the vertex stabilizer acts faithfully on
both the first step and the second step neighborhoods. Also a complete classification
is given of such graphs satisfying that the vertex stabilizer acts 2-transitively on the
second step neighborhood. Finally, we determine the unique 2-distance-primitive
graph which is locally cyclic.

Mathematics Subject Classifications: 05E18, 20B25

1 Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ,
we use V (Γ) and Aut(Γ) to denote its vertex set and automorphism group, respectively.

∗Corresponding author.
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For the group theoretic terminology not defined here we refer the reader to [9, 14]. The diameter of a graph $\Gamma$ is the maximum distance occurring over all pairs of vertices. Let $u \in V(\Gamma)$ and $i$ be a positive integer at most the diameter of $\Gamma$. We use $\Gamma_i(u)$ to denote the set of vertices at distance $i$ with vertex $u$ in $\Gamma$. Sometimes, $\Gamma_1(u)$ is also denoted by $\Gamma(u)$.

A transitive permutation group $G$ is said to be acting primitively on a set $\Omega$ if it has only trivial blocks in $\Omega$. If $G$ acts primitively on $\Omega$, then every nontrivial normal subgroup of $G$ is transitive on $\Omega$. There is a remarkable classification of finite primitive permutation groups mainly due to M. O’Nan and L. Scott, called the O’Nan-Scott Theorem for primitive permutation groups, see [26, 35]. They independently gave a classification of finite primitive groups, and proposed their result at the “Santa Cruz Conference in finite groups” in 1979. For more work on primitive groups, see [5, 21, 25, 32].

A graph $\Gamma$ is said to be 2-distance-transitive if, for each $i \leq 2$, the automorphism group of $\Gamma$ is transitive on the ordered pairs of vertices at distance $i$. The study of finite 2-distance-transitive graphs goes back to Higman’s paper [18] in which “groups of maximal diameter” were introduced. These are permutation groups which act distance-transitively on some graph. Then 2-distance-transitive graphs have been studied extensively, see [11, 12, 15, 20, 33, 34].

In this paper, we investigate a family of graphs which has stronger transitivity than the family of 2-distance-transitive graphs, namely 2-distance-primitive graphs. A non-complete vertex-transitive graph $\Gamma$ is said to be 2-distance-primitive if, for $i = 1, 2$ and for any vertex $u$, $A_u$ is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$ where $A := \text{Aut}(\Gamma)$. Clearly, every 2-distance-primitive graph is 2-distance-transitive. The converse is not true, for instance, the complete multipartite graph $K_{m[n]}$ with $m \geq 3, n \geq 2$ is 2-distance-transitive but not 2-distance-primitive. (Its vertex set consists of $m$ parts of size $n$, and it has edges between all pairs of vertices from distinct parts.) Hence the family of 2-distance-primitive graphs is properly contained in the family of 2-distance-transitive graphs. Many well-known graphs have the 2-distance-primitive property. For instance, the cyclic graph $C_n$ is 2-distance-primitive whenever $n \geq 4$; the icoshedron (the graph in Figure 1) is 2-distance-primitive of valency 5; the family of 2-geodesic-transitive but not 2-arc-transitive graphs of prime valency provides an infinite family of such examples, refer to [13]. This family of graphs is also related to the class of well-known ‘locally primitive graphs’, see [19, 22, 23, 24, 30].

Our first theorem is a structural result and it shows that if a 2-distance-primitive graph is neither a cycle nor a complete bipartite graph, then its girth is 3 or 4.

**Theorem 1.** Let $\Gamma$ be a 2-distance-primitive graph. Then either $\Gamma \cong C_n$ for some $n \geq 4$, or $\Gamma$ is a complete bipartite graph, or $\Gamma$ has girth at most 4 and the vertex stabilizer acts faithfully on both the first step and the second step neighborhoods.

The complement graph $\overline{\Gamma}$ of a graph $\Gamma$, is the graph with vertex $V(\Gamma)$, and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in $\Gamma$. Recall that a permutation group $G$ acting on $\Omega$ is said to be 2-transitive if it is transitive on the set of ordered pairs of distinct points in $\Omega$.

A $d$-cube is a graph with vertex set $\Delta^d = \{(x_1, x_2, \ldots, x_d) | x_i \in \Delta\}$, where $\Delta = \{0, 1\}$,
and two vertices \( v \) and \( v' \) are adjacent if and only if they differ in exactly one coordinate. Let \( Y_d \) denote the graph with vertex set the same as a \( d \)-cube \( \Gamma \), and two vertices are adjacent in \( Y_d \) if and only if they are at distance two in \( \Gamma \). While \( Y_d \) is not connected, it has two isomorphic components on \( 2^{n-1} \) vertices, each of which is called a halved \( d \)-cube.

For a 2-distance-primitive graph, if its vertex stabilizer acts 2-transitively on the first step neighborhood, then it is well-known that this graph is 2-arc-transitive, and those graphs have been studied extensively, see [1, 10, 16, 29, 36, 37]. Our second theorem classifies the family of 2-distance-primitive graphs whose vertex stabilizer acts 2-transitively on the second step neighborhood.

**Theorem 2.** Let \( \Gamma \) be a 2-distance-primitive graph of valency \( r \geq 2 \). Suppose that the vertex stabilizer of a vertex is 2-transitive on the second step neighborhood. Then \( \Gamma \) is one of the following graphs: \( C_n \) with \( n \geq 4 \), \( K_{r,r} \), \( K_{r+1,r+1} - (r+1)K_2 \) with \( r \geq 3 \), the halved 5-cube, the complement graph of the Higman-Sims graph and the complement graph of the Gewirtz graph.

A subgraph \( X \) of a graph \( \Gamma \) is an induced subgraph if two vertices of \( X \) are adjacent in \( X \) if and only if they are adjacent in \( \Gamma \). When \( U \subseteq V(\Gamma) \), we denote by \( [U] \) the subgraph of \( \Gamma \) induced by \( U \). A graph \( \Gamma \) is said to be locally cyclic if \( [\Gamma(u)] \) is a cycle for every vertex \( u \). In particular, the girth of a locally cyclic graph is 3. The following theorem determines the class of 2-distance-primitive graphs which are locally cyclic, and surprisingly, there is a unique such example.

**Theorem 3.** Let \( \Gamma \) be a connected, non-complete, locally cyclic graph. Then \( \Gamma \) is 2-distance-primitive if and only if \( \Gamma \) is the icosahedron.

## 2 Proof of Theorem 1

In the characterization of 2-distance-primitive graphs, the following constants are useful. Our definition is inspired by the concept of intersection arrays defined for the distance-regular graphs (see [4]).
Definition 4. Let $\Gamma$ be an $s$-distance-transitive graph, $u \in V(\Gamma)$, and let $v \in \Gamma_i(u)$, $i \leq s$. Then the number of edges from $v$ to $\Gamma_{i-1}(u)$, $\Gamma_i(u)$, and $\Gamma_{i+1}(u)$ does not depend on the choice of $v$ and these numbers are denoted, respectively, by $c_i$, $a_i$ and $b_i$.

Clearly we have that $a_i + b_i + c_i$ is equal to the valency of $\Gamma$ whenever the constants are well-defined. Note that for 2-distance-primitive graphs, the constants are always well-defined for $i = 1$, 2.

For a connected graph $\Gamma$ of diameter $d \geq 2$, we denote by $\Gamma_d$ the graph whose vertices are those of $\Gamma$ and whose edges are the 2-subsets of points at mutual distance $d$ in $\Gamma$. Then, $\Gamma$ is said to be antipodal if $\Gamma_d$ is a disjoint union of complete graphs.

