On volume functions of special flow polytopes associated to the root system of type \( A \)

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Submitted: Oct 16, 2019; Accepted: Dec 11, 2020; Published: Dec 24, 2020
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Abstract

In this paper, we consider the volume of a special kind of flow polytope. We show that its volume satisfies a certain system of differential equations, and conversely, the solution of the system of differential equations is unique up to a constant multiple. In addition, we give an inductive formula for the volume with respect to the rank of the root system of type \( A \).

Mathematics Subject Classifications: 52B20, 05A16

1 Introduction

The number of lattice points and the volume of a convex polytope are important and interesting objects and have been studied from various points of view (see, e.g., [4]). For example, the number of lattice points of a convex polytope associated to a root system is called the Kostant partition function, and it plays an important role in representation theory of Lie groups (see, e.g., [9]).

We consider a flow polytope associated to the root system of type \( A \). As explained in [2, 3], the cone spanned by the positive roots is divided into several polyhedral cones called chambers, and the combinatorial property of a flow polytope depends on a chamber. Moreover, there is a specific chamber called the nice chamber, which plays a significant role in [11]. In this paper, we call a flow polytope for the nice chamber a special flow polytope. Also in [2, 3], a number of theoretical results related to the Kostant partition function and the volume function of a flow polytope can be found. In particular, it is shown that these functions for the nice chamber are written as iterated residues ([3, Lemma 21]). We also refer to [1] for similar formulas for other chambers in more general settings. Moreover, we
mention that a generalization of the Lidskii formula is shown in [3, Theorem 38], there
is a geometric proof of the Lidskii formula in [12], and combinatorial applications of this
formula are given in [5, 7].

The purpose of this paper is to characterize the volume function of a flow polytope for
the nice chamber in terms of a system of differential equations, based on a result in [3]. In
order to state the main results, we give some notation. Let \( e_1, \ldots, e_{r+1} \) be the standard
basis of \( \mathbb{R}^{r+1} \) and let

\[ A_r^+ = \{ e_i - e_j | 1 \leq i < j \leq r + 1 \} \]

be the positive root system of type \( A \) with rank \( r \). We assign a positive integer \( m_{i,j} \) to
each \( i \) and \( j \) with \( 1 \leq i < j \leq r + 1 \). Let us set \( m = (m_{i,j}) \) and \( M = \sum_{1 \leq i < j \leq r + 1} m_{i,j} \).
For \( a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \in \mathbb{R}^{r+1} \), where \( a_i \in \mathbb{R}_{\geq 0} \) \((i = 1, \ldots, r)\), the
following polytope \( P_{A_r^+, m}(a) \) is called the flow polytope associated to the root system of
type \( A \):

\[ P_{A_r^+, m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \right| \begin{array}{l}
1 \leq i < j \leq r + 1, 1 \leq k \leq m_{i,j}, \ y_{i,j,k} \geq 0, \\
\sum_{1 \leq i < j \leq r + 1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k} (e_i - e_j) = a \\
\end{array} \right\}. \]

Note that the flow polytopes in [3] include the case that some of \( m_{i,j} \)'s are zero, whereas
we exclude such cases in this paper. We denote the volume of \( P_{A_r^+, m}(a) \) by \( v_{A_r^+, m}(a) \).

The open set

\[ \mathfrak{c}_{\text{nice}} := \{ a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \in \mathbb{R}^{r+1} | a_i > 0, i = 1, \ldots, r \} \]

in \( \mathbb{R}^{r+1} \) is called the nice chamber. We are interested in the volume \( v_{A_r^+, m}(a) \) when \( a \) is in
the closure of the nice chamber, and then it is written by \( v_{A_r^+, m, \mathfrak{c}_{\text{nice}}} \). It is a homogeneous
polynomial of degree \( M - r \). The first result of this paper is the following.

Theorem 1. Let \( a = \sum_{i=1}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}_{\text{nice}}} \), and let \( v_{A_r^+, m, \mathfrak{c}_{\text{nice}}}(a) \) be the volume
of \( P_{A_r^+, m}(a) \). Then \( v = v_{A_r^+, m, \mathfrak{c}_{\text{nice}}}(a) \) satisfies the system of differential equations as follows:

\[
\begin{align*}
\partial_{a_1}^{m_{r,r+1}} v &= 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v &= 0 \\
&\vdots \\
(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v &= 0,
\end{align*}
\]

where \( \partial_i = \frac{\partial}{\partial a_i} \) for \( i = 1, \ldots, r \). Conversely, the polynomial \( v = v(a) \) of degree \( M - r \)
satisfying the above equations is equal to a constant multiple of \( v_{A_r^+, m, \mathfrak{c}_{\text{nice}}}(a) \).

We remark that it is known that the volume function \( v_{A_r^+, m}(a) \) of \( P_{A_r^+, m}(a) \), as a
distribution on \( \mathbb{R}^r \), satisfies the differential equation

\[ L v_{A_r^+, m}(a) = \delta(a) \]

in general, where \( L = \prod_{1 < j} (\partial_i - \partial_j)^{m_{i,j}} \) and \( \delta(a) \) is the Dirac delta function on \( \mathbb{R}^r \) ([8, 11]). Note that \( \partial_{r+1} \) in the definition of \( L \) is supposed to be zero. The above theorem
characterizes the function \( v_{A^+} (a) \) on \( \overline{c_{\text{nice}}} \) more explicitly. It might be interesting to see what kind of properties of the volume can be derived from Theorem 1.

