On volume functions of special flow polytopes associated to the root system of type A

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Abstract

In this paper, we consider the volume of a special kind of flow polytope. We show that its volume satisfies a certain system of differential equations, and conversely, the solution of the system of differential equations is unique up to a constant multiple. In addition, we give an inductive formula for the volume with respect to the rank of the root system of type A.

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1 Introduction

The number of lattice points and the volume of a convex polytope are important and interesting objects and have been studied from various points of view (see, e.g., [4]). For example, the number of lattice points of a convex polytope associated to a root system is called the Kostant partition function, and it plays an important role in representation theory of Lie groups (see, e.g., [9]).

We consider a flow polytope associated to the root system of type A. As explained in [2, 3], the cone spanned by the positive roots is divided into several polyhedral cones called *chambers*, and the combinatorial property of a flow polytope depends on a chamber. Moreover, there is a specific chamber called the *nice chamber*, which plays a significant role in [11]. In this paper, we call a flow polytope for the nice chamber a *special flow polytope*. Also in [2, 3], a number of theoretical results related to the Kostant partition function and the volume function of a flow polytope can be found. In particular, it is shown that these functions for the nice chamber are written as iterated residues ([3, Lemma 21]). We also refer to [1] for similar formulas for other chambers in more general settings. Moreover, we mention that a generalization of the Lidskii formula is shown in [3, Theorem 38], there is a geometric proof of the Lidskii formula in [12], and combinatorial applications of this formula are given in [5, 7].

The purpose of this paper is to characterize the volume function of a flow polytope for the nice chamber in terms of a system of differential equations, based on a result in [3]. In order to state the main results, we give some notation. Let $e_1, \dots, e_{r+1}$ be the standard basis of $\mathbb{R}^{r+1}$ and let

$$A_r^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant r+1\}$$

be the positive root system of type A with rank r. We assign a positive integer $m_{i,j}$ to each i and j with $1 \leqslant i < j \leqslant r+1$. Let us set $m = (m_{i,j})$ and $M = \sum_{1\leqslant i<j\leqslant r+1} m_{i,j}$. For $a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \in \mathbb{R}^{r+1}$, where $a_i \in \mathbb{R}_{\geqslant 0}$ $(i = 1, \dots, r)$, the following polytope $P_{A_r^+,m}(a)$ is called the flow polytope associated to the root system of type A:

$$P_{A_r^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \;\middle|\; \begin{array}{l} 1 \leqslant i < j \leqslant r+1\,,\ 1 \leqslant k \leqslant m_{i,j}\,,\ y_{i,j,k} \geqslant 0\,, \\ \sum_{1\leqslant i<j\leqslant r+1} \sum_{1\leqslant k\leqslant m_{i,j}} y_{i,j,k}(e_i - e_j) = a \end{array} \right\}.$$

Note that the flow polytopes in [3] include the case that some of $m_{i,j}$'s are zero, whereas we exclude such cases in this paper. We denote the volume of $P_{A_r^+,m}(a)$ by $v_{A_r^+,m}(a)$.

The open set

$$\mathfrak{c}_{\text{nice}} := \{a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \in \mathbb{R}^{r+1} \mid a_i > 0, i = 1, \dots, r\}$$

in $\mathbb{R}^{r+1}$ is called the nice chamber. We are interested in the volume $v_{A_r^+,m}(a)$ when a is in the closure of the nice chamber, and then it is written by $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}$. It is a homogeneous polynomial of degree $M - r$. The first result of this paper is the following.

Theorem 1. *Let $a = \sum_{i=1}^r a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}_{\text{nice}}}$, and let $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ be the volume of $P_{A_r^+,m}(a)$. Then $v = v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ satisfies the system of differential equations as follows:*

$$\begin{cases} \partial_r^{m_{r,r+1}} v = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v = 0 \\ \quad \vdots \\ (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v = 0, \end{cases}$$

where $\partial_i = \frac{\partial}{\partial a_i}$ for $i = 1, \dots, r$. Conversely, the polynomial $v = v(a)$ of degree $M - r$ satisfying the above equations is equal to a constant multiple of $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$.