We prove our first theorem.

Proof of Theorem 1. If $\Gamma$ has valency 2, then $\Gamma \cong C_n$ for some $n \geq 4$. In the remainder, we suppose that $\Gamma$ has valency at least 3. Let $u \in V(\Gamma)$. Assume that $\Gamma$ has girth at least 5. Then $c_2 = 1$, so every vertex of $\Gamma_2(u)$ is adjacent to exactly one vertex of $\Gamma(u)$, it follows that for each $v \in \Gamma(u)$, $\Gamma_2(u) \cap \Gamma(v)$ is a block of the $A_u$-action on $\Gamma_2(u)$. Since $\Gamma$ has girth at least 3, $b_1 \geq 2$, and so $\Gamma_2(u) \cap \Gamma(v)$ is a nontrivial block, contradicting the fact that $A_u$ is primitive on $\Gamma_2(u)$. Thus $\Gamma$ has girth at most 4, that is, $\Gamma$ has girth 3 or 4.

Suppose that $\Gamma$ is not a complete bipartite graph. We denote by $A_u^*$ and $B_u^*$ the kernels of the $A_i$-action on $\Gamma(u)$ and $\Gamma_2(u)$, respectively. Then both $A_u^*$ and $B_u^*$ are normal subgroups of $A_u$. By the assumption, $A_u^*$ is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so $A_u^*$ acts either transitively or trivially on $\Gamma_2(u)$, and $B_u^*$ acts either transitively or trivially on $\Gamma(u)$.

(i) Suppose $A_u^*$ is transitive on $\Gamma_2(u)$. Note that for each $v \in \Gamma(u)$, $A_u^*$ fixes $\Gamma_2(u) \cap \Gamma(v)$ setwise, so $v$ is adjacent to all vertices of $\Gamma_2(u)$. Hence every vertex of $\Gamma(u)$ is adjacent to all vertices of $\Gamma_2(u)$, and so every vertex of $\Gamma_2(u)$ is also adjacent to all vertices of $\Gamma(u)$. Thus $\Gamma$ has diameter 2 and $[\Gamma_2(u)]$ is an empty graph.

Suppose first that $\Gamma$ has girth 3. Then $\Gamma$ is antipodal. In particular, $\Gamma_2(u) \cup \{u\}$ is an antipodal block of $A$ acting on $V(\Gamma)$, hence $|\Gamma_2(u)| + 1$ divides $|V(\Gamma)| = 1 + |\Gamma(u)| + |\Gamma_2(u)|$. Thus $\Gamma \cong K_m|b|$ with $m \geq 3$ and $b = 1 + |\Gamma(u)|$, contradicting the fact that $A_u$ is primitive on $\Gamma(u)$. Suppose next that $\Gamma$ has girth 4. By the previous argument, every vertex of $\Gamma(u)$ is adjacent to all vertices of $\Gamma_2(u)$, and every vertex of $\Gamma_2(u)$ is also adjacent to all vertices of $\Gamma(u)$. Hence $|\Gamma_2(u)| = |\Gamma(u)| - 1$, and the induced subgraph $[\Gamma(u) \cup \Gamma_2(u)]$ is a complete bipartite graph. Thus $\Gamma$ is a complete bipartite graph, contradicting our assumption that $\Gamma$ is not a complete bipartite graph.

Thus $A_u^*$ is not transitive on $\Gamma_2(u)$, so $A_u^*$ is trivial on $\Gamma_2(u)$. Then for any $v \in \Gamma(u)$, $A_u^*$ fixes each vertex of $\Gamma(v)$, hence $A_u^* \leq A_u^*$. As $\Gamma$ is connected, and by induction, $A_u^*$ fixes all vertices of $\Gamma$, so $A_u^* = 1$. Thus $A_u$ is faithful on $\Gamma(u)$.

(ii) Now we prove that $A_u$ is faithful on $\Gamma_2(u)$. Suppose $B_u^*$ is transitive $\Gamma(u)$. Note that for each $w \in \Gamma_2(u)$, $B_u^*$ fixes $\Gamma(u) \cap \Gamma(w)$ setwise. So $w$ is adjacent to all vertices of $\Gamma(u)$. Hence every vertex of $\Gamma_2(u)$ is adjacent to all vertices of $\Gamma(u)$. Thus $\Gamma$ has diameter 2 and $[\Gamma_2(u)]$ is an empty graph.

If $\Gamma$ has girth 4, then $\Gamma$ is complete bipartite, contradicting the assumption that $\Gamma$ is not a complete bipartite graph. If $\Gamma$ has girth 3, then $\Gamma$ is antipodal and $\Gamma_2(u) \cup \{u\}$ is...
an antipodal block, so \(|\Gamma_2(u)| + 1\) divides \(|V(\Gamma)| = 1 + |\Gamma(u)| + |\Gamma_2(u)|\). Thus \(\Gamma \cong K_m[b]\) with \(m \geq 3\) and \(b = 1 + |\Gamma_2(u)|\), so \(A_u\) is imprimitive on \(\Gamma(u)\), a contradiction. Thus \(B_u^*\) is trivial on \(\Gamma(u)\). Hence \(B_u^* \leq A_u^* = 1\). Therefore \(A_u\) acts faithfully on \(\Gamma_2(u)\). \(\square\)

3 Proof of Theorem 2

We prove Theorem 2 by a series of lemmas. The first lemma shows that a 2-distance-transitive graph of girth 4 is unique, if its first step neighbor and second step neighbor have the same number of vertices.

Lemma 5. Let \(\Gamma\) be a 2-distance-transitive graph of girth 4 and valency \(r \geq 3\). If \(|\Gamma_2(u)| = r\) for some \(u \in V(\Gamma)\), then \(\Gamma \cong K_{r+1,r+1} - (r + 1)K_2\).

Proof. Assume that \(|\Gamma_2(u)| = r\) for some \(u \in V(\Gamma)\). Let \((u, v, w, z)\) be a 3-geodesic. Since \(\Gamma\) is 2-distance-transitive with girth 4 and valency \(r\), there are \(r(r - 1)\) edges between \(\Gamma(u)\) and \(\Gamma_2(u)\), and so \(r(r - 1) = c_2 \cdot |\Gamma_2(u)|\). By the assumption, \(|\Gamma_2(u)| = r\), so we get \(c_2 = r - 1\). Hence \(|\Gamma(v) \cap \Gamma(z)| = c_2 = r - 1\), as \((v, w, z)\) is a 2-geodesic. Note that \(|\Gamma_2(u) \cap \Gamma(v)| = r - 1\) and \(\Gamma(v) \cap \Gamma(z) \subseteq \Gamma_2(u) \cap \Gamma(v)\). It follows that \(\Gamma_2(u) \cap \Gamma(v) = \Gamma(v) \cap \Gamma(z)\).

Since \(r \geq 3\), \(c_2 = r - 1 \geq 2\). Hence there exists a vertex \(v_2 \in \Gamma(u) \setminus \{v\}\) such that \((v_2, w, z)\) is a 2-geodesic. So \(|\Gamma(v_2) \cap \Gamma(z)| = r - 1\), this indicates that \(\Gamma_2(u) \cap \Gamma(v_2) = \Gamma(v_2) \cap \Gamma(z)\).