In addition, in Theorem 20, we show the volume \( v_{A^+} (a) \) is written by a linear combination of \( v_{A^+} (a') \) and its partial derivatives, where \( m' = (m_{i,j})_{2 \leq i < j \leq r+1} \), \( c_{\text{nice}}' \) is the nice chamber of \( A^+ \), and \( a' = \sum_{i=2}^{r+1} a_i (e_i - e_{r+1}) \in \overline{c_{\text{nice}}} \). It might be interesting to ask whether there is a relation between this theorem and the inductive formulas of Schmidt–Bincer [13, (4.1), (4.24)].

This paper is organized as follows. In Section 2, we recall the iterated residue, the Jeffrey-Kirwan residue, and the nice chamber based on [2], [3], [6] and [10]. Also, we give some examples of \( P_{A^+, m_nice} (a) \) and the calculations of the volume \( v_{A^+, m_nice} (a) \). In Section 3, we prove the main theorems.

2 Preliminaries

In this section, we set up the tools to prove the main theorems based on [2], [3], [6] and [10].

2.1 Flow polytopes and its volumes

Let \( e_1, \ldots, e_{r+1} \) be the standard basis of \( \mathbb{R}^{r+1} \), and let

\[
V = \left\{ a = \sum_{i=1}^{r+1} a_i e_i \in \mathbb{R}^{r+1} \left| \sum_{i=1}^{r+1} a_i = 0 \right. \right\}.
\]

We consider the positive root system of type \( A \) with rank \( r \) as follows:

\[
A^+_r = \{ e_i - e_j \mid 1 \leq i < j \leq r+1 \}.
\]

Let \( C(A^+_r) \) be the convex cone generated by \( A^+_r \):

\[
C(A^+_r) = \left\{ a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \mid a_1, \ldots, a_r \in \mathbb{R}_{\geq 0} \right\}.
\]

We assign a positive integer \( m_{i,j} \) to each \( i \) and \( j \) with \( 1 \leq i < j \leq r+1 \), and it is called a multiplicity. Let us set \( m = (m_{i,j}) \) and \( M = \sum_{1 \leq i < j \leq r+1} m_{i,j} \).

**Definition 2.** Let \( a = a_1 e_1 + \cdots + a_r e_r - (a_1 + \cdots + a_r) e_{r+1} \in C(A^+_r) \). We consider the following polytope:

\[
P_{A^+, m_nice} (a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \left| 1 \leq i < j \leq r+1, 1 \leq k \leq m_{i,j}, y_{i,j,k} \geq 0, \sum_{1 \leq i \leq j \leq r+1} \sum_{1 \leq k \leq m_{i,j}} y_{i,j,k} (e_i - e_j) = a \right. \right\},
\]

which is called the flow polytope associated to the root system of type \( A \).

**Remark 3.** The flow polytopes in [3] include the case that \( m_{i,j} = 0 \) for some \( i \) and \( j \).
The elements of $A_r^+$ generate a lattice $V_Z$ in $V$. The lattice $V_Z$ determines a measure $da$ on $V$.

Let $du$ be the Lebesgue measure on $\mathbb{R}^M$. Let $[\alpha_1, \ldots, \alpha_M]$ be a sequence of elements of $A_r^+$ with multiplicity $m_{i,j}$, and let $\varphi$ be the surjective linear map from $\mathbb{R}^M$ to $V$ defined by $\varphi(e_k) = \alpha_k$. The vector space $\ker(\varphi) = \varphi^{-1}(0)$ is of dimension $d = M - r$ and it is equipped with the quotient Lebesgue measure $du/da$. For $a \in V$, the affine space $\varphi^{-1}(a)$ is parallel to $\ker(\varphi)$, and thus also equipped with the Lebesgue measure $du/da$. Volumes of subsets of $\varphi^{-1}(a)$ are computed for this measure. In particular, we can consider the volume $v_{A_r^+,m}(a)$ of the polytope $P_{A_r^+,m}(a)$.

### 2.2 Total residue and iterated residue

Let $A_r = A_r^+ \cup (-A_r^+)$, and let $U$ be the dual vector space of $V$. We denote by $R_{A_r}$ the ring of rational functions $f(x_1, \ldots, x_r)$ on the complexification $U_C$ of $U$ with poles on the hyperplanes $x_i - x_j = 0$ ($1 \leq i < j \leq r + 1$) or $x_i = 0$ ($1 \leq i \leq r$). A subset $\sigma$ of $A_r$ is called a basis of $A_r$ if the elements $\alpha \in \sigma$ form a basis of $V$. In this case, we set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}$$

and call such a element a simple fraction. We denote by $S_{A_r}$ the linear subspace of $R_{A_r}$ spanned by simple fractions. The space $U$ acts on $R_{A_r}$ by differentiation: $(\partial(u)f)(x) = \left(\frac{d}{du}\right)f(x + \varepsilon u)|_{\varepsilon = 0}$. We denote by $\partial(U)R_{A_r}$ the space spanned by derivatives of functions in $R_{A_r}$. It is shown in [6, Proposition 7] that $R_{A_r} = \partial(U)R_{A_r} \oplus S_{A_r}$. The projection map $T_{res} : R_{A_r} \to S_{A_r}$, with respect to this decomposition is called the total residue map.