We remark that it is known that the volume function $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$, as a distribution on $\mathbb{R}^r$, satisfies the differential equation

$$Lv_{A_r^+,m}(a) = \delta(a)$$

in general, where $L = \prod_{i<j}(\partial_i - \partial_j)^{m_{i,j}}$ and $\delta(a)$ is the Dirac delta function on $\mathbb{R}^r$ ([8, 11]). Note that ∂_{r+1} in the definition of L is supposed to be zero. The above theorem

characterizes the function $v_{A_r^+,m,\mathfrak{c}_{\rm nice}}(a)$ on $\overline{\mathfrak{c}_{\rm nice}}$ more explicitly. It might be interesting to see what kind of properties of the volume can be derived from Theorem 1.

In addition, in Theorem 20, we show the volume $v_{A_r^+,m,\mathfrak{c}_{\rm nice}}(a)$ is written by a linear combination of $v_{A_{r-1}^+,m',\mathfrak{c}'_{\rm nice}}(a')$ and its partial derivatives, where $m' = (m_{i,j})_{2\leqslant i<j\leqslant r+1}$, $\mathfrak{c}'_{\rm nice}$ is the nice chamber of A_{r-1}^+, and $a' = \sum_{i=2}^{r} a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}'_{\rm nice}}$. It might be interesting to ask whether there is a relaton between this theorem and the inductive formulas of Schmidt–Bincer [13, (4.1), (4.24)].

This paper is organized as follows. In Section 2, we recall the iterated residue, the Jeffrey-Kirwan residue, and the nice chamber based on [2], [3], [6] and [10]. Also, we give some examples of $P_{A_r^+,m}(a)$ and the calculations of the volume $v_{A_r^+,m,\mathfrak{c}_{\rm nice}}(a)$. In Section 3, we prove the main theorems.

2 Preliminaries

In this section, we set up the tools to prove the main theorems based on [2], [3], [6] and [10].

2.1 Flow polytopes and its volumes

Let $e_1, \dots, e_{r+1}$ be the standard basis of $\mathbb{R}^{r+1}$, and let

$$V = \left\{ a = \sum_{i=1}^{r+1} a_i e_i \in \mathbb{R}^{r+1} \;\middle|\; \sum_{i=1}^{r+1} a_i = 0 \right\}.$$

We consider the positive root system of type A with rank r as follows:

$$A_r^+ = \{e_i - e_j \,|\, 1 \leqslant i < j \leqslant r+1\}.$$

Let $C(A_r^+)$ be the convex cone generated by A_r^+:

$$C(A_r^+) = \{a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \,|\, a_1, \dots, a_r \in \mathbb{R}_{\geqslant 0}\}.$$

We assign a positive integer $m_{i,j}$ to each i and j with $1 \leqslant i < j \leqslant r+1$, and it is called a multiplicity. Let us set $m = (m_{i,j})$ and $M = \sum_{1\leqslant i<j\leqslant r+1} m_{i,j}$.

Definition 2. Let $a = a_1e_1 + \cdots + a_re_r - (a_1 + \cdots + a_r)e_{r+1} \in C(A_r^+)$. We consider the following polytope:

$$P_{A_r^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^M \;\middle|\; \begin{array}{l} 1 \leqslant i < j \leqslant r+1\,,\ 1 \leqslant k \leqslant m_{i,j}\,,\ y_{i,j,k} \geqslant 0\,, \\ \sum_{1\leqslant i<j\leqslant r+1} \sum_{1\leqslant k\leqslant m_{i,j}} y_{i,j,k}(e_i - e_j) = a \end{array} \right\},$$

which is called the *flow polytope* associated to the root system of type A.

Remark 3. The flow polytopes in [3] include the case that $m_{i,j} = 0$ for some i and j.

The elements of A_r^+ generate a lattice $V_{\mathbb{Z}}$ in V. The lattice $V_{\mathbb{Z}}$ determines a measure da on V.

Let du be the Lebesgue measure on $\mathbb{R}^M$. Let $[\alpha_1, \dots, \alpha_M]$ be a sequence of elements of A_r^+ with multiplicity $m_{i,j}$, and let φ be the surjective linear map from $\mathbb{R}^M$ to V defined by $\varphi(e_k) = \alpha_k$. The vector space $\ker(\varphi) = \varphi^{-1}(0)$ is of dimension $d = M - r$ and it is equipped with the quotient Lebesgue measure du/da. For $a \in V$, the affine space $\varphi^{-1}(a)$ is parallel to $\ker(\varphi)$, and thus also equipped with the Lebesgue measure du/da. Volumes of subsets of $\varphi^{-1}(a)$ are computed for this measure. In particular, we can consider the volume $v_{A_r^+,m}(a)$ of the polytope $P_{A_r^+,m}(a)$.