Suppose that \(\Gamma_2(u) \cap \Gamma(v) = \Gamma_2(u) \cap \Gamma(v_2)\). Since \(\Gamma\) has girth 4, it follows that \((\Gamma_2(u) \cap \Gamma(v)) \cup \{u\} = \Gamma(v) \cap \Gamma(v_2)\), hence \(|\Gamma(v) \cap \Gamma(v_2)| = r\), contradicting the fact that \(|\Gamma(v) \cap \Gamma(v_2)| = c_2 = r - 1\), as \((v, u, v_2)\) is a 2-geodesic. Thus \(\Gamma_2(u) \cap \Gamma(v) \neq \Gamma_2(u) \cap \Gamma(v_2)\), so \((\Gamma_2(u) \cap \Gamma(v)) \cup \{u\} = \Gamma(v) \cap \Gamma(v_2)\) = \(\Gamma_2(u)\). By the previous argument, \(\Gamma_2(u) \cap \Gamma(v) = \Gamma(v) \cap \Gamma(z)\) and \(\Gamma_2(u) \cap \Gamma(v_2) = \Gamma(v_2) \cap \Gamma(z)\). Thus \(\Gamma_2(u) \subseteq \Gamma(z)\). Since \(r = |\Gamma_2(u)| \leq |\Gamma(z)| = r\), it follows that \(\Gamma_2(u) = \Gamma(z)\). Therefore, \(\Gamma_3(u) = \{z\}\) and \(\Gamma\) has diameter 3. Precisely, this graph is \(K_{r+1,r+1} - (r + 1)K_2\). \(\square\)

Lemma 6. Let \(\Gamma\) be a 2-arc-transitive graph of diameter 2 and girth 5. Then \(\Gamma\) is one of the following graphs: \(C_5\), the Petersen graph, or the Hoffman-Singleton graph.

Proof. Since \(\Gamma\) has diameter 2 and girth 5, \(\Gamma\) is a Moore graph. Then it follows from [4, Theorem 6.7.1] that \(\Gamma\) has valency 2, 3, 7 or 57. By [2] or [4, p.207, Remark (i)], the valency 57 case does not occur, and so \(\Gamma\) has valency 2, 3 or 7. Further, by [4, p.207, Remark (i)] or [17, p.206], if \(\Gamma\) has valency 2, then \(\Gamma = C_5\); if \(\Gamma\) has valency 3, then \(\Gamma\) is the Petersen graph; and if \(\Gamma\) has valency 7, then \(\Gamma\) is the Hoffman-Singleton graph. \(\square\)

The socle of a 2-transitive group is either elementary abelian or non-regular non-abelian simple, see [14, Theorem 4.1B], and in the latter case, the socle is primitive, see [14, p.244].

Lemma 7. Let \(\Gamma\) be a 2-distance-primitive graph of diameter 2 and girth 4. If \(\Gamma\) is 2-arc-transitive, then \(\Gamma\) is one of the following graphs: \(K_{m,m}\) with \(m \geq 2\), Higman-Sims graph, 2-cube, the Gewirtz graph or the folded 5-cube.
Proof. Suppose that $\Gamma$ is 2-arc-transitive. Let $A := \text{Aut}(\Gamma)$ and let $u \in V(\Gamma)$. Assume that $A$ is not primitive on $V(\Gamma)$. Then $A$ has some nontrivial blocks on $V(\Gamma)$, and say $\Delta_i$. Since the graph $\Gamma$ is arc-transitive, each $\Delta_i$ does not contain edges of $\Gamma$. Let $u, u' \in \Delta_i$. Then $u' \in \Gamma_2(u)$ and $\Delta_i \subseteq \{u\} \cup \Gamma_2(u)$, as $\Gamma$ has diameter 2. Since $A_u$ fixes the block $\Delta_i$ and it is also transitive on $\Gamma_2(u)$, it follows that $\{u\} \cup \Gamma_2(u) \subseteq \Delta_i$, so $\{u\} \cup \Gamma_2(u) = \Delta_i$. Thus $\{u\} \cup \Gamma_2(u)$ is a block of $\Gamma$. By the vertex-transitivity of $\Gamma$, we know that $\Gamma(u)$ is a union of some blocks. If $\Gamma(u)$ contains more than one block, then $\Gamma$ has girth 3, contradicting the fact that $\Gamma$ has girth 4. Thus $\Gamma(u)$ is a block of cardinality $|\Delta_i|$. Since $\Gamma$ has diameter 2, it follows that $\Gamma \cong K_{m,m}$ where $m = |\Delta_i| \geq 2$. In the remainder, we suppose that $A$ acts primitively on $V(\Gamma)$.

Since $\Gamma$ is 2-arc-transitive, the stabilizer $A_u$ is 2-transitive on $\Gamma(u)$, and it is well-known that this 2-transitive action is of type either affine or almost simple. Suppose that $A_u$ is an affine group. Since $A_u$ is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$, it follows that its socle is regular on both $\Gamma(u)$ and $\Gamma_2(u)$, and so $|\Gamma(u)| = |\Gamma_2(u)|$. Then by Lemma 5, $\Gamma \cong K_{r+1,r+1}-(r+1)K_2$ with diameter 3, contradicting the assumption that $\Gamma$ has diameter 2. Thus $A_u$ acts 2-transitively on $\Gamma(u)$ of almost simple type, and either $A_u \cong P_{\text{GL}}(2,8)$ or the socle of $A_u$ is 2-transitive. Again as $\Gamma$ is 2-arc-transitive of diameter 2, $A_u$ is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so $A$ is a primitive rank 3 group. Since $A_u$ is 2-transitive on $\Gamma(u)$, $A$ has a 2-transitive suborbit, it follows from [31, Theorem A] that $A$ is primitive of type either affine or almost simple. In particular, the socle of $A_u$ is 2-transitive.

Suppose that $A$ is an affine group. Then $A$ is completely listed in [27]. The stabilizer $A_u$ and subdegrees are given in Tables 12, 13 and 14 of [27]. The groups in Tables 12 and 14 are not 2-transitive. Hence $A_u$ is in Table 13. Then by Theorem (B) of [27], $R \leq A_u \leq N_{\text{GL}(d,p)}(R)$ where $R$ is an $r$-group, $A_u$ is not almost simple, a contradiction. Hence $A$ is not an affine group.

Thus $A$ is an almost simple primitive group. If $A = S_n$ or $A_n$, then by [7, Theorem 4.5] or [10, p.4], $\Gamma$ has parameter $c_2 = 2$, and $\Gamma$ is one of the following graphs: a cube, a folded $d$-cube, or the incidence graph of the Paley design on 11 points. Since $A$ is primitive on $V(\Gamma)$, $\Gamma$ is not a bipartite graph, so $\Gamma$ is a cube or a folded $d$-cube. Note that $\Gamma$ has diameter 2. Hence $\Gamma$ is the 2-cube or the folded 5-cube (folded $d$-cube has diameter $\lfloor d/2 \rfloor$).

The primitive rank 3 groups in which the socle is either an exceptional group of Lie type or a sporadic group are listed in [28]. Let $A$ be a primitive rank 3 group in [28] with socle $L$, and let $H$ be the stabilizer in $L$ of a vertex $u$. If $L$ is an exceptional simple group of Lie type, then $L,H$ and the subdegrees $k,l$ are listed in Table 1 of [28]. Since $L$ is the socle of $A$ and $H = L_u$, $H$ is a normal subgroup of $A_u$. Since $A_u$ is almost simple, if $H \neq 1$, then $H$ is the socle of $A_u$ and it is an non-abelian simple 2-transitive group. Thus $A$ is not in Table 1 of [28]. We inspect the groups in Table 2 of [28]. Then $(L,H) = (H,S, M_{22})$ is the unique candidate, and it provides the example Higman-Sims graph.

