Two-distance-primitive graphs

Wei Jin[*] [†]
School of Mathematics and Statistics
Central South University
Changsha, Hunan, 410075, P.R.China
School of Statistics
Jiangxi University of Finance and Economics
Nanchang, Jiangxi, 330013, P.R.China
jinweipei82@163.com

Ci Xuan Wu[‡]
School of Statistics and Mathematics
Yunnan University of Finance and Economics
Kunming, 650221, P.R.China
wucixuan@gmail.com

Jin Xin Zhou
Department of Mathematics
Beijing Jiaotong University
Beijing, 100044, P.R.China
jxzhou@bjtu.edu.cn

Submitted: Jul 21, 2019; Accepted: Nov 18, 2020; Published: Dec 24, 2020

Abstract

A 2-distance-primitive graph is a vertex-transitive graph whose vertex stabilizer is primitive on both the first step and the second step neighborhoods. Let Γ be such a graph. This paper shows that either Γ is a cyclic graph, or Γ is a complete bipartite graph, or Γ has girth at most 4 and the vertex stabilizer acts faithfully on both the first step and the second step neighborhoods. Also a complete classification is given of such graphs satisfying that the vertex stabilizer acts 2-transitively on the second step neighborhood. Finally, we determine the unique 2-distance-primitive graph which is locally cyclic.

Mathematics Subject Classifications: 05E18, 20B25

1 Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively.

[*]Corresponding author.

[†]Supported by NSFC (11661039,12061034) and NSF of Jiangxi (20192ACBL21007,2018ACB21001, GJJ190273).

[‡]Supported by NSFC (11901512) and NSF of Yunnan (2019FD116).

For the group theoretic terminology not defined here we refer the reader to [9, 14]. The *diameter* of a graph Γ is the maximum distance occurring over all pairs of vertices. Let $u \in V(\Gamma)$ and i be a positive integer at most the diameter of Γ. We use $\Gamma_i(u)$ to denote the set of vertices at distance i with vertex u in Γ. Sometimes, $\Gamma_1(u)$ is also denoted by $\Gamma(u)$.

A transitive permutation group G is said to be acting *primitively* on a set Ω if it has only trivial blocks in Ω. If G acts primitively on Ω, then every nontrivial normal subgroup of G is transitive on Ω. There is a remarkable classification of finite primitive permutation groups mainly due to M. O'Nan and L. Scott, called the *O'Nan-Scott Theorem for primitive permutation groups*, see [26, 35]. They independently gave a classification of finite primitive groups, and proposed their result at the "Santa Cruz Conference in finite groups" in 1979. For more work on primitive groups, see [5, 21, 25, 32].

A graph Γ is said to be 2-*distance-transitive* if, for each $i \leqslant 2$, the automorphism group of Γ is transitive on the ordered pairs of vertices at distance i. The study of finite 2-distance-transitive graphs goes back to Higman's paper [18] in which "groups of maximal diameter" were introduced. These are permutation groups which act distance-transitively on some graph. Then 2-distance-transitive graphs have been studied extensively, see [11, 12, 15, 20, 33, 34].

In this paper, we investigate a family of graphs which has stronger transitivity than the family of 2-distance-transitive graphs, namely 2-distance-primitive graphs. A non-complete vertex-transitive graph Γ is said to be 2-*distance-primitive* if, for $i = 1, 2$ and for any vertex u, A_u is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$ where $A := \mathrm{Aut}(\Gamma)$. Clearly, every 2-distance-primitive graph is 2-distance-transitive. The converse is not true, for instance, the complete multipartite graph $\mathrm{K}_{m[n]}$ with $m \geqslant 3, n \geqslant 2$ is 2-distance-transitive but not 2-distance-primitive. (Its vertex set consists of m parts of size n, and it has edges between all pairs of vertices from distinct parts.) Hence the family of 2-distance-primitive graphs is properly contained in the family of 2-distance-transitive graphs. Many well-known graphs have the 2-distance-primitive property. For instance, the cyclic graph C_n is 2-distance-primitive whenever $n \geqslant 4$; the icosahedron (the graph in Figure 1) is 2-distance-primitive of valency 5; the family of 2-geodesic-transitive but not 2-arc-transitive graphs of prime valency provides an infinite family of such examples, refer to [13]. This family of graphs is also related to the class of well-known 'locally primitive graphs', see [19, 22, 23, 24, 30].

Our first theorem is a structural result and it shows that if a 2-distance-primitive graph is neither a cycle nor a complete bipartite graph, then its girth is 3 or 4.

Theorem 1. *Let Γ be a 2-distance-primitive graph. Then either $\Gamma \cong C_n$ for some $n \geqslant 4$, or Γ is a complete bipartite graph, or Γ has girth at most 4 and the vertex stabilizer acts faithfully on both the first step and the second step neighborhoods.*

The *complement graph* $\overline{\Gamma}$ of a graph Γ, is the graph with vertex $V(\Gamma)$, and two vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ. Recall that a permutation group G acting on Ω is said to be 2-*transitive* if it is transitive on the set of ordered pairs of distinct points in Ω.

A *d-cube* is a graph with vertex set $\Delta^d = \{(x_1, x_2, \ldots, x_d) | x_i \in \Delta\}$, where $\Delta = \{0, 1\}$,

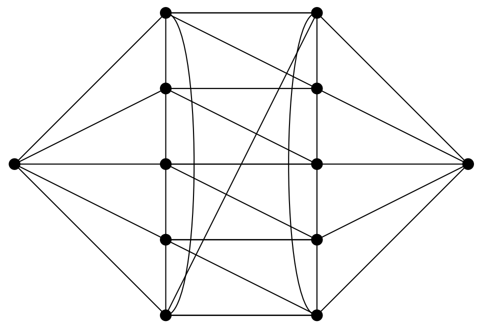

Figure 1: Icosahedron

and two vertices v and v' are adjacent if and only if they differ in exactly one coordinate. Let Y_d denote the graph with vertex set the same as a d-cube Γ, and two vertices are adjacent in Y_d if and only if they are at distance two in Γ. While Y_d is not connected, it has two isomorphic components on 2^{n-1} vertices, each of which is called a *halved d-cube*.

For a 2-distance-primitive graph, if its vertex stabilizer acts 2-transitively on the first step neighborhood, then it is well-known that this graph is 2-arc-transitive, and those graphs have been studied extensively, see [1, 10, 16, 29, 36, 37]. Our second theorem classifies the family of 2-distance-primitive graphs whose vertex stabilizer acts 2-transitively on the second step neighborhood.

Theorem 2. *Let Γ be a 2-distance-primitive graph of valency $r \geqslant 2$. Suppose that the vertex stabilizer of a vertex is 2-transitive on the second step neighborhood. Then Γ is one of the following graphs: C_n with $n \geqslant 4$, $\mathrm{K}_{r,r}$, $\mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$ with $r \geqslant 3$, the halved 5-cube, the complement graph of the Higman-Sims graph and the complement graph of the Gewirtz graph.*

A subgraph X of a graph Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U. A graph Γ is said to be *locally cyclic* if $[\Gamma(u)]$ is a cycle for every vertex u. In particular, the girth of a locally cyclic graph is 3. The following theorem determines the class of 2-distance-primitive graphs which are locally cyclic, and surprisingly, there is a unique such example.

Theorem 3. *Let Γ be a connected, non-complete, locally cyclic graph. Then Γ is 2-distance-primitive if and only if Γ is the icosahedron.*

2 Proof of Theorem 1

In the characterization of 2-distance-primitive graphs, the following constants are useful. Our definition is inspired by the concept of intersection arrays defined for the distance-regular graphs (see [4]).

Definition 4. Let Γ be an s-distance-transitive graph, $u \in V(\Gamma)$, and let $v \in \Gamma_i(u)$, $i \leqslant s$. Then the number of edges from v to $\Gamma_{i-1}(u)$, $\Gamma_i(u)$, and $\Gamma_{i+1}(u)$ does not depend on the choice of v and these numbers are denoted, respectively, by c_i, a_i and b_i.

Clearly we have that $a_i + b_i + c_i$ is equal to the valency of Γ whenever the constants are well-defined. Note that for 2-distance-primitive graphs, the constants are always well-defined for $i = 1,\ 2$.