2.2 Total residue and iterated residue

Let $A_r = A_r^+ \cup (-A_r^+)$, and let U be the dual vector space of V. We denote by R_{A_r} the ring of rational functions $f(x_1, \dots, x_r)$ on the complexification $U_{\mathbb{C}}$ of U with poles on the hyperplanes $x_i - x_j = 0$ $(1 \leqslant i < j \leqslant r+1)$ or $x_i = 0$ $(1 \leqslant i \leqslant r)$. A subset σ of A_r is called a *basis* of A_r if the elements $\alpha \in \sigma$ form a basis of V. In this case, we set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}$$

and call such a element a *simple fraction*. We denote by S_{A_r} the linear subspace of R_{A_r} spanned by simple fractions. The space U acts on R_{A_r} by differentiation: $(\partial(u)f)(x) = (\frac{d}{d\varepsilon})f(x + \varepsilon u)|_{\varepsilon=0}$. We denote by $\partial(U)R_{A_r}$ the space spanned by derivatives of functions in R_{A_r}. It is shown in [6, Proposition 7] that $R_{A_r} = \partial(U)R_{A_r} \oplus S_{A_r}$. The projection map $\mathrm{Tres}_{A_r} : R_{A_r} \to S_{A_r}$ with respect to this decomposition is called the *total residue map*.

We extend the definition of the total residue to the space $\hat{R}_{A_r}$ consisting of functions P/Q where Q is a finite product of powers of the linear forms $\alpha \in A_r$ and $P = \sum_{k=0}^{\infty} P_k$ is a formal power series with P_k of degree k. As the total residue vanishes outside the homogeneous component of degree $-r$ of A_r, we can define $\mathrm{Tres}_{A_r}(P/Q) = \mathrm{Tres}_{A_r}(P_{q-r}/Q)$, where q is degree of Q. For $a \in V$ and multiplicities $m = (m_{i,j}) \in (\mathbb{Z}_{\geqslant 0})^M$ of elements of A_r^+, the function

$$F := \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{i=1}^r x_i^{m_{i,r+1}} \prod_{1 \leqslant i < j \leqslant r} (x_i - x_j)^{m_{i,j}}}$$

is in $\hat{R}_{A_r}$. We define $J_{A_r^+,m}(a) \in S_{A_r}$ by

$$J_{A_r^+,m}(a) = \mathrm{Tres}_{A_r} F.$$

Next, we describe the iterated residue.

Definition 4. For $f \in R_{A_r}$, we define the *iterated residue* by

$$\mathrm{Ires}_{x=0} f = \mathrm{Res}_{x_1=0} \mathrm{Res}_{x_2=0} \cdots \mathrm{Res}_{x_r=0} f(x_1, \dots, x_r).$$

Since the iterated residue $\mathrm{Ires}_{x=0} f$ vanishes on the space $\partial(U)R_{A_r}$ as in [3], we have

$$\mathrm{Ires}_{x=0} J_{A_r^+,m}(a) = \mathrm{Ires}_{x=0} F. \tag{1}$$

2.3 Chambers and Jeffrey–Kirwan residue

Definition 5. Let $C(\nu)$ be the closed cone generated by ν for any subset ν of A_r^+ and let $C(A_r^+)_{\mathrm{sing}}$ be the union of the cones $C(\nu)$ where ν is any subset of A_r^+ of cardinal strictly less than $r = \dim V$. By definition, the set $C(A_r^+)_{\mathrm{reg}}$ of A_r^+-regular elements is the complement of $C(A_r^+)_{\mathrm{sing}}$. A connected component of $C(A_r^+)_{\mathrm{reg}}$ is called a *chamber*.

The Jeffrey–Kirwan residue [10] associated to a chamber $\mathfrak{c}$ of $C(A_r^+)$ is a linear form $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$ on the vector space S_{A_r} of simple fractions. Any function f in S_{A_r} can be written as a linear combination of functions f_σ, with a basis σ of A_r contained in A_r^+. To determine the linear map $f \mapsto \langle\langle \mathfrak{c}, f\rangle\rangle$, it is enough to determine it on this set of functions f_σ. So we assume that σ is a basis of A_r contained in A_r^+.