Finally, suppose that $A$ is an almost simple group of classical type. Then $A$ is investigated in [6]. Since $A$ is primitive and $A_u$ acts primitively on both $\Gamma(u)$ and $\Gamma_2(u)$,
\(A\) is completely determined in [6, Theorem 1.1]. As \(A_u\) is almost simple, we can easily conclude that the two possible cases are that \((\text{soc}(A), \text{soc}(A_u)) = (\text{PSL}(3, 4), A_u)\) and \((\text{soc}(A), \text{soc}(A_u)) = (\text{PSU}(4, 3), \text{PSL}(3, 4))\). For the former case, by Magma [3], the two nontrivial subdegrees of \(A\) are 10 and 45. This produces the Gewirtz graph. For the latter case, again by Magma [3], the two nontrivial subdegrees of \(A\) are 56 and 105, and hence \(A_u\) does not provide any 2-transitive representation on each suborbit, which is not possible. \(\square\)

Lemma 8. Let \(\Gamma\) be a 2-distance-primitive graph. If \(a_2 = 0\), then either \(\Gamma \cong C_n\) with \(n \geq 6\) or \(\Gamma\) has girth 4.

Proof. Let \(u \in V(\Gamma), i \in \{1, 2\}\) and let \(A := \text{Aut}(\Gamma)\). Assume that the induced subgraph \([\Gamma_i(u)]\) is disconnected. Then each disconnected component \(\Delta\) of \([\Gamma_i(u)]\) is a block of the \(A_u\)-action on \(\Gamma_i(u)\). Since \(A_u\) is primitive on \(\Gamma_i(u)\), it follows that \(\Delta\) is a trivial block, that is, \(\Delta\) has size 1. Thus \([\Gamma_i(u)]\) is an empty graph. Therefore, \([\Gamma_i(u)]\) is either connected or empty.

Suppose that \(a_2 = 0\). Let \((u, v)\) be an arc. Then the two induced subgraphs \([\Gamma_2(u)]\) and \([\Gamma_2(v)]\) are empty graphs. Hence \([\Gamma(u) \cap \Gamma_2(v)]\) is an empty graph. Assume that \(\Gamma\) has girth 3. Then \([\Gamma(u)]\) is not an empty graph, and by the previous argument \([\Gamma(u)]\) is connected. Set \([\Gamma(u) \cap \Gamma_i(v)] = x \geq 1\). Note that \(\Gamma(u) = \{v\} \cup (\Gamma(u) \cap \Gamma(v)) \cup (\Gamma(u) \cap \Gamma_2(v))\). Hence every vertex \(v'\) of \(\Gamma(u) \cap \Gamma_2(v)\) is adjacent to \(x\) vertices of \(\Gamma(u) \cap \Gamma(v)\), so \(\Gamma(u) \cap \Gamma(v) = \Gamma(u) \cap \Gamma(v')\). Since \([\Gamma(u)]\) is vertex-transitive, it follows that \(\{v\} \cup (\Gamma(u) \cap \Gamma_2(v))\) is a nontrivial block of the \(A_u\)-action on \(\Gamma(u)\), which is a contradiction, as \(A_u\) is primitive on \(\Gamma(u)\). Thus \(\Gamma\) has girth at least 4, and by Theorem 1, either \(\Gamma \cong C_n\) with \(n \geq 6\) or \(\Gamma\) has girth exactly 4. \(\square\)

Lemma 9. Let \(\Gamma\) be a 2-distance-primitive graph of girth 3. Let \(A := \text{Aut}(\Gamma)\) and let \(u \in V(\Gamma)\). Suppose that \(A_u\) is 2-transitive on \(\Gamma_2(u)\). Then \(\Gamma\) is one of the following graphs: the halved 5-cube, the complement of the Gewirtz graph or the complement of the Higman-Sims graph.

Proof. Since \(A_u\) is 2-transitive on \(\Gamma_2(u)\), it follows that the induced subgraph \([\Gamma_2(u)]\) is either a complete graph or an empty graph. If \([\Gamma_2(u)]\) is an empty graph, then \(a_2 = 0\). Since \(\Gamma\) has girth 3, \(\Gamma \cong C_n\) for any \(n \geq 6\), and by Lemma 8, \(\Gamma\) has girth 4, a contradiction. Hence \([\Gamma_2(u)]\) is a complete graph. Let \((u, v, w)\) be a 2-geodesic. Assume that \(\Gamma\) has diameter at least 3. Let \(z \in \Gamma_3(u) \cap \Gamma(w)\). Then \(z \in \Gamma_2(v)\). However, \(z\) is not adjacent to any vertex of \(\Gamma(u) \cap \Gamma_2(v)\), contradicting the fact that \([\Gamma_2(v)]\) is a complete graph. Thus \(\Gamma\) has diameter 2.

Suppose that \(A\) is not primitive on \(V(\Gamma)\). Then \(A\) has some nontrivial blocks on \(V(\Gamma)\), and say \(\Delta_i\). Since \(\Gamma\) is arc-transitive, each \(\Delta_i\) does not contain edges of \(\Gamma\). Let \(u, u' \in \Delta_1\). Note that \(\Gamma\) has diameter 2. Then \(u' \in \Gamma_2(u)\). Since \(A_u\) fixes the block \(\Delta_1\) and also it acts transitively on \(\Gamma_2(u)\), it follows that \(\{u\} \cup \Gamma_2(u) \subseteq \Delta_1\). As \(\Delta_1\) does not contain any edge, it follows that \(\{u\} \cup \Gamma_2(u) = \Delta_1\). Thus \(\{u\} \cup \Gamma_2(u)\) is a block of the \(A\)-action on \(V(\Gamma)\) and \(|\Gamma_2(u)| = 1\), as \([\Gamma_2(u)]\) is a complete graph. Since \(\Gamma\) is 2-distance-transitive of
diameter 2, it follows that $\Gamma \cong K_{m[2]}$ for some $m \geq 3$, contradicting that $A_u$ is primitive on $\Gamma(u)$. Thus $A$ is primitive on $V(\Gamma)$.

Assume that $\Gamma_2(u) \subseteq \Gamma(v)$. Then as $A_u$ is transitive on $\Gamma(u)$, each vertex of $\Gamma(u)$ is adjacent to all vertices of $\Gamma_2(u)$, and so each $w_i \in \Gamma_2(u)$ is adjacent to all vertices of $\Gamma(u)$. Hence $|\Gamma(w_i)| \geq |\Gamma(u)| + |\Gamma_2(u)| - 1$, as $|\Gamma_2(u)|$ is a complete graph. Since $|\Gamma(w_i)| = |\Gamma(u)|$, it follows that $|\Gamma_2(u)| = 1$. Thus $\{u\} \cup \Gamma_2(u)$ is a block of the $A$-action on $V(\Gamma)$, contradicting that $A$ is primitive on $V(\Gamma)$. Hence $\Gamma_2(u) \not\subseteq \Gamma(v)$, and there exists a vertex of $\Gamma_2(u)$ that is not adjacent to $v$. Therefore, $\Gamma$ also has diameter 2.