For a connected graph Γ of diameter $d \geqslant 2$, we denote by Γ_d the graph whose vertices are those of Γ and whose edges are the 2-subsets of points at mutual distance d in Γ. Then, Γ is said to be *antipodal* if Γ_d is a disjoint union of complete graphs.

We prove our first theorem.

Proof of Theorem 1. If Γ has valency 2, then $\Gamma \cong C_n$ for some $n \geqslant 4$. In the remainder, we suppose that Γ has valency at least 3. Let $u \in V(\Gamma)$. Assume that Γ has girth at least 5. Then $c_2 = 1$, so every vertex of $\Gamma_2(u)$ is adjacent to exactly one vertex of $\Gamma(u)$, it follows that for each $v \in \Gamma(u)$, $\Gamma_2(u) \cap \Gamma(v)$ is a block of the A_u-action on $\Gamma_2(u)$. Since Γ has valency at least 3, $b_1 \geqslant 2$, and so $\Gamma_2(u) \cap \Gamma(v)$ is a nontrivial block, contradicting the fact that A_u is primitive on $\Gamma_2(u)$. Thus Γ has girth at most 4, that is, Γ has girth 3 or 4.

Suppose that Γ is not a complete bipartite graph. We denote by A_u^* and B_u^* the kernels of the A_u-action on $\Gamma(u)$ and $\Gamma_2(u)$, respectively. Then both A_u^* and B_u^* are normal subgroups of A_u. By the assumption, A_u is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so A_u^* acts either transitively or trivially on $\Gamma_2(u)$, and B_u^* acts either transitively or trivially on $\Gamma(u)$.

(i) Suppose A_u^* is transitive on $\Gamma_2(u)$. Note that for each $v \in \Gamma(u)$, A_u^* fixes $\Gamma_2(u) \cap \Gamma(v)$ setwise, so v is adjacent to all vertices of $\Gamma_2(u)$. Hence every vertex of $\Gamma(u)$ is adjacent to all vertices of $\Gamma_2(u)$, and so every vertex of $\Gamma_2(u)$ is also adjacent to all vertices of $\Gamma(u)$. Thus Γ has diameter 2 and $[\Gamma_2(u)]$ is an empty graph.

Suppose first that Γ has girth 3. Then Γ is antipodal. In particular, $\Gamma_2(u) \cup \{u\}$ is an antipodal block of A acting on $V(\Gamma)$, hence $|\Gamma_2(u)|+1$ divides $|V(\Gamma)| = 1+|\Gamma(u)|+|\Gamma_2(u)|$. Thus $\Gamma \cong \mathrm{K}_{m[b]}$ with $m \geqslant 3$ and $b = 1+|\Gamma_2(u)|$, contradicting the fact that A_u is primitive on $\Gamma(u)$. Suppose next that Γ has girth 4. By the previous argument, every vertex of $\Gamma(u)$ is adjacent to all vertices of $\Gamma_2(u)$, and every vertex of $\Gamma_2(u)$ is also adjacent to all vertices of $\Gamma(u)$. Hence $|\Gamma_2(u)| = |\Gamma(u)| - 1$, and the induced subgraph $[\Gamma(u) \cup \Gamma_2(u)]$ is a complete bipartite graph. Thus Γ is a complete bipartite graph, contradicts our assumption that Γ is not a complete bipartite graph.

Thus A_u^* is not transitive on $\Gamma_2(u)$, so A_u^* is trivial on $\Gamma_2(u)$. Then for any $v \in \Gamma(u)$, A_u^* fixes each vertex of $\Gamma(v)$, hence $A_u^* \leqslant A_v^*$. As Γ is connected, and by induction, A_u^* fixes all vertices of Γ, so $A_u^* = 1$. Thus A_u is faithful on $\Gamma(u)$.

(ii) Now we prove that A_u is faithful on $\Gamma_2(u)$. Suppose B_u^* is transitive $\Gamma(u)$. Note that for each $w \in \Gamma_2(u)$, B_u^* fixes $\Gamma(u) \cap \Gamma(w)$ setwise. So w is adjacent to all vertices of $\Gamma(u)$. Hence every vertex of $\Gamma_2(u)$ is adjacent to all vertices of $\Gamma(u)$. Thus Γ has diameter 2 and $[\Gamma_2(u)]$ is an empty graph.

If Γ has girth 4, then Γ is complete bipartite, contradicting the assumption that Γ is not a complete bipartite graph. If Γ has girth 3, then Γ is antipodal and $\Gamma_2(u) \cup \{u\}$ is an antipodal block, so $|\Gamma_2(u)| + 1$ divides $|V(\Gamma)| = 1 + |\Gamma(u)| + |\Gamma_2(u)|$. Thus $\Gamma \cong \mathrm{K}_{m[b]}$ with $m \geqslant 3$ and $b = 1 + |\Gamma_2(u)|$, so A_u is imprimitive on $\Gamma(u)$, a contradiction. Thus B_u^* is trivial on $\Gamma(u)$. Hence $B_u^* \leqslant A_u^* = 1$. Therefore A_u acts faithfully on $\Gamma_2(u)$. □

3 Proof of Theorem 2

We prove Theorem 2 by a series of lemmas. The first lemma shows that a 2-distance-transitive graph of girth 4 is unique, if its first step neighbor and second step neighbor have the same number of vertices.

Lemma 5. *Let Γ be a 2-distance-transitive graph of girth 4 and valency $r \geqslant 3$. If $|\Gamma_2(u)| = r$ for some $u \in V(\Gamma)$, then $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$.*

Proof. Assume that $|\Gamma_2(u)| = r$ for some $u \in V(\Gamma)$. Let (u, v, w, z) be a 3-geodesic. Since Γ is 2-distance-transitive with girth 4 and valency r, there are $r(r-1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$, and so $r(r-1) = c_2 \cdot |\Gamma_2(u)|$. By the assumption, $|\Gamma_2(u)| = r$, so we get $c_2 = r - 1$. Hence $|\Gamma(v) \cap \Gamma(z)| = c_2 = r - 1$, as (v, w, z) is a 2-geodesic. Note that $|\Gamma_2(u) \cap \Gamma(v)| = r - 1$ and $\Gamma(v) \cap \Gamma(z) \subseteq \Gamma_2(u) \cap \Gamma(v)$. It follows that $\Gamma_2(u) \cap \Gamma(v) = \Gamma(v) \cap \Gamma(z)$.

Since $r \geqslant 3$, $c_2 = r - 1 \geqslant 2$. Hence there exists a vertex $v_2 \in \Gamma(u) \setminus \{v\}$ such that (v_2, w, z) is a 2-geodesic. So $|\Gamma(v_2) \cap \Gamma(z)| = r - 1$, this indicates that $\Gamma_2(u) \cap \Gamma(v_2) = \Gamma(v_2) \cap \Gamma(z)$.

Suppose that $\Gamma_2(u) \cap \Gamma(v) = \Gamma_2(u) \cap \Gamma(v_2)$. Since Γ has girth 4, it follows that $(\Gamma_2(u) \cap \Gamma(v)) \cup \{u\} = \Gamma(v) \cap \Gamma(v_2)$, hence $|\Gamma(v) \cap \Gamma(v_2)| = r$, contradicting the fact that $|\Gamma(v) \cap \Gamma(v_2)| = c_2 = r - 1$, as (v, u, v_2) is a 2-geodesic. Thus $\Gamma_2(u) \cap \Gamma(v) \neq \Gamma_2(u) \cap \Gamma(v_2)$, so $(\Gamma_2(u) \cap \Gamma(v)) \cup (\Gamma_2(u) \cap \Gamma(v_2)) = \Gamma_2(u)$. By the previous argument, $\Gamma_2(u) \cap \Gamma(v) = \Gamma(v) \cap \Gamma(z)$ and $\Gamma_2(u) \cap \Gamma(v_2) = \Gamma(v_2) \cap \Gamma(z)$. Thus $\Gamma_2(u) \subseteq \Gamma(z)$. Since $r = |\Gamma_2(u)| \subseteq |\Gamma(z)| = r$, it follows that $\Gamma_2(u) = \Gamma(z)$. Therefore, $\Gamma_3(u) = \{z\}$ and Γ has diameter 3. Precisely, this graph is $\mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$. □

Lemma 6. *Let Γ be a 2-arc-transitive graph of diameter 2 and girth 5. Then Γ is one of the following graphs: C_5, the Petersen graph, or the Hoffman-Singleton graph.*

Proof. Since Γ has diameter 2 and girth 5, Γ is a Moore graph. Then it follows from [4, Theorem 6.7.1] that Γ has valency 2, 3, 7 or 57. By [2] or [4, p.207, Remark (i)], the valency 57 case does not occur, and so Γ has valency 2, 3 or 7. Further, by [4, p.207, Remark (i)] or [17, p.206], if Γ has valency 2, then Γ is C_5; if Γ has valency 3, then Γ is the Petersen graph; and if Γ has valency 7, then Γ is the Hoffman-Singleton graph. □

The socle of a 2-transitive group is either elementary abelian or non-regular non-abelian simple, see [14, Theorem 4.1B], and in the latter case, the socle is primitive, see [14, p.244].