Definition 6. For a chamber $\mathfrak{c}$ and $f_\sigma \in S_{A_r}$, we define the Jeffrey–Kirwan residue $\langle\langle \mathfrak{c}, f_\sigma\rangle\rangle$ associated to a chamber $\mathfrak{c}$ as follows:

- If $\mathfrak{c} \subset C(\sigma)$, then $\langle\langle \mathfrak{c}, f_\sigma\rangle\rangle = 1$.
- If $\mathfrak{c} \cap C(\sigma) = \emptyset$, then $\langle\langle \mathfrak{c}, f_\sigma\rangle\rangle = 0$,

where $C(\sigma)$ is the convex cone generated by σ.

Remark 7. More generally, as in [3, Definition 11], the Jeffrey–Kirwan residue $\langle\langle \mathfrak{c}, f_\sigma\rangle\rangle$ is defined to be $\frac{1}{\mathrm{vol}(\sigma)}$ if $\mathfrak{c} \subset C(\sigma)$, where $\mathrm{vol}(\sigma)$ is the volume of the parallelepiped $\oplus_{\alpha\in\sigma}[0,1]\alpha$, relative to our Lebesgue measure da. In our case, the volume $\mathrm{vol}(\sigma)$ is equal to 1 since A_r is unimodular.

The volume $v_{A_r^+,m}(a)$ of the flow polytope $P_{A_r^+,m}(a)$ is written by the function $J_{A_r^+,m}(a)$ and the Jeffrey–Kirwan residue in the following.

Theorem 8 (Baldoni–Vergne [3])**.** *Let $\mathfrak{c}$ be a chamber of $C(A_r^+)$. Then, for $a \in \bar{\mathfrak{c}}$, the volume $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$ is given by*

$$v_{A_r^+,m}(a) = \langle\langle \mathfrak{c}, J_{A_r^+,m}(a)\rangle\rangle.$$

We denote by $v_{A_r^+,m,\mathfrak{c}}(a)$ the polynomial function of a coinciding with $v_{A_r^+,m}(a)$ when $a \in \bar{\mathfrak{c}}$. It is a homogeneous polynomial of degree $M - r$.

2.4 Nice chamber

Definition 9. The open subset $\mathfrak{c}_{\mathrm{nice}}$ of $C(A_r^+)$ is defined by

$$\mathfrak{c}_{\mathrm{nice}} = \{a \in C(A_r^+) \mid a_i > 0 \ (i = 1, \ldots, r)\}.$$

The set $\mathfrak{c}_{\mathrm{nice}}$ is in fact a chamber for the root system A_r^+ ([3]). The chamber $\mathfrak{c}_{\mathrm{nice}}$ is called the *nice chamber*.

Lemma 10 (Baldoni–Vergne [3])**.** *For the nice chamber $\mathfrak{c}_{\mathrm{nice}}$ of A_r^+ and $f \in S_{A_r}$, we have*

$$\langle\langle \mathfrak{c}_{\mathrm{nice}}, f\rangle\rangle = \mathrm{Ires}_{x=0} f.$$

From Theorem 8, Lemma 10 and (1), we have the following corollary.

Corollary 11 (Lidskii formula [3]). *Let $a \in \overline{\mathfrak{c}_{\text{nice}}}$. Then the volume function $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ is given by*

$$v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Ires}_{x=0} F.$$

2.5 Examples

In this subsection, we give some examples of the flow polytopes for A_1, A_2, and A_3, and calculate their volumes.

Example 12. When $r = 1$, the nice chamber of A_1^+ is $\mathfrak{c}_{\text{nice}} = \{a = a_1(e_1 - e_2) \mid a_1 > 0\}$. For $a = a_1(e_1 - e_2) \in \overline{\mathfrak{c}_{\text{nice}}}$,

$$P_{A_1^+,m}(a) = \left\{(y_{i,j,k}) \in \mathbb{R}^{m_{1,2}} \mid y_{i,j,k} \geqslant 0\,,\ y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,m_{1,2}} = a_1\right\}.$$

From Corollary 11, we have

$$v_{A_1^+,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Res}_{x_1=0}\left(\frac{e^{a_1x_1}}{x_1^{m_{1,2}}}\right) = \frac{1}{(m_{1,2}-1)!}a_1^{m_{1,2}-1}.$$

Example 13. When $r = 2$, there are two chambers $\mathfrak{c}_1, \mathfrak{c}_2$ of A_2^+ as below, and the nice chamber $\mathfrak{c}_{\text{nice}}$ of A_2^+ is $\mathfrak{c}_1$.