Since $A_u$ is 2-transitive on $\Gamma_2(u)$ and $\Gamma(u) = \Gamma_2(u)$, it follows that $\Gamma$ is a 2-arc-transitive graph. By the previous argument, $\Gamma$ has diameter 2, so $\Gamma$ has girth 4 or 5. If $\Gamma$ has girth 5, then by Lemma 6, $\Gamma$ is one of: $C_5$, Petersen graph or Hoffman-Singleton graph. If $\Gamma$ is $C_5$, then $\Gamma$ is the Petersen graph or the Hoffman-Singleton graph. Then $|\Gamma_2(u) \cap \Gamma(v)| = k - 1$ where $|\Gamma(u)| = k$, and so $\Gamma_2(u) \cap \Gamma(v)$ is a block of the $A_u$ action on $\Gamma_2(u)$, $A_u$ is not primitive on $\Gamma_2(u)$. Since $\Gamma(u) = \Gamma_2(u)$, $A_u$ is not primitive on $\Gamma(u)$, a contradiction. Thus $\Gamma$ has girth 4. Then it follows from Lemma 7 that $\Gamma$ is one of the following graphs: $K_{m,m}$ with $m \geq 2$, Higman-Sims graph, the Gewirtz graph, 2-cube or the folded 5-cube. Since the complement graphs of both the 2-cube and $K_{m,m}$ with $m \geq 2$ are disconnected, $\Gamma$ is neither of those two graphs, and so $\Gamma$ is the Higman-Sims graph, the Gewirtz graph or the folded 5-cube. Thus $\Gamma$ is the halved 5-cube, the complement of the Gewirtz graph or the complement of the Higman-Sims graph.

We cite two lemmas which will be used in the remaining.

**Lemma 10.** ([8, p.9, Notes (1)]) Let $G$ be a non-abelian simple group. Suppose that $G$ has more than one 2-transitive permutation representation. Then $G$ and its degree $n$ are in one line of Table 1.

**Table 1:** Nonsolvable 2-transitive groups with two representations

<table>
<thead>
<tr>
<th>$T$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5 \cong PSL(2,4) \cong PSL(2,5)$</td>
<td>5, 6</td>
</tr>
<tr>
<td>$A_6 \cong PSL(2,9)$</td>
<td>6, 10</td>
</tr>
<tr>
<td>$PSL(2,7) \cong PSL(3,2)$</td>
<td>7, 8</td>
</tr>
<tr>
<td>$A_7$</td>
<td>7, 15</td>
</tr>
<tr>
<td>$A_8 \cong PSL(4,2)$</td>
<td>8, 15</td>
</tr>
<tr>
<td>$PSL(2,8)$</td>
<td>9, 28</td>
</tr>
<tr>
<td>$PSL(2,11)$</td>
<td>11, 12</td>
</tr>
<tr>
<td>$M_{11}$</td>
<td>11, 12</td>
</tr>
<tr>
<td>$PSp(2d,2), d &gt; 2$</td>
<td>$2^{2d-1} + 2^{d-1}, 2^{2d-1} - 2^{d-1}$</td>
</tr>
</tbody>
</table>

The following well-known result is mainly due to Burnside.
Lemma 11. ([14, Theorem 3.5B]) A primitive permutation group $G$ of prime degree $p$ is either 2-transitive, or solvable and $G \leq AGL(1,p)$.

Lemma 12. Let $\Gamma$ be a 2-distance-transitive graph of prime valency $p$. Let $u \in V(\Gamma)$ and $A := \text{Aut}(\Gamma)$. Suppose that $A_u$ is 2-transitive on $\Gamma_2(u)$. Then the socle of $A$ as $A$ is also primitive on $\Gamma_2(u)$. It follows from Theorem 1 that either $\Gamma \cong K_{p,p}$ or $A_u$ is faithful on both $\Gamma(u)$ and $\Gamma_2(u)$. Suppose that $\Gamma \not\cong K_{p,p}$. Then $A_u \cong A_u^{\Gamma(u)} \cong A_u^{\Gamma_2(u)}$.

Assume that $A_u$ is not 2-transitive on $\Gamma(u)$. Then by Lemma 11, $A_u \cong \mathbb{Z}_p : \mathbb{Z}_r$ where $r | p - 1$ and $r < p - 1$. Since $A_u$ is 2-transitive on $\Gamma_2(u)$, it follows that the normal subgroup $\mathbb{Z}_p$ is transitive on $\Gamma_2(u)$, and so $\mathbb{Z}_p$ is regular on $\Gamma_2(u)$. Hence $|\Gamma_2(u)| = p$. However, as $r < p - 1$, $\mathbb{Z}_p : \mathbb{Z}_r$ does not have a 2-transitive representation on $p$ letters, which is a contradiction.

Thus $A_u$ is 2-transitive on $\Gamma(u)$, and so $\Gamma$ has girth 4. Assume first that $A_u$ is solvable. Then the socle of $A_u$ is regular on both $\Gamma(u)$ and $\Gamma_2(u)$, and so $|\Gamma(u)| = |\Gamma_2(u)| = p$. It follows from Lemma 5 that $\Gamma \cong K_{p+1,p+1} - (p+1)K_2$.

Now assume that $A_u$ is non-solvable. Suppose $A_u$ has more than one 2-transitive representation. Then by Lemma 10, the socle $T$ of $A_u$ and its degree $n$ are listed in Table 1. Note that neither $2^{2d-1} + 2^{d-1} = 2d-1(2^d + 1)$ nor $2^{2d-1} - 2^{d-1} = 2^{d-1}(2^d - 1)$ is a prime whenever $d > 2$. Hence $T$ and its degree $n$ are listed in Table 2.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5 \cong \text{PSL}(2,4) \cong \text{PSL}(2,5)$</td>
<td>5, 6</td>
</tr>
<tr>
<td>$\text{PSL}(2,7) \cong \text{PSL}(3,2)$</td>
<td>7, 8</td>
</tr>
<tr>
<td>$A_7$</td>
<td>7, 15</td>
</tr>
<tr>
<td>$\text{PSL}(2,11)$</td>
<td>11, 12</td>
</tr>
<tr>
<td>$M_{11}$</td>
<td>11, 12</td>
</tr>
</tbody>
</table>

Since $A_u$ is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, it follows that $p(p - 1) = c_2 \cdot |\Gamma_2(u)|$. Hence $|\Gamma_2(u)|$ is a divisor of $p(p - 1)$. Since $p$ is a prime, by Table 2, $(p, |\Gamma_2(u)|) \in \{(5, 6), (7, 8), (11, 12), (7, 15)\}$. However, for any such a pair $(p, |\Gamma_2(u)|)$, the integer $|\Gamma_2(u)|$ is not a divisor of $p(p - 1)$, which is a contradiction. Therefore, $A_u$ has exactly one 2-transitive representation, so $|\Gamma(u)| = |\Gamma_2(u)| = p$. Again, by Lemma 5, $\Gamma \cong K_{p+1,p+1} - (p+1)K_2$.

Lemma 13. Let $\Gamma$ be a 2-arc-transitive graph of valency 6. Then $(a_1, c_2) \neq (0,3)$.

Proof. Suppose that $(a_1, c_2) = (0,3)$. Then $b_1 = 5$, and $|\Gamma_2(u)| = 10$. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3, w_4, w_5\}$. We suppose that $\Gamma(u) \cap$
\(\Gamma(w_1) = \{v_1, v_2, v_3\}\), as \(c_2 = 3\). Since \((v_1, u, v_2)\) is a 2-arc, \(|\Gamma(v_1) \cap \Gamma(v_2)| = 3\), set \(\Gamma(v_1) \cap \Gamma(v_2) = \{u, w_1, w_2\}\). Then \(|\Delta_1| = 3\) where \(\Delta_1 = \Gamma(u) \setminus \Gamma(v_2)\). Therefore \(\Gamma(u) \cap \Gamma(w_2) = \{v_1, v_2, v_3, x\}\). In particular, each \(y \in \{v_4, v_5, v_6\}\setminus\{x\}\) is adjacent to neither \(w_1\) nor \(w_2\). As \(\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \{w_1, w_2\}\), it follows that \(|\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(y)| = 0\).