Lemma 7. *Let Γ be a 2-distance-primitive graph of diameter 2 and girth 4. If Γ is 2-arc-transitive, then Γ is one of the following graphs: $\mathrm{K}_{m,m}$ with $m \geqslant 2$, Higman-Sims graph, 2-cube, the Gewirtz graph or the folded 5-cube.*

Proof. Suppose that Γ is 2-arc-transitive. Let $A := \mathrm{Aut}(\Gamma)$ and let $u \in V(\Gamma)$. Assume that A is not primitive on $V(\Gamma)$. Then A has some nontrivial blocks on $V(\Gamma)$, and say Δ_i. Since the graph Γ is arc-transitive, each Δ_i does not contain edges of Γ. Let $u, u' \in \Delta_1$. Then $u' \in \Gamma_2(u)$ and $\Delta_1 \subseteq \{u\} \cup \Gamma_2(u)$, as Γ has diameter 2. Since A_u fixes the block Δ_1 and it is also transitive on $\Gamma_2(u)$, it follows that $\{u\} \cup \Gamma_2(u) \subseteq \Delta_1$, so $\{u\} \cup \Gamma_2(u) = \Delta_1$. Thus $\{u\} \cup \Gamma_2(u)$ is a block of Γ. By the vertex-transitivity of Γ, we know that $\Gamma(u)$ is a union of some blocks. If $\Gamma(u)$ contains more than one block, then Γ has girth 3, contradicting the fact that Γ has girth 4. Thus $\Gamma(u)$ is a block of cardinality $|\Delta_1|$. Since Γ has diameter 2, it follows that $\Gamma \cong \mathrm{K}_{m,m}$ where $m = |\Delta_1| \geqslant 2$. In the remainder, we suppose that A acts primitively on $V(\Gamma)$.

Since Γ is 2-arc-transitive, the stabilizer A_u is 2-transitive on $\Gamma(u)$, and it is well-known that this 2-transitive action is of type either affine or almost simple. Suppose that A_u is an affine group. Since A_u is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$, it follows that its socle is regular on both $\Gamma(u)$ and $\Gamma_2(u)$, and so $|\Gamma(u)| = |\Gamma_2(u)|$. Then by Lemma 5, $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$ with diameter 3, contradicting the assumption that Γ has diameter 2. Thus A_u acts 2-transitively on $\Gamma(u)$ of almost simple type, and either $A_u \cong P\Gamma L(2, 8)$ or the socle of A_u is 2-transitive. Again as Γ is 2-arc-transitive of diameter 2, A_u is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, so A is a primitive rank 3 group. Since A_u is 2-transitive on $\Gamma(u)$, A has a 2-transitive suborbit, it follows from [31, Theorem A] that A is primitive of type either affine or almost simple. In particular, the socle of A_u is 2-transitive.

Suppose that A is an affine group. Then A is completely listed in [27]. The stabilizer A_u and subdegrees are given in Tables 12, 13 and 14 of [27]. The groups in Tables 12 and 14 are not 2-transitive. Hence A_u is in Table 13. Then by Theorem (B) of [27], $R \leqslant A_u \leqslant N_{GL(d,p)}(R)$ where R is an r-group, A_u is not almost simple, a contradiction. Hence A is not an affine group.

Thus A is an almost simple primitive group. If $A = S_n$ or A_n, then by [7, Theorem 4.5] or [10, p.4], Γ has parameter $c_2 = 2$, and Γ is one of the following graphs: a cube, a folded d-cube, or the incidence graph of the Paley design on 11 points. Since A is primitive on $V(\Gamma)$, Γ is not a bipartite graph, so Γ is a cube or a folded d-cube. Note that Γ has diameter 2. Hence Γ is the 2-cube or the folded 5-cube (folded d-cube has diameter $[d/2]$).

The primitive rank 3 groups in which the socle is either an exceptional group of Lie type or a sporadic group are listed in [28]. Let A be a primitive rank 3 group in [28] with socle L, and let H be the stabilizer in L of a vertex u. If L is an exceptional simple group of Lie type, then L, H and the subdegrees k, l are listed in Table 1 of [28]. Since L is the socle of A and $H = L_u$, H is a normal subgroup of A_u. Since A_u is almost simple, if $H \neq 1$, then H is the socle of A_u and it is an non-abelian simple 2-transitive group. Thus A is not in Table 1 of [28]. We inspect the groups in Table 2 of [28]. Then $(L, H) = (HS, M_{22})$ is the unique candidate, and it provides the example Higman-Sims graph.

Finally, suppose that A is an almost simple group of classical type. Then A is investigated in [6]. Since A is primitive and A_u acts primitively on both $\Gamma(u)$ and $\Gamma_2(u)$,

A is completely determined in [6, Theorem 1.1]. As A_u is almost simple, we can easily conclude that the two possible cases are that $(soc(A), soc(A_u)) = (PSL(3,4), A_6)$ and $(soc(A), soc(A_u)) = (PSU(4,3), PSL(3,4))$. For the former case, by Magma [3], the two nontrivial subdegrees of A are 10 and 45. This produces the Gewirtz graph. For the latter case, again by Magma [3], the two nontrivial subdegrees of A are 56 and 105, and hence A_u does not provide any 2-transitive representation on each suborbit, which is not possible. $\square$

Lemma 8. *Let Γ be a 2-distance-primitive graph. If $a_2 = 0$, then either $\Gamma \cong C_n$ with $n \geqslant 6$ or Γ has girth 4.*

Proof. Let $u \in V(\Gamma)$, $i \in \{1,2\}$ and let $A := \mathrm{Aut}(\Gamma)$. Assume that the induced subgraph $[\Gamma_i(u)]$ is disconnected. Then each disconnected component Δ of $[\Gamma_i(u)]$ is a block of the A_u-action on $\Gamma_i(u)$. Since A_u is primitive on $\Gamma_i(u)$, it follows that Δ is a trivial block, that is, Δ has size 1. Thus $[\Gamma_i(u)]$ is an empty graph. Therefore, $[\Gamma_i(u)]$ is either connected or empty.