We extend the definition of the total residue to the space $\hat{R}_{A_r}$ consisting of functions $P/Q$ where $Q$ is a finite product of powers of the linear forms $\alpha \in A_r$ and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with $P_k$ of degree $k$. As the total residue vanishes outside the homogeneous component of degree $-r$ of $A_r$, we can define $T_{res}_{A_r}(P/Q) = T_{res}_{A_r}(P_{q-r}/Q)$, where $q$ is degree of $Q$. For $a \in V$ and multiplicities $m = (m_{i,j}) \in (\mathbb{Z}_{\geq 0})^M$ of elements of $A_r^+$, the function

$$F := \prod_{i=1}^{r} x_i^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}$$

is in $\hat{R}_{A_r}$. We define $J_{A_r^+,m}(a) \in S_{A_r}$ by

$$J_{A_r^+,m}(a) = T_{res}_{A_r}F.$$ 

Next, we describe the iterated residue.

**Definition 4.** For $f \in R_{A_r}$, we define the iterated residue by

$$I_{resx=0}f = \text{Res}_{x_1=0}\text{Res}_{x_2=0}\cdots\text{Res}_{x_r=0}\ f(x_1, \ldots, x_r).$$

Since the iterated residue $I_{resx=0}f$ vanishes on the space $\partial(U)R_{A_r}$ as in [3], we have

$$I_{resx=0}J_{A_r^+,m}(a) = I_{resx=0}F. \quad (1)$$

*The Electronic Journal of Combinatorics* 27(4) (2020), #P4.56
2.3 Chambers and Jeffrey–Kirwan residue

**Definition 5.** Let $C(\nu)$ be the closed cone generated by $\nu$ for any subset $\nu$ of $A_r^+$ and let $C(A_r^+)^{\text{sing}}$ be the union of the cones $C(\nu)$ where $\nu$ is any subset of $A_r^+$ of cardinal strictly less than $r = \dim V$. By definition, the set $C(A_r^+)^{\text{reg}}$ of $A_r^+$-regular elements is the complement of $C(A_r^+)^{\text{sing}}$. A connected component of $C(A_r^+)^{\text{reg}}$ is called a chamber.

The Jeffrey–Kirwan residue [10] associated to a chamber $\mathfrak{c}$ of $C(A_r^+)$ is a linear form $f \mapsto \langle\langle \mathfrak{c}, f \rangle\rangle$ on the vector space $S_{A_r}$ of simple fractions. Any function $f$ in $S_{A_r}$ can be written as a linear combination of functions $f_\sigma$, with a basis $\sigma$ of $A_r$ contained in $A_r^+$. To determine the linear map $f \mapsto \langle\langle \mathfrak{c}, f \rangle\rangle$, it is enough to determine it on this set of functions $f_\sigma$. So we assume that $\sigma$ is a basis of $A_r$ contained in $A_r^+$.

**Definition 6.** For a chamber $\mathfrak{c}$ and $f_\sigma \in S_{A_r}$, we define the Jeffrey–Kirwan residue $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle$ associated to a chamber $\mathfrak{c}$ as follows:

- If $\mathfrak{c} \subseteq C(\sigma)$, then $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 1$.
- If $\mathfrak{c} \cap C(\sigma) = \emptyset$, then $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle = 0$,

where $C(\sigma)$ is the convex cone generated by $\sigma$.

**Remark 7.** More generally, as in [3, Definition 11], the Jeffrey–Kirwan residue $\langle\langle \mathfrak{c}, f_\sigma \rangle\rangle$ is defined to be $\frac{1}{\text{vol}(\sigma)}$ if $\mathfrak{c} \subseteq C(\sigma)$, where $\text{vol}(\sigma)$ is the volume of the parallelepiped $\bigoplus_{\alpha \in \sigma}[0,1]\alpha$, relative to our Lebesgue measure $d\alpha$. In our case, the volume $\text{vol}(\sigma)$ is equal to 1 since $A_r$ is unimodular.

The volume $v_{A_r^+,m}(a)$ of the flow polytope $P_{A_r^+,m}(a)$ is written by the function $J_{A_r^+,m}(a)$ and the Jeffrey–Kirwan residue in the following.

**Theorem 8** (Baldoni–Vergne [3]). Let $\mathfrak{c}$ be a chamber of $C(A_r^+)$. Then, for $a \in \mathfrak{c}$, the volume $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$ is given by

$$v_{A_r^+,m}(a) = \langle\langle \mathfrak{c}, J_{A_r^+,m}(a) \rangle\rangle.$$

We denote by $v_{A_r^+,m}(a)$ the polynomial function of $a$ coinciding with $v_{A_r^+,m}(a)$ when $a \in \mathfrak{c}$. It is a homogeneous polynomial of degree $M - r$.