Let \(A := \text{Aut}(\Gamma)\). As \(|\Gamma(u)| = 6\), it is well-known that there are only four 2-transitive permutation groups of degree 6, namely \(A_5\), \(S_5\), \(A_6\) and \(S_6\), see for instance [14, p.59-60]. Further, all these four permutation groups are 3-transitive on \(\Gamma(v)\). Thus \(A_{u,v_2}^{\Gamma(u)}\) is transitive between sets \(\{v_2, v_3\}\) and \(\{v_2, y\}\). Recall that \(|\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(y)| = 0\).

If \(|\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)| = 0\). However, \(\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) = \{w_1\}\), a contradiction. Therefore, \((a_1, c_2) \neq (0, 3)\). \(\square\)

**Lemma 14.** Let \(\Gamma\) be a 2-distance-primitive graph of valency \(r\) and girth at least 4. Let \(A := \text{Aut}(\Gamma)\) and let \(u \in \text{V}(\Gamma)\). Suppose that \(A_u\) is 2-transitive on \(\Gamma_2(u)\). Then \(\Gamma \cong C_n\) with \(n \geq 4\), \(K_{r,r}\), or \(K_{r+1,r+1}-(r+1)K_2\) with \(r \geq 3\).

**Proof.** If \(r = 2\), then \(\Gamma \cong C_n\) with \(n \geq 4\). In the remainder, we assume that \(r \geq 3\). Let \((u, v, w)\) be a 2-geodesic. Since \(\Gamma\) has girth at least 4, the induced subgraph \([\Gamma(u)]\) is an empty graph. By the assumption, \(A_u\) is 2-transitive on \(\Gamma_2(u)\), so \([\Gamma_2(u)]\) is either complete or empty. Assume that \([\Gamma_2(u)]\) is a complete graph. Then \([\Gamma_2(u) \cap \Gamma(v)]\) is a complete graph. Since \(\Gamma\) has valency at least 3 and girth at least 4, \(b_1 = \Gamma_2(u) \cap \Gamma(v) \geq 2\), so \((v, x, y)\) is a triangle for any two distinct vertices \(x, y \in \Gamma_2(u) \cap \Gamma(v)\), contradicting the fact that \(\Gamma\) has girth at least 4.

Thus \([\Gamma_2(u)]\) is an empty graph. Since \(b_1 \geq 2\), there exists a vertex \(w_1 \in \Gamma_2(u) \cap \Gamma(v)\) such that \(w_1 \neq w\). Then \((v, w, w_1)\) is a 2-geodesic. Since \(A_{u,w}\) is transitive on \(\Gamma_2(u) \setminus \{w\}\), it follows that for any \(w' \in \Gamma_2(u) \setminus \{w\}\), \(A_{u,w}\) is transitive between \(w'\) and \(w_1\), and so \(w' \in \Gamma_2(w)\). Hence \(\Gamma_2(u) \setminus \{w\} \subseteq \Gamma_2(w)\). As \(|\Gamma_2(u) \setminus \{w\}| = |\Gamma_2(w)| - 1\), it follows that \(|\{w\} \cup \Gamma_2(w) = \{u\} \cup \Gamma_2(u)\). (**)

If \(\Gamma\) has diameter at least 4, then there exists a vertex \(z \in \Gamma_4(u) \cap \Gamma_2(w)\), contradicting (**). Thus \(\Gamma\) has diameter at most 3.

Assume that \(\Gamma\) has diameter 2. Recall that there exists both \([\Gamma(u)]\) and \([\Gamma_2(u)]\) are empty graphs. Hence every vertex of \(\Gamma_2(u)\) is adjacent to all vertices of \(\Gamma(u)\), and so \(\Gamma \cong K_{r,r}\).

Now suppose that \(\Gamma\) has diameter 3. Let \(z \in \Gamma_3(u) \cap \Gamma(w)\). Then \((u, v, w, z)\) is a 3-geodesic. Assume \(b_2 = 1\). Then \(|\Gamma_3(u) \cap \Gamma(w)| = 1\). Since \([\Gamma_2(u)]\) is an empty graph, it follows that \(|\Gamma(u) \cap \Gamma(w)| = r - 1\). Note that there are \(r(r - 1)\) edges between \(\Gamma(u)\) and \(\Gamma_2(u)\). Thus \(|\Gamma_2(u)| = r\). It follows from Lemma 5 that \(\Gamma \cong K_{r+1,r+1}-(r+1)K_2\).

Now assume that \(b_2 \geq 2\). Then \(|\Gamma_3(u)| \geq 2\). If \(z\) is adjacent to some \(z' \in \Gamma_3(u)\), then \(z' \in \Gamma_2(w) \cup \Gamma(w)\). By (**), \(z' \not\in \Gamma_2(w)\), so \(z' \in \Gamma(w)\), hence \((z, w, z')\) is a triangle,
contradicting the fact that $\Gamma$ has girth at least 4. Thus $\Gamma_3(u) \cap \Gamma(z) = \emptyset$. Since $\Gamma$ has diameter 3, it follows that $\Gamma(z) \subseteq \Gamma_2(u)$. As $w$ is any vertex of $\Gamma_2(u)$ and $z$ is any vertex of $\Gamma_3(u) \cap \Gamma(w)$, it follows that $[\Gamma_3(u)]$ is an empty graph. Therefore,

$$\Gamma$$ is a diameter 3 bipartite graph. \hfill (**) \hfill

Setting the two biparts of $\Gamma$ are $\Delta_1 = \{u\} \cup \Gamma_2(u)$ and $\Delta_2 = \Gamma(u) \cup \Gamma_3(u)$. Since $A_u$ is 2-transitive on $\Gamma_2(u)$, $A_{\Delta_1}^{\Delta_1}$ is 3-transitive on $\Delta_1$. Since $\Gamma$ is vertex-transitive, also $A_{\Delta_2}^{\Delta_2}$ is 3-transitive on $\Delta_2$. It is well-known that a 2-transitive permutation group is type either affine or almost simple. Assume first that the $A_{\Delta_1}^{\Delta_1}$-action on $\Delta_1$ is the affine type. Suppose $A_{\Delta_1}^{\Delta_1}$ is solvable. Then $(A_{\Delta_1}^{\Delta_1})_u$ is solvable. As $A_u$ is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$, it follows that the socle of $A_u$ is regular on both $\Gamma(u)$ and $\Gamma_2(u)$, hence $|\Gamma(u)| = |\Gamma_2(u)|$. By Lemma 5, $\Gamma \cong K_{r+1, r+1} - (r+1)K_2$, contradicting that $b_2 \geq 2$. Suppose $A_{\Delta_1}^{\Delta_1}$ is non-solvable. Then as $A_{\Delta_1}^{\Delta_1}$ is 3-transitive on $\Delta_1$ of affine type, it follows that $|\Delta_1|$ and $(A_{\Delta_1}^{\Delta_1})_u$ are listed in [9, p.195], and inspecting the candidates, $|\Delta_i|$ and $(A_{\Delta_i}^{\Delta_i})_u$ are one of the following cases: 1) $|\Delta_i| = q^d$ and $SL(d, q) \vartriangleleft (A_{\Delta_i}^{\Delta_i})_u \leq \Gamma L(d, q)$; 2) $|\Delta_i| = q^{2d}$ and $Sp(d, q) \vartriangleleft (A_{\Delta_i}^{\Delta_i})_u$, $d \geq 2$; 3) $|\Delta_i| = q^5$ and $G_2(q) \vartriangleleft (A_{\Delta_i}^{\Delta_i})_u \leq \Gamma L(d, q)$, $q$ is even. In these cases, the socle of $(A_{\Delta_i}^{\Delta_i})_u$ is non-solvable. Since $(A_{\Delta_1}^{\Delta_1})_u$ is a 2-transitive group, we know that $(A_{\Delta_i}^{\Delta_i})_u$ is 2-transitive of almost simple type. Thus $|\Delta_i| - 1$ and the socle of $(A_{\Delta_i}^{\Delta_i})_u$ are listed in [9, p.197], by inspecting the candidates, they do not occur.