Suppose that $a_2 = 0$. Let (u, v) be an arc. Then the two induced subgraphs $[\Gamma_2(u)]$ and $[\Gamma_2(v)]$ are empty graphs. Hence $[\Gamma(u) \cap \Gamma_2(v)]$ is an empty graph. Assume that Γ has girth 3. Then $[\Gamma(u)]$ is not an empty graph, and by the previous argument $[\Gamma(u)]$ is connected. Set $|\Gamma(u) \cap \Gamma(v)| = x \geqslant 1$. Note that $\Gamma(u) = \{v\} \cup (\Gamma(u) \cap \Gamma(v)) \cup (\Gamma(u) \cap \Gamma_2(v))$. Hence every vertex v' of $\Gamma(u) \cap \Gamma_2(v)$ is adjacent to x vertices of $\Gamma(u) \cap \Gamma(v)$, so $\Gamma(u) \cap \Gamma(v) = \Gamma(u) \cap \Gamma(v')$. Since $[\Gamma(u)]$ is vertex-transitive, it follows that $\{v\} \cup (\Gamma(u) \cap \Gamma_2(v))$ is a nontrivial block of the A_u-action on $\Gamma(u)$, which is a contradiction, as A_u is primitive on $\Gamma(u)$. Thus Γ has girth at least 4, and by Theorem 1, either $\Gamma \cong C_n$ with $n \geqslant 6$ or Γ has girth exactly 4. $\square$

Lemma 9. *Let Γ be a 2-distance-primitive graph of girth 3. Let $A := \mathrm{Aut}(\Gamma)$ and let $u \in V(\Gamma)$. Suppose that A_u is 2-transitive on $\Gamma_2(u)$. Then Γ is one of the following graphs: the halved 5-cube, the complement of the Gewirtz graph or the complement of the Higman-Sims graph.*

Proof. Since A_u is 2-transitive on $\Gamma_2(u)$, it follows that the induced subgraph $[\Gamma_2(u)]$ is either a complete graph or an empty graph. If $[\Gamma_2(u)]$ is an empty graph, then $a_2 = 0$. Since Γ has girth 3, $\Gamma \ncong C_n$ for any $n \geqslant 6$, and by Lemma 8, Γ has girth 4, a contradiction. Hence $[\Gamma_2(u)]$ is a complete graph.

Let (u, v, w) be a 2-geodesic. Assume that Γ has diameter at least 3. Let $z \in \Gamma_3(u) \cap \Gamma(w)$. Then $z \in \Gamma_2(v)$. However, z is not adjacent to any vertex of $\Gamma(u) \cap \Gamma_2(v)$, contradicting the fact that $[\Gamma_2(v)]$ is a complete graph. Thus Γ has diameter 2.

Suppose that A is not primitive on $V(\Gamma)$. Then A has some nontrivial blocks on $V(\Gamma)$, and say Δ_i. Since Γ is arc-transitive, each Δ_i does not contain edges of Γ. Let $u, u' \in \Delta_1$. Note that Γ has diameter 2. Then $u' \in \Gamma_2(u)$. Since A_u fixes the block Δ_1 and also it acts transitively on $\Gamma_2(u)$, it follows that $\{u\} \cup \Gamma_2(u) \subseteq \Delta_1$. As Δ_1 does not contain any edge, it follows that $\{u\} \cup \Gamma_2(u) = \Delta_1$. Thus $\{u\} \cup \Gamma_2(u)$ is a block of the A-action on $V(\Gamma)$ and $|\Gamma_2(u)| = 1$, as $[\Gamma_2(u)]$ is a complete graph. Since Γ is 2-distance-transitive of diameter 2, it follows that $\Gamma \cong \mathrm{K}_{m[2]}$ for some $m \geqslant 3$, contradicting that A_u is primitive on $\Gamma(u)$. Thus A is primitive on $V(\Gamma)$.

Assume that $\Gamma_2(u) \subseteq \Gamma(v)$. Then as A_u is transitive on $\Gamma(u)$, each vertex of $\Gamma(u)$ is adjacent to all vertices of $\Gamma_2(u)$, and so each $w_i \in \Gamma_2(u)$ is adjacent to all vertices of $\Gamma(u)$. Hence $|\Gamma(w_i)| \geqslant |\Gamma(u)| + |\Gamma_2(u)| - 1$, as $[\Gamma_2(u)]$ is a complete graph. Since $|\Gamma(w_i)| = |\Gamma(u)|$, it follows that $|\Gamma_2(u)| = 1$. Thus $\{u\} \cup \Gamma_2(u)$ is a block of the A-action on $V(\Gamma)$, contradicting that A is primitive on $V(\Gamma)$. Hence $\Gamma_2(u) \nsubseteq \Gamma(v)$, and there exists a vertex of $\Gamma_2(u)$ that is not adjacent to v. Therefore, $\overline{\Gamma}$ also has diameter 2.

Since A_u is 2-transitive on $\Gamma_2(u)$ and $\overline{\Gamma}(u) = \Gamma_2(u)$, it follows that $\overline{\Gamma}$ is a 2-arc-transitive graph. By the previous argument, $\overline{\Gamma}$ has diameter 2, so $\overline{\Gamma}$ has girth 4 or 5. If $\overline{\Gamma}$ has girth 5, then by Lemma 6, $\overline{\Gamma}$ is one of: C_5, Petersen graph or Hoffman-Singleton graph. If $\overline{\Gamma}$ is C_5, then Γ is C_5, contradicting that Γ has girth 3. Assume that $\overline{\Gamma}$ is the Petersen graph or the Hoffman-Singleton graph. Then $|\overline{\Gamma}_2(u) \cap \overline{\Gamma}(v)| = k-1$ where $|\overline{\Gamma}(u)| = k$, and so $\overline{\Gamma}_2(u) \cap \overline{\Gamma}(v)$ is a block of the A_u action on $\overline{\Gamma}_2(u)$, A_u is not primitive on $\overline{\Gamma}_2(u)$. Since $\Gamma(u) = \overline{\Gamma}_2(u)$, A_u is not primitive on $\Gamma(u)$, a contradiction. Thus $\overline{\Gamma}$ has girth 4. Then it follows from Lemma 7 that $\overline{\Gamma}$ is one of the following graphs: $\mathrm{K}_{m,m}$ with $m \geqslant 2$, Higman-Sims graph, the Gewirtz graph, 2-cube or the folded 5-cube. Since the complement graphs of both the 2-cube and $\mathrm{K}_{m,m}$ with $m \geqslant 2$ are disconnected, $\overline{\Gamma}$ is neither of those two graphs, and so $\overline{\Gamma}$ is the Higman-Sims graph, the Gewirtz graph or the folded 5-cube. Thus Γ is the halved 5-cube, the complement of the Gewirtz graph or the complement of the Higman-Sims graph. □

We cite two lemmas which will be used in the remaining.

Lemma 10. ([8, p.9, Notes (1)]) *Let G be a non-abelian simple group. Suppose that G has more than one 2-transitive permutation representation. Then G and its degree n are in one line of Table 1.*

Table 1: Nonsolvable 2-transitive groups with two representations

T	n
$A_5 \cong PSL(2,4) \cong PSL(2,5)$	5, 6
$A_6 \cong PSL(2,9)$	6, 10
$PSL(2,7) \cong PSL(3,2)$	7, 8
A_7	7, 15
$A_8 \cong PSL(4,2)$	8, 15
$PSL(2,8)$	9, 28
$PSL(2,11)$	11, 12
M_{11}	11, 12
$PSp(2d,2), d > 2$	$2^{2d-1} + 2^{d-1}$, $2^{2d-1} - 2^{d-1}$

The following well-known result is mainly due to Burnside.

Lemma 11. ([14, Theorem 3.5B]) *A primitive permutation group G of prime degree p is either 2-transitive, or solvable and $G \leqslant AGL(1,p)$.*

Lemma 12. *Let Γ be a 2-distance-transitive graph of prime valency p. Let $u \in V(\Gamma)$ and $A := \mathrm{Aut}(\Gamma)$. Suppose that A_u is 2-transitive on $\Gamma_2(u)$. Then $\Gamma \cong \mathrm{K}_{p+1,p+1} - (p+1)\mathrm{K}_2$, $\mathrm{K}_{p,p}$ with $p \geqslant 3$ or C_n with $n \geqslant 4$.*

Proof. If $p = 2$, then $\Gamma \cong C_n$ for some $n \geqslant 4$. In the remainder, we suppose that $p \geqslant 3$. Since Γ has prime valency, A_u is primitive on $\Gamma(u)$. Since A_u is 2-transitive on $\Gamma_2(u)$, A_u is also primitive on $\Gamma_2(u)$. It follows from Theorem 1 that either $\Gamma \cong \mathrm{K}_{p,p}$ or A_u is faithful on both $\Gamma(u)$ and $\Gamma_2(u)$. Suppose that $\Gamma \ncong \mathrm{K}_{p,p}$. Then $A_u \cong A_u^{\Gamma(u)} \cong A_u^{\Gamma_2(u)}$.