2.4 Nice chamber

**Definition 9.** The open subset $\mathfrak{c}_{\text{nice}}$ of $C(A_r^+)$ is defined by

$$\mathfrak{c}_{\text{nice}} = \{ a \in C(A_r^+) \mid a_i > 0 \ (i = 1, \ldots, r) \}.$$

The set $\mathfrak{c}_{\text{nice}}$ is in fact a chamber for the root system $A_r^+$ ([3]). The chamber $\mathfrak{c}_{\text{nice}}$ is called the nice chamber.

**Lemma 10** (Baldoni–Vergne [3]). For the nice chamber $\mathfrak{c}_{\text{nice}}$ of $A_r^+$ and $f \in S_{A_r}$, we have

$$\langle\langle \mathfrak{c}_{\text{nice}}, f \rangle\rangle = \text{Ires}_{x=0} f.$$
From Theorem 8, Lemma 10 and (1), we have the following corollary.

**Corollary 11** (Lidskii formula [3]). Let \( a \in \mathcal{C}_{\text{nice}} \). Then the volume function \( v_{A_+^+,m,\mathcal{C}_{\text{nice}}} (a) \) is given by

\[
v_{A_+^+,m,\mathcal{C}_{\text{nice}}} (a) = \text{Res}_{x=0} F.
\]

### 2.5 Examples

In this subsection, we give some examples of the flow polytopes for \( A_1, A_2, \) and \( A_3 \), and calculate their volumes.

**Example 12.** When \( r = 1 \), the nice chamber of \( A_1^+ \) is \( \mathcal{C}_{\text{nice}} = \{ a = a_1(e_1 - e_2) | a_1 > 0 \} \). For \( a = a_1(e_1 - e_2) \in \mathcal{C}_{\text{nice}}, \)

\[
P_{A_1^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{m_1,2} | y_{i,j,k} \geq 0, y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,m_1,2} = a_1 \right\}.
\]

From Corollary 11, we have

\[
v_{A_1^+,m,\mathcal{C}_{\text{nice}}} (a) = \text{Res}_{x=0} \left( \frac{e^{a_1 x_1}}{x_1^{m_1,2}} \right) = \frac{1}{(m_1,2 - 1)!} a_1^{m_1,2 - 1}.
\]

**Example 13.** When \( r = 2 \), there are two chambers \( \mathcal{C}_1, \mathcal{C}_2 \) of \( A_2^+ \) as below, and the nice chamber \( \mathcal{C}_{\text{nice}} \) of \( A_2^+ \) is \( \mathcal{C}_1 \).

For example, we set \( m_{1,2} = n \ (n \in \mathbb{Z}_{>0}), m_{1,3} = 1, \) and \( m_{2,3} = 1 \). For \( a = a_1 e_1 + a_2 e_2 - (a_1 + a_2) e_3 \in \mathcal{C}_{\text{nice}}, \)

\[
P_{A_2^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{n+2} | \begin{align*}
y_{i,j,k} & \geq 0 \\
y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,n} + y_{1,3,1} = a_1 \\
y_{2,1,1} - y_{1,2,2} - \cdots - y_{1,2,n} + y_{2,3,1} = a_2
\end{align*} \right\}.
\]

From Corollary 11, we have

\[
v_{A_2^+,m,\mathcal{C}_{\text{nice}}} (a) = \text{Res}_{x=0} \left( \frac{e^{a_1 x_1 + a_2 x_2}}{x_1 x_2 (x_1 - x_2)^n} \right) = \text{Res}_{x=0} \text{Res}_{x=2} \left( \frac{e^{a_1 x_1 + a_2 x_2}}{x_1 x_2 (x_1 - x_2)^n} \right) = \frac{1}{n!} a_1^n.
\]
**Example 14.** When \( r = 3 \), there are seven chambers of \( A_3^+ \) as below ([1]), and the nice chamber \( c_{\text{nice}} \) of \( A_3^+ \) is \( c_1 \).

![Diagram of chamber of \( A_3^+ \)](image)

For example, we set \( m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2, \) and \( m_{3,4} = 2 \). For \( a = \sum_{i=1}^{3} a_i(e_i - e_4) \in c_{\text{nice}} \),

\[
P_{A_3^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^9 \mid \begin{align*}
y_{i,j,k} & \geq 0 \\
y_{1,2,1} + y_{1,3,1} + y_{1,4,1} + y_{1,4,2} & = a_1 \\
y_{1,1,2,1} + y_{2,3,1} + y_{2,4,1} + y_{2,4,2} & = a_2 \\
y_{1,3,1} - y_{2,3,1} + y_{3,4,1} + y_{3,4,2} & = a_3
\end{align*} \right\}.
\]

From Corollary 11, we have

\[
v_{A_3^+,m,c_{\text{nice}}}(a) = \text{Ires}_{x=0} \left( \frac{e^{a_1x_1+a_2x_2+a_3x_3}}{x_1^2x_2^2x_3^3(x_1-x_2)(x_1-x_3)(x_2-x_3)} \right)
\]

\[
= \frac{1}{360} a_1^3(a_1^3 + 6a_1^2a_2 + 3a_1^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^3 + 30a_2^2a_3).
\]

### 3 Main theorems

In this section, we prove the main theorems of this paper. Let \( c_{\text{nice}} \) be the nice chamber of \( A^+_r \) and let \( a = \sum_{i=1}^{3} a_i(e_i - e_{r+1}) \in c_{\text{nice}} \).