| Table 3: Non-solvable $k$-transitive groups with $k \geq 3$ |
|----------------------|------------------|
| $M$                  | degree $t$       |
| $A_t$, $t \geq 5$    | $t$              |
| $PSL(2, q)$, $q$ is a prime power, $q \neq 2, 3$ | $q + 1$          |
| $M_{11}$             | 11               |
| $M_{12}$             | 12               |
| $M_{22}$             | 22               |
| $M_{23}$             | 23               |
| $M_{24}$             | 24               |

Thus the 2-transitive action of $A_{\Delta_1}^{\Delta_1}$ on $\Delta_1$ is the almost simple type. By [9, p.196-197], the socle $M$ of $A_{\Delta_1}^{\Delta_1}$ and $|\Delta_i| = t$ are in one of the lines of Table 3. Since $A_u$ is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, there are $r(r - 1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$, and so

$$r(r - 1) = c_2 \cdot |\Gamma_2(u)| = c_2(t - 1). \quad (1)$$

Recall that $3 \leq r \leq t - 2$. Suppose $t = 1$ is a prime integer. Then by equation (1), $t - 1 | r(r - 1)$, a contradiction. Thus $t - 1$ is not a prime. Hence $t \neq 12, 24$.

Suppose that $A_u$ is 2-transitive on $\Gamma(u)$. If $A_u$ has exactly one 2-transitive permutation representation, then $|\Gamma(u)| = |\Gamma_2(u)|$, and by Lemma 5, $\Gamma \cong K_{r+1, r+1} - (r + 1)K_2$, contradicts that $b_2 \geq 2$. Thus $A_u$ has more than one 2-transitive permutation representation. Then by Lemma 10, the socle of $A_u$ and its degree $n$ are in one line of Table 1. If
r is a prime, then by Lemma 12, $\Gamma \cong K_{r+1,r+1} - (r + 1)K_2$ with $r \geq 3$, a contradiction. Thus $r$ is not a prime. By equation (1), $r(r-1) = c_2|\Gamma_2(u)|$. Since $\Gamma \not\cong K_{r,r}$, $c_2 \neq r$, so $c_2 \leq r - 1$. If $c_2 = r - 1$, then $|\Gamma_2(u)| = r$, and by Lemma 5, $\Gamma \cong K_{r+1,r+1} - (r + 1)K_2$, contradicts that $b_2 \geq 2$. Assume $c_2 < r - 1$. Then $t - 1 = |\Gamma_2(u)| > r$. By checking Tables 1 and 3, the pair $(r, |\Gamma_2(u)|) \in \{(6, 10), (8, 15), (9, 28)\}$. It follows from Lemma 13 that $(a_1, c_2) \neq (0, 3)$, so $(r, |\Gamma_2(u)|) \neq (6, 10)$. However, if $(r, |\Gamma_2(u)|) = (8, 15)$ or $(9, 28)$, then $|\Gamma_2(u)|$ is not a divisor of $r(r - 1)$, a contradiction. Therefore, $A_u$ is not 2-transitive on $\Gamma(u)$.

Suppose $(M, t) = (M_{11}, 11)$. Then $t - 1 = 10 = \frac{r(r-1)}{c_2}$, where $3 \leq r \leq 9$. Hence $r = 5$ or 6. If $r = 5$, then $c_2 = 2$; if $r = 6$, then $c_2 = 3$. Recall that $A_u$ is primitive but not 2-transitive on $\Gamma(u)$. Then $r \neq 6$. If $r = 5$, then $A_u \cong Z_5 : Z_k$ where $k < 5$ and $k|4$, this contradicts that $A_u$ is 2-transitive on $\Gamma_2(u)$, as $|\Gamma_2(u)| = 10$.

Suppose $(M, t) = (M_{22}, 22)$. Then $t - 1 = 21 = \frac{r(r-1)}{c_2}$. The stabilizer of $M_{22}$ is $PSL(3, 4)$. Since $21|r(r-1)$, it follows that $r = 7$ or 15. Since $A_u$ is primitive on $\Gamma(u)$, $M_u$ is transitive on $\Gamma(u)$. However, $PSL(3, 4)$ does not have a transitive representation on 7 or 15 vertices, a contradiction.

Suppose $(M, t) = (PSL(2, q), q + 1)$. Then $t - 1 = q = \frac{r(r-1)}{c_2}$. However, in this case, the stabilizer $A_u$ does not have a 2-transitive representation of degree $q$ where $q$ is a prime power, except $q = 5$. Assume $q = 5$. Then $|\Gamma_2(u)| = 5 = \frac{r(r-1)}{c_2}$. Recall that $3 \leq r \leq t - 2$. So $r = 3$, which is impossible.

Suppose $(M, t) = (M_{23}, 23)$. Then $t - 1 = 22 = \frac{r(r-1)}{c_2}$ and $M_u \cong M_{22}$. Since $11|r(r-1)$, it follows that $r = 11$ or 12. However, $M_{22}$ does not have a transitive representation on 11 or 12 vertices, a contradiction.

Finally, suppose $(M, t) = (A_n, n)$. Then $|\Gamma_2(u)| = n - 1 = \frac{r(r-1)}{c_2}$ where $3 \leq r \leq n - 2$. Since $M_u = A_{n-1}$ is transitive on $\Gamma(u)$, but $|\Gamma(u)| = r \leq n - 2$, which is impossible. □

We are ready to prove our second theorem.

**Proof of Theorem 2.** If $\Gamma$ has girth at least 4, then by Lemma 14, $\Gamma \cong C_n$ for some $n \geq 4$, $K_{r,r}$, or $K_{r+1,r+1} - (r + 1)K_2$ with $r \geq 3$. If $\Gamma$ has girth 3, then by Lemma 9, $\Gamma$ is either the halved 5-cube or the complement of the Higman-Sims graph. We complete the proof. □

### 4 Locally cyclic graphs

In this section, we prove Theorem 3, that is, determine the unique 2-distance-primitive graph which is locally cyclic.

**Proof of Theorem 3.** Suppose first that $\Gamma$ is a non-complete, connected, locally cyclic 2-distance-primitive graph of valency $n \geq 3$. Then $|\Gamma(u)| \cong C_n$ for each $u \in V(\Gamma)$. If $n = 3$, then $|\Gamma(u)| \cong C_3$, so $\Gamma \cong K_4$, contradicting that $\Gamma$ is non-complete. Hence $n \geq 4$. Since $\Gamma$ is 2-distance-primitive, the stabilizer $A_u$ is primitive on $\Gamma(u)$ where $A := \text{Aut}(\Gamma)$, and so the $A_u$-action on $\Gamma(u)$ does not have nontrivial blocks. As $|\Gamma(u)| \cong C_n$, it follows that $n$ is an odd integer, and so $n \geq 5$. 