Assume that A_u is not 2-transitive on $\Gamma(u)$. Then by Lemma 11, $A_u \cong \mathbb{Z}_p : \mathbb{Z}_r$ where $r|p-1$ and $r < p-1$. Since A_u is 2-transitive on $\Gamma_2(u)$, it follows that the normal subgroup $\mathbb{Z}_p$ is transitive on $\Gamma_2(u)$, and so $\mathbb{Z}_p$ is regular on $\Gamma_2(u)$. Hence $|\Gamma_2(u)| = p$. However, as $r < p-1$, $\mathbb{Z}_p : \mathbb{Z}_r$ does not have a 2-transitive representation on p letters, which is a contradiction.

Thus A_u is 2-transitive on $\Gamma(u)$, and so Γ has girth 4. Assume first that A_u is solvable. Then the socle of A_u is regular on both $\Gamma(u)$ and $\Gamma_2(u)$, and so $|\Gamma(u)| = |\Gamma_2(u)| = p$. It follows from Lemma 5 that $\Gamma \cong \mathrm{K}_{p+1,p+1} - (p+1)\mathrm{K}_2$.

Now assume that A_u is non-solvable. Suppose A_u has more than one 2-transitive representation. Then by Lemma 10, the socle T of A_u and its degree n are listed in Table 1. Note that neither $2^{2d-1} + 2^{d-1} = 2^{d-1}(2^d + 1)$ nor $2^{2d-1} - 2^{d-1} = 2^{d-1}(2^d - 1)$ is a prime whenever $d > 2$. Hence T and its degree n are listed in Table 2.

Table 2:

T	n
$A_5 \cong PSL(2,4) \cong PSL(2,5)$	5, 6
$PSL(2,7) \cong PSL(3,2)$	7, 8
A_7	7, 15
$PSL(2,11)$	11, 12
M_{11}	11, 12

Since A_u is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, it follows that $p(p-1) = c_2 \cdot |\Gamma_2(u)|$. Hence $|\Gamma_2(u)|$ is a divisor of $p(p-1)$. Since p is a prime, by Table 2, $(p, |\Gamma_2(u)|) \in \{(5,6),(7,8),(11,12),(7,15)\}$. However, for any such a pair $(p, |\Gamma_2(u)|)$, the integer $|\Gamma_2(u)|$ is not a divisor of $p(p-1)$, which is a contradiction. Therefore, A_u has exactly one 2-transitive representation, so $|\Gamma(u)| = |\Gamma_2(u)| = p$. Again, by Lemma 5, $\Gamma \cong \mathrm{K}_{p+1,p+1} - (p+1)\mathrm{K}_2$. □

Lemma 13. *Let Γ be a 2-arc-transitive graph of valency 6. Then $(a_1, c_2) \neq (0, 3)$.*

Proof. Suppose that $(a_1, c_2) = (0, 3)$. Then $b_1 = 5$, and $|\Gamma_2(u)| = 10$. Set $\Gamma(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\Gamma_2(u) \cap \Gamma(v_1) = \{w_1, w_2, w_3, w_4, w_5\}$. We suppose that $\Gamma(u) \cap$

$\Gamma(w_1) = \{v_1, v_2, v_3\}$, as $c_2 = 3$. Since (v_1, u, v_2) is a 2-arc, $|\Gamma(v_1) \cap \Gamma(v_2)| = 3$, set $\Gamma(v_1) \cap \Gamma(v_2) = \{u, w_1, w_2\}$. Then $|\Delta_1| = 3$ where $\Delta_1 = (\Gamma_2(u) \cap \Gamma(v_2)) \setminus \Gamma(v_1)$.

Assume that v_3 and w_2 are adjacent. Then $\Gamma(v_1) \cap \Gamma(v_3) = \{u, w_1, w_2\} = \Gamma(v_2) \cap \Gamma(v_3)$. Thus $|\Delta_2| = 3$ where $\Delta_2 = (\Gamma_2(u) \cap \Gamma(v_3)) \setminus (\Gamma(v_1) \cup \Gamma(v_2))$. Note that $\Gamma_2(u) \cap \Gamma(v_1), \Delta_1$ and Δ_2 pair-wise have empty intersection and $(\Gamma_2(u) \cap \Gamma(v_1)) \cup \Delta_1 \cup \Delta_2 \subseteq \Gamma_2(u)$, so $|\Gamma_2(u)| \geqslant |\Gamma_2(u) \cap \Gamma(v_1)| + |\Delta_1| + |\Delta_2| = 11$, contradicting the fact that $|\Gamma_2(u)| = 10$. Hence v_3 and w_2 are non-adjacent.

Therefore $\Gamma(u) \cap \Gamma(w_2) = \{v_1, v_2, x\}$ for some $x \in \{v_4, v_5, v_6\}$, and $\Gamma(u) \cap (\Gamma(w_1) \cup \Gamma(w_2)) = \{v_1, v_2, v_3, x\}$. In particular, each $y \in \{v_4, v_5, v_6\} \setminus \{x\}$ is adjacent to neither w_1 nor w_2. As $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) = \{w_1, w_2\}$, it follows that $|\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(y)| = 0$. Let $A := \mathrm{Aut}(\Gamma)$. As $|\Gamma(u)| = 6$, it is well-known that there are only four 2-transitive permutation groups of degree 6, namely A_5, S_5, A_6 and S_6, see for instance [14, p.59-60]. Further, all these four permutation groups are 3-transitive on $\Gamma(u)$. Thus $A_{u,v_1}^{\Gamma(u)}$ is transitive between sets $\{v_2, v_3\}$ and $\{v_2, y\}$. Recall that $|\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(y)| = 0$. It follows that $|\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3)| = 0$. However, $\Gamma_2(u) \cap \Gamma(v_1) \cap \Gamma(v_2) \cap \Gamma(v_3) = \{w_1\}$, a contradiction. Therefore, $(a_1, c_2) \neq (0, 3)$. □

Lemma 14. *Let Γ be a 2-distance-primitive graph of valency r and girth at least 4. Let $A := \mathrm{Aut}(\Gamma)$ and let $u \in V(\Gamma)$. Suppose that A_u is 2-transitive on $\Gamma_2(u)$. Then $\Gamma \cong C_n$ with $n \geqslant 4$, $\mathrm{K}_{r,r}$, or $\mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$ with $r \geqslant 3$.*

Proof. If $r = 2$, then $\Gamma \cong C_n$ with $n \geqslant 4$. In the remainder, we assume that $r \geqslant 3$. Let (u, v, w) be a 2-geodesic. Since Γ has girth at least 4, the induced subgraph $[\Gamma(u)]$ is an empty graph. By the assumption, A_u is 2-transitive on $\Gamma_2(u)$, so $[\Gamma_2(u)]$ is either complete or empty. Assume that $[\Gamma_2(u)]$ is a complete graph. Then $[\Gamma_2(u) \cap \Gamma(v)]$ is a complete graph. Since Γ has valency at least 3 and girth at least 4, $b_1 = |\Gamma_2(u) \cap \Gamma(v)| \geqslant 2$, so (v, x, y) is a triangle for any two distinct vertices $x, y \in \Gamma_2(u) \cap \Gamma(v)$, contradicting the fact that Γ has girth at least 4.

Thus $[\Gamma_2(u)]$ is an empty graph. Since $b_1 \geqslant 2$, there exists a vertex $w_1 \in \Gamma_2(u) \cap \Gamma(v)$ such that $w_1 \neq w$. Then (w, v, w_1) is a 2-geodesic. Since $A_{u,w}$ is transitive on $\Gamma_2(u) \setminus \{w\}$, it follows that for any $w' \in \Gamma_2(u) \setminus \{w\}$, $A_{u,w}$ is transitive between w' and w_1, and so $w' \in \Gamma_2(w)$. Hence $\Gamma_2(u) \setminus \{w\} \subseteq \Gamma_2(w)$. As $|\Gamma_2(u) \setminus \{w\}| = |\Gamma_2(w)| - 1$, it follows that

$$\{w\} \cup \Gamma_2(w) = \{u\} \cup \Gamma_2(u). \quad (*)$$

If Γ has diameter at least 4, then there exists a vertex $z \in \Gamma_4(u) \cap \Gamma_2(w)$, contradicting $(*)$. Thus Γ has diameter at most 3.