**Theorem 15.** For \( a \in c_{\text{nice}} \), let \( P_{A_r^+,m}(a) \) be the flow polytope as in Definition 2 and let \( v_{A_r^+,m,c_{\text{nice}}}(a) \) be the volume of \( P_{A_r^+,m}(a) \). Then \( v = v_{A_r^+,m,c_{\text{nice}}}(a) \) satisfies the system of differential equations as follows:

\[
\begin{align*}
\partial_r^{m_r,r+1} v &= 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1},r} \partial_{r-1}^{m_{r-1},r+1} v &= 0 \\
&\vdots \\
(\partial_1 - \partial_2)^{m_2,2} (\partial_1 - \partial_3)^{m_1,3} \cdots (\partial_1 - \partial_r)^{m_1,r} \partial_1^{m_{1},r+1} v &= 0,
\end{align*}
\]

(2)
where \( \partial_i = \frac{\partial}{\partial a_i} \) for \( i = 1, \ldots, r \).

**Proof.** We will prove the first two relations. Let \( F = \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{i=1}^k x_i^{m_{i,r}+1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \). It is easy to see that

\[
P(\partial_1, \ldots, \partial_r)(\text{Ires}_{x=0} F) = \text{Ires}_{x=0}(P(\partial_1, \ldots, \partial_r)F) = \text{Ires}_{x=0}(P(x_1, \ldots, x_r)F),
\]

where \( P \) is a polynomial. Since \( e^{e_1 x_1 + \cdots + e_k x_k} \) is holomorphic at \( x_k = 0 \),

\[
\text{Res}_{x_k=0}\left( \frac{e^{e_1 x_1 + \cdots + e_k x_k}}{\prod_{i=1}^{k-1} x_i^{m_{i,r}+1} \prod_{1 \leq i < j \leq k} (x_i - x_j)^{m_{i,j}}} \right) = 0
\]

for \( k = 1, \ldots, r \). Therefore, from Corollary 11, (3) and (4), we obtain

\[
\partial_r^{m_{r,r+1} v} = \partial_r^{m_{r,r+1}} \text{Ires}_{x=0} F = \text{Ires}_{x=0} \partial_r^{m_{r,r+1} v}
\]

and

\[
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_r^{m_{r-1,r+1} v}
= \text{Ires}_{x=0}(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_r^{m_{r-1,r+1} v}
\]

\[
= \text{Ires}_{x=0}(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{x_r^{m_{r,r+1}} \prod_{i=1}^{r-2} x_i^{m_{i,r}+1} \prod_{1 \leq i < j \leq r, (i,j) \neq (r-1,r)} (x_i - x_j)^{m_{i,j}}} \right)
\]

\[
= \text{Ires}_{x=0}\left( \frac{x_r^{m_{r,r+1}} \prod_{i=1}^{r-2} x_i^{m_{i,r}+1} \prod_{1 \leq i < j \leq r, (i,j) \neq (r-1,r)} (x_i - x_j)^{m_{i,j}}}{e^{a_1 x_1 + \cdots + a_r x_r}} \right)
\]

Similarly, we can verify the remaining expressions. \( \square \)

**Remark 16.** In general, it is known that the volume function \( v_{A_r^+, m}(a) \) of \( P_{A_r^+, m}(a) \), as a distribution on \( V \), satisfies the differential equation

\[
L v_{A_r^+, m}(a) = \delta(a),
\]

where \( L = \prod_{i<j} (\partial_i - \partial_j)^{m_{i,j}} \) and \( \delta(a) \) is the Dirac delta function on \( V ([8, 11]) \). Note that \( \partial_{r+1} \) in the definition of \( L \) is supposed to be zero. Theorem 15 above, together with Proposition 17 and Theorem 18 as below, characterizes the function \( v_{A_r^+, m, \text{nice}}(a) \) on \( \overline{\text{nice}} \) more explicitly.
Let $M_\ell = \sum_{i=\ell+1}^{\ell+r} m_{\ell,i}$ for $\ell = 1, \ldots, r$. Then we have the following proposition.

**Proposition 17.** The coefficient of $a_1^{M_1-1}a_2^{M_2-1} \cdots a_r^{M_r-1}$ in the volume function $v_{n,m}(a)$ is given by

$$\frac{1}{(M_1 - 1)!(M_2 - 1)! \cdots (M_{r-1} - 1)!(M_r - 1)!}.$$

**Proof.** From the Lidskii formula in Corollary 11, we have

$$v_{n,m}(a) = \sum_{|\ell| = l-r} a_1^{i_1} \cdots a_r^{i_r} \text{Ires}_{x=0} \left( \frac{x_1^{i_1} \cdots x_r^{i_r}}{\prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right),$$

where $|\ell| = i_1 + \cdots + i_r$. When $i_\ell = M_{\ell} - 1$ for $\ell = 1, \ldots, r$,

$$\text{Ires}_{x=0} \left( \frac{x_1^{M_1-1} \cdots x_r^{M_r-1}}{\prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right) = \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-1}=0} \text{Res}_{x_r=0} \left( \frac{x_1^{(\sum_{i=2}^{r} m_{i,1})-1} \cdots x_r^{m_{r-1},r-1-1}}{x_r \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{i,j}}} \right)$$

$$= \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-1}=0} \left( x_1^{(\sum_{i=2}^{r-1} m_{i,1})-1} \cdots x_r^{m_{r-2},r-1-1} \right) = \text{Res}_{x_1=0} \frac{1}{x_1} = 1.$$