By Theorem 1, $A_u$ acts faithfully on $\Gamma(u)$. As $[\Gamma(u)] \cong C_n$, $A_u = A_u^{\Gamma(u)} \leq \text{Aut}(C_n) = D_{2n} = \mathbb{Z}_n : \mathbb{Z}_2$. In particular, $\mathbb{Z}_n \leq A_u$ as $n$ is an odd integer and $A_u$ is transitive on $\Gamma(u)$. Further, since $A_u$ is primitive on $\Gamma_2(u)$, the normal subgroup $\mathbb{Z}_n$ is transitive and so regular on $\Gamma_2(u)$, so $|\Gamma_2(u)| = n$.

Let $(u, v, w)$ be a 2-geodesic. Since $\Gamma$ is non-complete, $[\Gamma(u)]$ is a non-complete graph, and so $|\Gamma(u) \cap \Gamma_2(v)| \geq 1$. If $|\Gamma(u) \cap \Gamma_2(v)| = 1$, then $n = 4$, as $[\Gamma(u)] \cong C_n$, contradicting that $n \geq 5$. Hence $|\Gamma(u) \cap \Gamma_2(v)| \geq 2$. Since $[\Gamma(u)] \cong C_n$ and $\Gamma(u) = \{v\} \cup (\Gamma(u) \cap \Gamma(v)) \cup (\Gamma(u) \cap \Gamma_2(v))$, it follows that the induced subgraph $[\Gamma(u) \cap \Gamma_2(v)]$ contains edges, and so $[\Gamma_2(v)]$ contains edges. Hence $[\Gamma_2(u)]$ contains edges. Recall that $n$ is odd, so $[\Gamma_2(u)]$ has even valency. Since $c_2 = n - 3$, $a_2 \leq 3$, so $a_2 = 2$, that is, $[\Gamma_2(u)]$ has valency 2. As $A_u$ is primitive on $\Gamma_2(u)$, it follows that

$$[\Gamma_2(u)] \cong C_n.$$

Let $z \in \Gamma_3(u) \cap \Gamma(u)$. Then $(u, v, w, z)$ is a 3-geodesic. Recall that $c_2 = n - 3$ and $a_2 = 2$, it follows that $b_2 = 1$, so $|\Gamma_3(u) \cap \Gamma(w)| = 1$, hence $\Gamma_3(u) \cap \Gamma(w) = \{z\}$. Since $(v, w, z)$ is a 2-geodesic, $[\Gamma(v) \cap \Gamma(z)] = n - 3$. Note that $\Gamma(v) = \{v\} \cup (\Gamma(u) \cap \Gamma(v)) \cup (\Gamma_2(u) \cap \Gamma(v))$, $|\Gamma_2(u) \cap \Gamma(v)| = n - 3$ and $(\{u\} \cup (\Gamma(u) \cap \Gamma(v)) \cap \Gamma(z) = \emptyset$. It follows that $\Gamma_2(u) \cap \Gamma(v) = \Gamma(v) \cap \Gamma(z)$. Hence $n - 3 = |\Gamma_2(u) \cap \Gamma(v)| = |\Gamma(v) \cap \Gamma(z)| \leq |\Gamma_2(u) \cap \Gamma(z)| \leq n$.

Since $\Gamma$ is 2-distance-transitive and $[\Gamma_3(u) \cap \Gamma(w)] = 1$, it follows that $\Gamma$ is 3-distance-transitive. Thus $[\Gamma_2(u) \cap \Gamma(z)] = c_3$, so $n - 3 \leq c_3 \leq n$. Counting the number of edges between $\Gamma_3(u)$ and $\Gamma_3(u)$, we get $n = c_3|\Gamma_3(u)|$. Hence $c_3$ divides $n$. Since $n - 3 \leq c_3 \leq n$, it follows that $c_3 = n - 3, n - 2, n - 1$ or $n$. Since $n - 1$ and $n$ are coprime and $c_3$ is a divisor of $n$, $c_3 \neq n - 1$. If $c_3 = n - 2$, then as $c_3|n$, $n = 3$ or $4$, contradicting that $n \geq 5$. If $c_3 = n - 3$, then as $c_3|n$, $n = 4$ or $6$, which is impossible, as $n \geq 5$ is odd. Therefore, $c_3 = n$, and so

$$|\Gamma_3(u)| = 1.$$

Thus $\Gamma_3(u) = \{z\}$.

Let $\Delta_1 = \Gamma(v) \cap \Gamma_2(u)$ and $\Delta_2 = \Gamma_2(u) \setminus \Delta_1$. Then $|\Delta_1| = n - 3$ and $|\Delta_2| = 3$. Set $\Gamma(u) = \{v_1 = v, v_2, \ldots, v_n\}$ and $\Gamma_2(u) = \{w_1 = w, w_2, \ldots, w_n\}$. Assume $(v_1, v_3, v_4, \ldots, v_n, v_3, v_1) \cong C_n$. Then $|\Gamma(v_1) \cap \Gamma(v_2)| = 2$. Suppose $\Gamma(v_1) \cap \Gamma(v_2) = \{u, w_1\}$. Then $\Gamma(v_2) \cap \Delta_1 = \{w_1\}$. Since $|\Gamma_2(u) \cap \Gamma(v_2)| = n - 3$, it follows that $|\Gamma(v_2) \cap \Delta_2| = n - 4 \leq 3$, and so $n \leq 7$. Thus $n = 5$ or $7$, as $n \geq 5$ is odd.

Suppose $n = 7$. Then $|\Delta_1| = 4$, $|\Delta_2| = 3$, and $\Delta_2 \subseteq \Gamma(v_3)$. Similarly, $\Delta_2 \subseteq \Gamma(v_3)$, as $(v_1, v_3)$ is also an arc. Thus $\Delta_2 \subseteq \Gamma(v_2) \cap \Gamma(v_3)$. Assume $\Delta_1 = \{w_1, w_2, w_3, w_4\}$ and $\Delta_2 = \{w_5, w_6, w_7\}$. Then $\Gamma(v_1) = \{u, v_2, v_3\} \cup \Delta_1$. Suppose $(u, v_2, w_1, w_2, w_3, w_4, v_3) \cong C_7 \cong \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$. Then $\Gamma(v_3) = \{u, v_1, v_3, w_4\} \cup \Delta_2$. Since $[\Gamma(v_3)] \cong C_7$ and $(v_4, u, v_1, v_4, w_5, w_6, w_7)$ is a 6-arc, it follows that $v_4$ is adjacent to $w_7$. Since $v_4 \in \Gamma_2(v_1)$, $[\Gamma(v_1) \cap \Gamma(v_4)] = 4$, so $[\Gamma(v_4) \cap \Delta_1] = 2$, hence $[\Gamma(v_3) \cap \Delta_2] = 2$, say $\Gamma(v_3) \cap \Delta_2 = \{w_7, w_j\}$. Note that $(v_5, u, v_3, w_7)$ is a 4-arc and $\Delta_2 \subseteq \Gamma(v_3)$. Hence $v_3$ is adjacent to both $w_7$ and $w_j$, contradicting that $[\Gamma(v_4)] \cong C_7$. Thus $n \neq 7$, and so $n = 5$, and $\Gamma$ is the icosahedron.

Conversely, assume that $\Gamma$ is the icosahedron. Then $[\Gamma(u)] \cong [\Gamma_2(u)] \cong C_5$ for each $u \in V(\Gamma)$. By Theorem 1.2 of [13], $\Gamma$ is 2-geodesic-transitive, and so it is 2-distance-primitive.

□

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