Assume that Γ has diameter 2. Recall that both $[\Gamma(u)]$ and $[\Gamma_2(u)]$ are empty graphs. Hence every vertex of $\Gamma_2(u)$ is adjacent to all vertices of $\Gamma(u)$, and so $\Gamma \cong \mathrm{K}_{r,r}$.

Now suppose that Γ has diameter 3. Let $z \in \Gamma_3(u) \cap \Gamma(w)$. Then (u, v, w, z) is a 3-geodesic. Assume $b_2 = 1$. Then $|\Gamma_3(u) \cap \Gamma(w)| = 1$. Since $[\Gamma_2(u)]$ is an empty graph, it follows that $|\Gamma(u) \cap \Gamma(w)| = r - 1$. Note that there are $r(r-1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$. Thus $|\Gamma_2(u)| = r$. It follows from Lemma 5 that $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$.

Now assume that $b_2 \geqslant 2$. Then $|\Gamma_3(u)| \geqslant 2$. If z is adjacent to some $z' \in \Gamma_3(u)$, then $z' \in \Gamma_2(w) \cup \Gamma(w)$. By $(*)$, $z' \notin \Gamma_2(w)$, so $z' \in \Gamma(w)$, hence (z, w, z') is a triangle, contradicting the fact that Γ has girth at least 4. Thus $\Gamma_3(u) \cap \Gamma(z) = \emptyset$. Since Γ has diameter 3, it follows that $\Gamma(z) \subseteq \Gamma_2(u)$. As w is any vertex of $\Gamma_2(u)$ and z is any vertex of $\Gamma_3(u) \cap \Gamma(w)$, it follows that $[\Gamma_3(u)]$ is an empty graph. Therefore,

$$\Gamma \text{ is a diameter 3 bipartite graph.} \quad (**)$$

Setting the two biparts of Γ are $\Delta_1 = \{u\} \cup \Gamma_2(u)$ and $\Delta_2 = \Gamma(u) \cup \Gamma_3(u)$. Since A_u is 2-transitive on $\Gamma_2(u)$, $A_{\Delta_1}^{\Delta_1}$ is 3-transitive on Δ_1. Since Γ is vertex-transitive, also $A_{\Delta_2}^{\Delta_2}$ is 3-transitive on Δ_2. It is well-known that a 2-transitive permutation group is type either affine or almost simple. Assume first that the $A_{\Delta_i}^{\Delta_i}$-action on Δ_i is the affine type. Suppose $A_{\Delta_i}^{\Delta_i}$ is solvable. Then $(A_{\Delta_i}^{\Delta_i})_u$ is solvable. As A_u is primitive on both $\Gamma(u)$ and $\Gamma_2(u)$, it follows that the socle of A_u is regular on both $\Gamma(u)$ and $\Gamma_2(u)$, hence $|\Gamma(u)| = |\Gamma_2(u)|$. By Lemma 5, $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$, contradicting that $b_2 \geqslant 2$. Suppose $A_{\Delta_i}^{\Delta_i}$ is non-solvable. Then as $A_{\Delta_i}^{\Delta_i}$ is 3-transitive on Δ_i of affine type, it follows that $|\Delta_i|$ and $(A_{\Delta_i}^{\Delta_i})_u$ are listed in [9, p.195], and inspecting the candidates, $|\Delta_i|$ and $(A_{\Delta_i}^{\Delta_i})_u$ are one of the following cases: 1) $|\Delta_i| = q^d$ and $SL(d,q) \lhd (A_{\Delta_i}^{\Delta_i})_u \leqslant \Gamma L(d,q)$; 2) $|\Delta_i| = q^{2d}$ and $Sp(d,q) \lhd (A_{\Delta_i}^{\Delta_i})_u$, $d \geqslant 2$; 3) $|\Delta_i| = q^6$ and $G_2(q) \lhd (A_{\Delta_i}^{\Delta_i})_u \leqslant \Gamma L(d,q)$, q is even. In those cases, the socle of $(A_{\Delta_i}^{\Delta_i})_u$ is non-solvable. Since $(A_{\Delta_i}^{\Delta_i})_u$ is a 2-transitive group, we know that $(A_{\Delta_i}^{\Delta_i})_u$ is 2-transitive of almost simple type. Thus $|\Delta_i| - 1$ and the socle of $(A_{\Delta_i}^{\Delta_i})_u$ are listed in [9, p.197], by inspecting the candidates, they do not occur.

Table 3: Non-solvable k-transitive groups with $k \geqslant 3$

M	degree t
A_t, $t \geqslant 5$	t
$PSL(2,q)$, q is a prime power, $q \neq 2,3$	$q+1$
M_{11}	11
M_{11}	12
M_{12}	12
M_{22}	22
M_{23}	23
M_{24}	24

Thus the 2-transitive action of $A_{\Delta_i}^{\Delta_i}$ on Δ_i is the almost simple type. By [9, p.196-197], the socle M of $A_{\Delta_i}^{\Delta_i}$ and $|\Delta_i| = t$ are in one of the lines of Table 3. Since A_u is transitive on both $\Gamma(u)$ and $\Gamma_2(u)$, there are $r(r-1)$ edges between $\Gamma(u)$ and $\Gamma_2(u)$, and so

$$r(r-1) = c_2 \cdot |\Gamma_2(u)| = c_2(t-1). \quad (1)$$

Recall that $3 \leqslant r \leqslant t-2$. Suppose $t-1$ is a prime integer. Then by equation (1), $t-1|r(r-1)$, a contradiction. Thus $t-1$ is not a prime. Hence $t \neq 12, 24$.

Suppose that A_u is 2-transitive on $\Gamma(u)$. If A_u has exactly one 2-transitive permutation representation, then $|\Gamma(u)| = |\Gamma_2(u)|$, and by Lemma 5, $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$, contradicts that $b_2 \geqslant 2$. Thus A_u has more than one 2-transitive permutation representation. Then by Lemma 10, the socle of A_u and its degree n are in one line of Table 1. If

r is a prime, then by Lemma 12, $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$ with $r \geqslant 3$, a contradiction. Thus r is not a prime. By equation (1), $r(r-1) = c_2|\Gamma_2(u)|$. Since $\Gamma \ncong \mathrm{K}_{r,r}$, $c_2 \neq r$, so $c_2 \leqslant r-1$. If $c_2 = r-1$, then $|\Gamma_2(u)| = r$, and by Lemma 5, $\Gamma \cong \mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$, contradicts that $b_2 \geqslant 2$. Assume $c_2 < r-1$. Then $t - 1 = |\Gamma_2(u)| > r$. By checking Tables 1 and 3, the pair $(r, |\Gamma_2(u)|) \in \{(6, 10), (8, 15), (9, 28)\}$. It follows from Lemma 13 that $(a_1, c_2) \neq (0, 3)$, so $(r, |\Gamma_2(u)|) \neq (6, 10)$. However, if $(r, |\Gamma_2(u)|) = (8, 15)$ or $(9, 28)$, then $|\Gamma_2(u)|$ is not a divisor of $r(r-1)$, a contradiction. Therefore, A_u is not 2-transitive on $\Gamma(u)$.

Suppose $(M, t) = (M_{11}, 11)$. Then $t - 1 = 10 = \frac{r(r-1)}{c_2}$, where $3 \leqslant r \leqslant 9$. Hence $r = 5$ or 6. If $r = 5$, then $c_2 = 2$; if $r = 6$, then $c_2 = 3$. Recall that A_u is primitive but not 2-transitive on $\Gamma(u)$. Then $r \neq 6$. If $r = 5$, then $A_u \cong \mathbb{Z}_5 : \mathbb{Z}_k$ where $k < 5$ and $k|4$, this contradicts that A_u is 2-transitive on $\Gamma_2(u)$, as $|\Gamma_2(u)| = 10$.

Suppose $(M, t) = (M_{22}, 22)$. Then $t - 1 = 21 = \frac{r(r-1)}{c_2}$. The stabilizer of M_{22} is $PSL(3, 4)$. Since $21|r(r-1)$, it follows that $r = 7$ or 15. Since A_u is primitive on $\Gamma(u)$, M_u is transitive on $\Gamma(u)$. However, $PSL(3, 4)$ does not have a transitive representation on 7 or 15 vertices, a contradiction.