Thus we obtain the proposition. \qed

**Theorem 18.** Let $\phi_r = \phi(a_1, \ldots, a_r)$ be a homogeneous polynomial of $a_1, \ldots, a_r$ with degree $d$ and let $M = \sum_{1 \leq i < j \leq r+1} m_{i,j}$. Suppose $\phi_r$ satisfies the system of differential equations as follows:

$$\begin{cases}
\partial_r^{m_{r,r+1}} \phi_r = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1},r} \partial_{r-1}^{m_{r-1},r+1} \phi_r = 0 \\
\vdots \\
(\partial_1 - \partial_r)^{m_{1,2}}(\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r = 0.
\end{cases}\quad(5)
$$

(i) If $M - r < d$, then $\phi_r = 0$.

(ii) If $0 \leq d \leq M - r$, then there is a non trivial homogeneous polynomial $\phi_r$ satisfying (4).

(iii) If $d = M - r$ in particular, $\phi_r$ is equal to a constant multiple of $v = v_{n,m}(a)$. 

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Proof. We argue by induction on \( r \). In the case that \( r = 1 \), we write
\[
\phi_1 = \phi(a_1) = pa_1^d,
\]
where \( p \) is a constant. If \( m_{1,2} - 1 < d \) and \( \phi_1 \) satisfies the differential equation \( \partial_1^{m_{1,2}} \phi_1 = 0 \), then \( p = 0 \) and hence \( \phi_1 = 0 \). If \( 0 \leq d \leq m_{1,2} - 1 \), then for any \( p \neq 0 \), \( \partial_1^{m_{1,2}} \phi_1 = 0 \).

Also, if \( d = m_{1,2} - 1 \), in particular, then \( \phi_1 = pa_1^{m_{1,2} - 1} \), while \( v = \frac{1}{(m_{1,2} - 1)!} a_1^{m_{1,2} - 1} \) as in Example 12. Hence \( \phi_1 \) is equal to a constant multiple of \( v \).

We assume that the statement of this theorem holds for \( r - 1 \). We write \( \phi_r \) as
\[
\phi_r = \phi(a_1, \ldots, a_r) = g_d(a_2, \ldots, a_r) + a_1 g_{d-1}(a_2, \ldots, a_r) + \cdots + a_1^d g_0(a_2, \ldots, a_r),
\]
where \( g_k \) is a homogeneous polynomial of \( a_2, \ldots, a_r \) with degree \( k \) for \( k = 0, 1, \ldots, d \). Then for \( k = 0, 1, \ldots, d \), \( g_k \) satisfies the differential equations as follows:
\[
\begin{aligned}
&\partial_r^{m_{r+1}} g_k = 0 \\
&(\partial_r - \partial_{r+1})^{m_{r+1} - 1} \partial_{r+1}^{m_{r+1} - 1} g_k = 0 \\
&\quad \vdots \\
&(\partial_2 - \partial_1)^{m_{2,3}} (\partial_2 - \partial_1)^{m_{2,4}} \cdots (\partial_2 - \partial_r)^{m_{2,r+1}} g_k = 0.
\end{aligned}
\]

We set \( h = (\sum_{2 \leq i < j \leq r+1} m_{i,j}) - (r - 1) \). From the inductive assumption, if \( 0 \leq k \leq h \), then \( g_k \) is a homogeneous polynomial. On the other hand, if \( h + 1 \leq k \leq d \), then \( g_k = 0 \), namely,
\[
g_d(a_2, \ldots, a_r) = g_{d-1}(a_2, \ldots, a_r) = \cdots = g_{h+1}(a_2, \ldots, a_r) = 0.
\]

(i) We consider the case of \( M - r < d \). Let \( M_1 = \sum_{i=2}^{r+1} m_{1,i} \). Now we compare the coefficients of \( a_1^{d-h-M_1+n} \) in \( (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r+1}} \partial_{1}^{m_{1,r+1}} \phi_r \) for \( n = 0, \ldots, h \). For \( q = 1, \ldots, M_1 - m_{1,r+1} \), we define
\[
D_q = \sum_{2 \leq i_1 \leq r} \left( \begin{array}{c} m_{1,i_1} \\ q \end{array} \right) \partial_1^{q} + \cdots + \sum_{2 \leq i_1 < \cdots < i_q \leq r} \left( \prod_{1 \leq l \leq k} \left( \begin{array}{c} m_{1,i_l} \\ p_l \end{array} \right) \right) \partial_{i_1}^{p_1} \partial_{i_2}^{p_2} \cdots \partial_{i_q}^{p_q} + \cdots + \sum_{2 \leq i_1 < \cdots < i_q \leq r} \left( \prod_{1 \leq l \leq q} \left( \begin{array}{c} m_{1,i_l} \\ 1 \end{array} \right) \right) \partial_{i_1} \partial_{i_2} \cdots \partial_{i_q}.
\]
Then we have the following equation:
\[
\begin{aligned}
&\frac{(d - h + n)!}{(d - h - M_1 + n)!} g_{h-n}(a_2, \ldots, a_r) - \frac{(d - h + n - 1)!}{(d - h - M_1 + n)!} D_1 g_{h-n+1}(a_2, \ldots, a_r) \\
&+ \cdots + (-1)^j \frac{(d - h + n - j)!}{(d - h - M_1 + n)!} D_j g_{h-n+j}(a_2, \ldots, a_r) \\
&+ \cdots + (-1)^{M_1 - m_{1,r+1}} \frac{(d - h + n - (M_1 - m_{1,r+1}))!}{(d - h - M_1 + n)!} D_{M_1 - m_{1,r+1}} g_{h-n+M_1,r}(a_2, \ldots, a_r)
\end{aligned} = 0.
\]
When $n = 0$, from (7) and (8), we have
\[ g_h(a_2, \ldots, a_r) = 0. \]