Suppose $(M, t) = (PSL(2, q), q + 1)$. Then $t - 1 = q = \frac{r(r-1)}{c_2}$. However, in this case, the stabilizer A_u does not have a 2-transitive representation of degree q where q is a prime power, except $q = 5$. Assume $q = 5$. Then $|\Gamma_2(u)| = 5 = \frac{r(r-1)}{c_2}$. Recall that $3 \leqslant r \leqslant t-2$. So $r = 3$, which is impossible.

Suppose $(M, t) = (M_{23}, 23)$. Then $t-1 = 22 = \frac{r(r-1)}{c_2}$ and $M_u \cong M_{22}$. Since $11|r(r-1)$, it follows that $r = 11$ or 12. However, M_{22} does not have a transitive representation on 11 or 12 vertices, a contradiction.

Finally, suppose $(M, t) = (A_n, n)$. Then $|\Gamma_2(u)| = n - 1 = \frac{r(r-1)}{c_2}$ where $3 \leqslant r \leqslant n-2$. Since $M_u = A_{n-1}$ is transitive on $\Gamma(u)$, but $|\Gamma(u)| = r \leqslant n-2$, which is impossible. □

We are ready to prove our second theorem.

Proof of Theorem 2. If Γ has girth at least 4, then by Lemma 14, $\Gamma \cong C_n$ for some $n \geqslant 4$, $\mathrm{K}_{r,r}$, or $\mathrm{K}_{r+1,r+1} - (r+1)\mathrm{K}_2$ with $r \geqslant 3$. If Γ has girth 3, then by Lemma 9, Γ is either the halved 5-cube or the complement of the Higman-Sims graph. We complete the proof. □

4 Locally cyclic graphs

In this section, we prove Theorem 3, that is, determine the unique 2-distance-primitive graph which is locally cyclic.

Proof of Theorem 3. Suppose first that Γ is a non-complete, connected, locally cyclic 2-distance-primitive graph of valency $n \geqslant 3$. Then $[\Gamma(u)] \cong C_n$ for each $u \in V(\Gamma)$. If $n = 3$, then $[\Gamma(u)] \cong C_3$, so $\Gamma \cong \mathrm{K}_4$, contradicting that Γ is non-complete. Hence $n \geqslant 4$. Since Γ is 2-distance-primitive, the stabilizer A_u is primitive on $\Gamma(u)$ where $A := \mathrm{Aut}(\Gamma)$, and so the A_u-action on $\Gamma(u)$ does not have nontrivial blocks. As $[\Gamma(u)] \cong C_n$, it follows that n is an odd integer, and so $n \geqslant 5$.

By Theorem 1, A_u acts faithfully on $\Gamma(u)$. As $[\Gamma(u)] \cong C_n$, $A_u = A_u^{\Gamma(u)} \leqslant \mathrm{Aut}(C_n) = D_{2n} = \mathbb{Z}_n : \mathbb{Z}_2$. In particular, $\mathbb{Z}_n \leqslant A_u$ as n is an odd integer and A_u is transitive on $\Gamma(u)$. Further, since A_u is primitive on $\Gamma_2(u)$, the normal subgroup $\mathbb{Z}_n$ is transitive and so regular on $\Gamma_2(u)$, so $|\Gamma_2(u)| = n$.

Let (u, v, w) be a 2-geodesic. Since Γ is non-complete, $[\Gamma(u)]$ is a non-complete graph, and so $|\Gamma(u) \cap \Gamma_2(v)| \geqslant 1$. If $|\Gamma(u) \cap \Gamma_2(v)| = 1$, then $n = 4$, as $[\Gamma(u)] \cong C_n$, contradicting that $n \geqslant 5$. Hence $|\Gamma(u) \cap \Gamma_2(v)| \geqslant 2$. Since $[\Gamma(u)] \cong C_n$ and $\Gamma(u) = \{v\} \cup (\Gamma(u) \cap \Gamma(v))) \cup (\Gamma(u) \cap \Gamma_2(v))$, it follows that the induced subgraph $[\Gamma(u) \cap \Gamma_2(v)]$ contains edges, and so $[\Gamma_2(v)]$ contains edges. Hence $[\Gamma_2(u)]$ contains edges. Recall that n is odd, so $[\Gamma_2(u)]$ has even valency. Since $c_2 = n - 3$, $a_2 \leqslant 3$, so $a_2 = 2$, that is, $[\Gamma_2(u)]$ has valency 2. As A_u is primitive on $\Gamma_2(u)$, it follows that

$$[\Gamma_2(u)] \cong C_n.$$

Let $z \in \Gamma_3(u) \cap \Gamma(w)$. Then (u, v, w, z) is a 3-geodesic. Recall that $c_2 = n-3$ and $a_2 = 2$, it follows that $b_2 = 1$, so $|\Gamma_3(u) \cap \Gamma(w)| = 1$, hence $\Gamma_3(u) \cap \Gamma(w) = \{z\}$. Since (v, w, z) is a 2-geodesic, $|\Gamma(v) \cap \Gamma(z)| = n - 3$. Note that $\Gamma(v) = \{u\} \cup (\Gamma(u) \cap \Gamma(v)) \cup (\Gamma_2(u) \cap \Gamma(v))$, $|\Gamma_2(u) \cap \Gamma(v)| = n - 3$ and $(\{u\} \cup (\Gamma(u) \cap \Gamma(v))) \cap \Gamma(z) = \emptyset$. It follows that $\Gamma_2(u) \cap \Gamma(v) = \Gamma(v) \cap \Gamma(z)$. Hence $n - 3 = |\Gamma_2(u) \cap \Gamma(v)| = |\Gamma(v) \cap \Gamma(z)| \leqslant |\Gamma_2(u) \cap \Gamma(z)| \leqslant n$.

Since Γ is 2-distance-transitive and $|\Gamma_3(u) \cap \Gamma(w)| = 1$, it follows that Γ is 3-distance-transitive. Thus $|\Gamma_2(u) \cap \Gamma(z)| = c_3$, so $n - 3 \leqslant c_3 \leqslant n$. Counting the number of edges between $\Gamma_2(u)$ and $\Gamma_3(u)$, we get $n = c_3|\Gamma_3(u)|$. Hence c_3 divides n. Since $n - 3 \leqslant c_3 \leqslant n$, it follows that $c_3 = n - 3, n - 2, n - 1$ or n. Since $n - 1$ and n are coprime and c_3 is a divisor of n, $c_3 \neq n - 1$. If $c_3 = n - 2$, then as $c_3 \big| n$, $n = 3$ or 4, contradicting that $n \geqslant 5$. If $c_3 = n - 3$, then as $c_3 \big| n$, $n = 4$ or 6, which is impossible, as $n \geqslant 5$ is odd. Therefore, $c_3 = n$, and so

$$|\Gamma_3(u)| = 1.$$

Thus $\Gamma_3(u) = \{z\}$.

Let $\Delta_1 = \Gamma(v) \cap \Gamma_2(u)$ and $\Delta_2 = \Gamma_2(u) \setminus \Delta_1$. Then $|\Delta_1| = n - 3$ and $|\Delta_2| = 3$. Set $\Gamma(u) = \{v_1 = v, v_2, \ldots, v_n\}$ and $\Gamma_2(u) = \{w_1 = w, w_2, \ldots, w_n\}$. Assume $(v_1, v_3, v_4, \ldots, v_n, v_2, v_1) \cong C_n$. Then $|\Gamma(v_1) \cap \Gamma(v_2)| = 2$. Suppose $\Gamma(v_1) \cap \Gamma(v_2) = \{u, w_1\}$. Then $\Gamma(v_2) \cap \Delta_1 = \{w_1\}$. Since $|\Gamma_2(u) \cap \Gamma(v_2)| = n - 3$, it follows that $|\Gamma(v_2) \cap \Delta_2| = n - 4 \leqslant 3$, and so $n \leqslant 7$. Thus $n = 5$ or 7, as $n \geqslant 5$ is odd.

Suppose $n = 7$. Then $|\Delta_1| = 4$, $|\Delta_2| = 3$, and $\Delta_2 \subset \Gamma(v_2)$. Similarly, $\Delta_2 \subset \Gamma(v_3)$, as (v_1, v_3) is also an arc. Thus $\Delta_2 \subset \Gamma(v_2) \cap \Gamma(v_3)$. Assume $\Delta_1 = \{w_1, w_2, w_3, w_4\}$ and $\Delta_2 = \{w_5, w_6, w_7\}$. Then $\Gamma(v_1) = \{u, v_2, v_3\} \cup \Delta_1$. Suppose $(u, v_2, w_1, w_2, w_3, w_4, v_3) \cong C_7 \cong (w_1, w_2, w_3, w_4, w_5, w_6, w_7)$. Then $\Gamma(v_3) = \{u, v_1, v_4, w_4\} \cup \Delta_2$. Since $[\Gamma(v_3)] \cong C_7$ and $(v_4, u, v_1, w_4, w_5, w_6, w_7)$ is a 6-arc, it follows that v_4 is adjacent to w_7. Since $v_4 \in \Gamma_2(v_1)$, $|\Gamma(v_1) \cap \Gamma(v_4)| = 4$, so $|\Gamma(v_4) \cap \Delta_1| = 2$, hence $|\Gamma(v_4) \cap \Delta_2| = 2$, say $\Gamma(v_4) \cap \Delta_2 = \{w_7, w_j\}$. Note that (v_5, u, v_3, w_7) is a 4-arc and $\Delta_2 \subseteq \Gamma(v_3)$. Hence v_3 is adjacent to both w_7 and w_j, contradicting that $[\Gamma(v_4)] \cong C_7$. Thus $n \neq 7$, and so $n = 5$, and Γ is the icosahedron.

Conversely, assume that Γ is the icosahedron. Then $[\Gamma(u)] \cong [\Gamma_2(u)] \cong C_5$ for each $u \in V(\Gamma)$. By Theorem 1.2 of [13], Γ is 2-geodesic-transitive, and so it is 2-distance-primitive. □

Acknowledgements

The authors are grateful to the anonymous referees for valuable suggestions and comments.

References

[1] B. Alspach, M. Conder, D. Marušič, and M. Y. Xu. A classification of 2-arc transitive circulants, *J. Algebraic Combin.*, 5:83–86, 1996.

[2] M. Aschbacher. The nonexistence of rank three permutation groups of degree 3250 and subgree 57, *J. Algebra*, 19:538–540, 1971.

[3] W. Bosma, C. Cannon, and C. Playoust. The MAGMA algebra system I: The user language, *J. Symbolic Comput.*, 24:235-265, 1997.

[4] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, Springer Verlag, Berlin, Heidelberg, New York, (1989).

[5] T. C. Burness, M. Giudici, and R. Wilson. Prime order derangements in primitive permutation groups, *J. Algebra*, 341:158–178, 2011.

[6] T. C. Burness, C. E. Praeger, and Á. Seress. Extremely primitive classical groups, *J. Pure and Applied Algebra*, 216(7):1580–1610, 2012.

[7] P. J. Cameron. Suborbits in transitive permutation groups, *Combinatorics*, M. Hall, Jr. and J. H. Van lint, p.419-450, Reidel, Dordrecht, (1975).

[8] P. J. Cameron. Finite permutation groups and finite simple groups, *Bull. London Math. Soc.*, 13:1–22, 1981.

[9] P. J. Cameron. *Permutation Groups*, volume 45 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, (1999).

[10] P. J. Cameron and C. E. Praeger. On 2-arc transitive graphs of girth 4, *J. Combin. Theory, Ser. B*, 35:1–11, 1983.

[11] A. Devillers, M. Giudici, C. H. Li, and C. E. Praeger. Locally s-distance transitive graphs, *J. Graph Theory*, 69(2):176–197, 2012.

[12] A. Devillers, W. Jin, C. H. Li, and C. E. Praeger. Local 2-geodesic transitivity and clique graphs, *J. Combin. Theory, Ser. A*, 120:500–508, 2013.

[13] A. Devillers, W. Jin, C. H. Li, and C. E. Praeger. Finite 2-geodesic transitive graphs of prime valency, *J. Graph Theory*, 80:18–27, 2015.

[14] J. D. Dixon and B. Mortimer. *Permutation groups*, Springer, New York, (1996).

[15] Y. Q. Feng and J. H. Kwak. Cubic s-regular graphs of order $2p^3$, *J. Graph Theory*, 52:341–352, 2006.

[16] M. Giudici, C. H. Li, and C. E. Praeger. Analysing finite locally s-arc transitive graphs, *Trans. Amer. Math. Soc.*, 356:291–317, 2003.

[17] C. D. Godsil and G. F. Royle. *Algebraic Graph Theory*, Springer, New York, Berlin, Heidelberg, (2001).

[18] D. G. Higman. Intersection matrices for finite permutation groups, *J. Algebra*, 6:22–42, 1967.

[19] A. A. Ivanov and C. E. Praeger. On finite affine 2-arc transitive graphs, *European J. Combin.*, 14:421–444, 1993.

[20] W. Jin and L. Tan. Finite two-distance-transitive graphs of valency 6, *Ars Math. Contemp.*, 11:49–58, 2016.

[21] C. H. Li. Permutation groups with a cyclic regular subgroup and arc transitive circulants, *J. Algebraic Combin.*, 21:131–136, 2005.

[22] C. H. Li, B. G. Luo, and J. M. Pan. Finite locally primitive abelian Cayley graphs, *Sci. China Math.*, 54(4):845–854, 2011.

[23] C. H. Li, J. M. Pan, and L. Ma. Locally primitive graphs of prime power order, *J. Aust. Math. Soc.*, 86:111–122, 2009.

[24] C. H. Li, C. E. Praeger, A. Venkatesh, and S. M. Zhou. Finite locally-primitive graphs, *Discrete Math.*, 246:197–218, 2002.

[25] C. H. Li and Á. Seress. The primitive permutation groups of squarefree degree, *Bull. London Math. Soc.*, 35:635–644, 2003.

[26] M. W. Liebeck, C. E. Praeger, and J. Saxl. On the O'Nan-Scott theorem for finite primitive permutation groups, *J. Austral. Math. Soc. Ser. A*, 44:389–396, 1988.

[27] M. W. Liebeck. The affine permutation groups of rank three, *Proc. London Math. Soc.*, 54(3):477–516, 1987.

[28] M. W. Liebeck and J. Saxl. The finite primitive permutation groups of rank three, *Bull. London Math. Soc.*, 18(2):165–172, 1986.

[29] J. M. Pan, Z. Liu, and Z. W. Yang. On 2-arc-transitive representations of the groups of fourth-power-free order, *Discrete Math.*, 310:1949–1955, 2010.

[30] P. Potočnik. On 2-arc transitive Cayley graphs of abelian groups, *Discrete Math.*, 244(1-3):417–421, 2002.

[31] C. E. Praeger. Primitive permutation groups with a doubly transitive subconstituent, *J. Austral Math. Soc. (Series A)*, 45:66–77, 1988.

[32] C. E. Praeger. Finite Transitive Permutation Groups and Finite Vertex-Transitive graphs, Graph Symmetry: Algebraic Methods and Applications, *NATO Adv. Sci. Inst. Ser.C Math. Phys. Sci.* 497:277–318, 1997.

[33] C. E. Praeger, J. Saxl and K. Yokohama. Distance transitive graphs and finite simple groups, *Proc. London Math. Soc.*, 55(3):1–21, 1987.

[34] Z. Qiao, S. F. Du, and J. Koolen. 2-Walk-regular dihedrants from group divisible designs, *Electronic J. Combin.*, 23(2):#P2.51, 2016.

[35] L. L. Scott. Representations in characteristic p, Santa Cruz conference on finite groups, *Proc. Sympos. Pure Math.*, 37:318–331, 1980.

[36] W. T. Tutte. A family of cubical graphs, *Proc. Cambridge Philos. Soc.*, 43:459–474, 1947.

[37] W. T. Tutte. On the symmetry of cubic graphs, *Canad. J. Math.*, 11:621–624, 1959.