When $n = 1$, we have
\[ \frac{(d - h + 1)!}{(d - h - M_1 + 1)!} g_{h-1}(a_2, \ldots, a_r) - \frac{(d - h)!}{(d - h - M_1 + 1)!} D_1 g_h(a_2, \ldots, a_r) = 0. \]

Thus we have
\[ g_{h-1}(a_2, \ldots, a_r) = 0. \]

Similarly, for $n = 1$, we have
\[ g_{h-2}(a_2, \ldots, a_r) = g_{h-3}(a_2, \ldots, a_r) = \cdots = g_0(a_2, \ldots, a_r) = 0 \]
and hence $\phi_r = 0$.

(ii) We consider the case of $0 \leq d \leq M - r$. By the inductive assumption, there is a non-trivial homogeneous polynomial $g_{h-n_1+i}$ satisfying (6) for $i = 1, \ldots, n_1$, where $n_1 = M - r - d + 1$. We can take
\[ g_{h-n_1+i}(a_2, \ldots, a_r) \neq 0. \]

When $n = n_1$, from (7) and (8),
\[ g_{h-n_1}(a_2, \ldots, a_r) = \frac{(d - h + n_1 - 1)!}{(d - h + n_1)!} D_1 g_{h-n_1+1}(a_2, \ldots, a_r) \]
\[ - \frac{(d - h + n_1 - 2)!}{(d - h + n_1)!} D_2 g_{h-n_1+2}(a_2, \ldots, a_r) \]
\[ + \cdots + (-1)^{n_1-1} \frac{(d - h)!}{(d - h + n_1)!} D_n g_h(a_2, \ldots, a_r). \]

When $n = n_1 + 1$,
\[ g_{h-(n_1+1)}(a_2, \ldots, a_r) = \frac{(d - h + n_1)!}{(d - h + n_1 + 1)!} D_1 g_{h-n_1}(a_2, \ldots, a_r) \]
\[ - \frac{(d - h + n_1 - 1)!}{(d - h + n_1 + 1)!} D_2 g_{h-n_1+1}(a_2, \ldots, a_r) \]
\[ + \cdots + (-1)^{n_1} \frac{(d - h)!}{(d - h + n_1 + 1)!} D_{n+1} g_h(a_2, \ldots, a_r). \]

Similarly, for $n = n_1 + 2, \ldots, h$, we can express $g_{h-j}(a_2, \ldots, a_r)$ ($j = n_1, n_1 + 1, \ldots, h$) in terms of $g_{h-j+i}(a_2, \ldots, a_r)$ ($i = 1, \ldots, j$) and their partial derivatives. Namely, we can express $\phi_r$ in terms of $g_{h-n_1+i}(a_2, \ldots, a_r)$ and their partial derivatives. It follows that $\phi_r \neq 0$ when $0 \leq d \leq M - r$.

(iii) If $d = M - r$ in particular, then $n_1 = 1$, and $g_{h-j}(j = 1, \ldots, h)$ becomes the linear combination of $g_h$ and their partial derivatives. Therefore $\phi_r$ is uniquely determined by
$g_h$. Moreover, from the inductive assumption, $g_h = C \cdot v_{A_{r-1}^+, m', \mathbf{c}_{\text{nice}}'}$, where $C$ is a constant, $m' = (m_{i,j})_{2 \leq i < j \leq r+1}$, and $\mathbf{c}_{\text{nice}}$ is a nice chamber of $A_{r-1}^+$. Hence the solution of (5) is unique up to a constant multiple. On the other hand, by Theorem 15, $v_{A_{r}^+, m, \mathbf{c}_{\text{nice}}}$ satisfies the system of differential equations (5). Hence $\phi_r$ is equal to a constant multiple of $v_{A_{r}^+, m, \mathbf{c}_{\text{nice}}}$.

Recall that in the proof of Theorem 18, we have defined the operator

$$D_q = \sum_{2 \leq l_1 < \cdots < l_q \leq r} \binom{m_{1, l_1}}{q} \partial_{l_1}^q + \cdots + \sum_{2 \leq l_1 < \cdots < l_q \leq r} \left( \prod_{1 \leq l \leq k} \binom{m_{1, l}}{p_l} \right) \partial_{l_1}^p \partial_{l_2}^p \cdots \partial_{l_k}^p$$

$$+ \cdots + \sum_{2 \leq l_1 < \cdots < l_q \leq r} \left( \prod_{1 \leq l \leq q} \binom{m_{1, l}}{1} \right) \partial_{l_1} \partial_{l_2} \cdots \partial_{l_q}$$

for $q = 1, \ldots, M_1 - m_{1, r+1}$.

**Remark 19.** Let $M_1 = \sum_{i=2}^{r+1} m_{1, i}$. Then $d = M - r$, from the proof of Theorem 18 (iii), $g_{h-j}$ ($j = 1, \ldots, h$) is uniquely determined as follows:

$$\begin{align*}
g_{h-1} &= \frac{(M_1 - 1)!}{M_1!} D_1 g_h \\
g_{h-2} &= \frac{(M_1 - 1)!}{(M_1 + 1)!} (D_1^2 - D_2) g_h \\
g_{h-3} &= \frac{(M_1 - 1)!}{(M_1 + 2)!} (D_1^3 - 2D_1 D_2 + D_3) g_h \\
 & \vdots \\
g_0 &= \frac{(M_1 - 1)!}{(M - r)!} (D_1^h - (h - 1)D_1^{h-2} D_2 + \cdots + (-1)^{h-1} D_h) g_h.
\end{align*}$$

Let $m' = (m_{i,j})_{2 \leq i < j \leq r+1}$, $\mathbf{c}_{\text{nice}}'$ a nice chamber of $A_{r-1}^+$ and $a' = \sum_{i=2}^{r} a_i (e_i - e_{r+1}) \in \overline{\mathbf{c}_{\text{nice}}'}$. From Proposition 17 and Remark 19, we obtain the following theorem.

**Theorem 20.** Let $h = (\sum_{2 \leq i < j \leq r+1} m_{i,j}) - (r - 1)$ and let $D_q$ ($q = 1, \ldots, h$) be as in (9). Then $v_{A_{r}^+, m, \mathbf{c}_{\text{nice}}} (a)$ is written by the linear combination of $v_{A_{r-1}^+, m', \mathbf{c}_{\text{nice}}'} (a')$ and its partial derivatives as follows:

$$v_{A_{r}^+, m, \mathbf{c}_{\text{nice}}} (a) = \left\{ \frac{a_1 M_1 - 1}{(M_1 - 1)!} D_1 + \frac{a_1 M_1}{M_1!} (D_1^2 - D_2) + \frac{a_1 M_1 + 2}{(M_1 + 2)!} (D_1^3 - 2D_1 D_2 + D_3) + \cdots + \frac{a_1 D^{h-r}}{(M - r)!} (D_1^h - (h - 1)D_1^{h-2} D_2 + \cdots + (-1)^{h-1} D_h) \right\} v_{A_{r-1}^+, m', \mathbf{c}_{\text{nice}}'} (a').$$

**Example 21.** Let $r = 3$, let $a = \sum_{i=1}^{3} a_i (e_i - e_4) \in \overline{\mathbf{c}_{\text{nice}}}$ and let $a' = \sum_{i=2}^{3} a_i (e_i - e_4) \in \overline{\mathbf{c}_{\text{nice}}'}$. We set $m_{1,2} = 1$, $m_{1,3} = 1$, $m_{1,4} = 2$, $m_{2,3} = 1$, $m_{2,4} = 2$ and $m_{3,4} = 2$ as in Example 14. Then we have

$$v_{A_{3}^+, m, \mathbf{c}_{\text{nice}}} (a) = \frac{1}{360} a_1^3 (a_1^2 + 6a_1 a_2 + 3a_2^2 a_3 + 15a_1 a_2^2 + 15a_1 a_2 a_3 + 10a_3^2 + 30a_2^2 a_3).$$
We can check that $v = v_{A^3, m, c_{nice}}(a)$ satisfies the system of differential equations as follows:

$$\begin{cases}
\partial_3^2 v = 0 \\
(\partial_2 - \partial_3)\partial_2^2 v = 0 \\
(\partial_1 - \partial_2)(\partial_1 - \partial_3)\partial_1^2 v = 0.
\end{cases}$$

Also, from Proposition 17, the coefficient of the term $a_1^3 a_2^2 a_3$ is $\frac{1}{31224} = \frac{1}{12}$. When $r = 2$,

$$v_{A^3, m', c_{nice}}(a') = \frac{1}{6} a_2^2 (a_2 + 3a_3).$$

Therefore, we have

$$\left\{ \frac{a_1^3}{6} + \frac{a_1^4}{24} D_1 + \frac{a_1^5}{120} (D_1^2 - D_2) + \frac{a_1^6}{720} (D_1^3 - 2D_1 D_2 + D_3) \right\} v_{A^3, m', c_{nice}}(a')$$

$$= \frac{a_1^3 a_2^3}{36} + \frac{a_1^3 a_3^2}{12} + \frac{a_1^4 a_2^2}{24} + \frac{a_1^4 a_3^2}{24} + \frac{a_1^5 a_2}{60} + \frac{a_1^5 a_3}{120} + \frac{a_1^6}{360} = v_{A^3, m, c_{nice}}(a)$$

as in (10).

Acknowledgements

The third author was supported by JSPS KAKENHI Grant Numbers JP16K05137 and JP19K03475. The authors thank the referee for valuable comments.

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