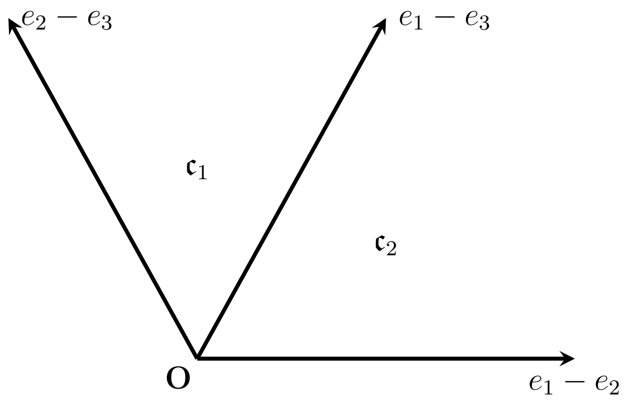

Figure 1 : The chamber of A_2^+.

For example, we set $m_{1,2} = n$ ($n \in \mathbb{Z}_{>0}$), $m_{1,3} = 1$, and $m_{2,3} = 1$. For $a = a_1e_1 + a_2e_2 - (a_1 + a_2)e_3 \in \overline{\mathfrak{c}_{\text{nice}}}$,

$$P_{A_2^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^{n+2} \;\middle|\; \begin{array}{l} y_{i,j,k} \geqslant 0 \\ y_{1,2,1} + y_{1,2,2} + \cdots + y_{1,2,n} + y_{1,3,1} = a_1 \\ -y_{1,2,1} - y_{1,2,2} - \cdots - y_{1,2,n} + y_{2,3,1} = a_2 \end{array} \right\}.$$

From Corollary 11, we have

$$v_{A_2^+,m,\mathfrak{c}_{\text{nice}}}(a) = \text{Ires}_{x=0}\left(\frac{e^{a_1x_1+a_2x_2}}{x_1x_2(x_1-x_2)^n}\right) = \text{Res}_{x_1=0}\text{Res}_{x_2=0}\left(\frac{e^{a_1x_1+a_2x_2}}{x_1x_2(x_1-x_2)^n}\right) = \frac{1}{n!}a_1^n.$$

Example 14. When $r = 3$, there are seven chambers of A_3^+ as below ([1]), and the nice chamber $\mathfrak{c}_{\text{nice}}$ of A_3^+ is $\mathfrak{c}_1$.

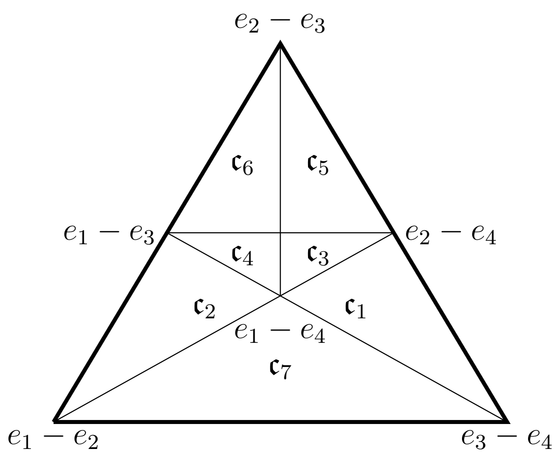

Figure 2 : The chamber of A_3^+.

For example, we set $m_{1,2} = 1$, $m_{1,3} = 1$, $m_{1,4} = 2$, $m_{2,3} = 1$, $m_{2,4} = 2$, and $m_{3,4} = 2$. For $a = \sum_{i=1}^{3} a_i(e_i - e_4) \in \overline{\mathfrak{c}_{\text{nice}}}$,

$$P_{A_3^+,m}(a) = \left\{ (y_{i,j,k}) \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} y_{i,j,k} \geqslant 0 \\ y_{1,2,1} + y_{1,3,1} + y_{1,4,1} + y_{1,4,2} = a_1 \\ -y_{1,2,1} + y_{2,3,1} + y_{2,4,1} + y_{2,4,2} = a_2 \\ -y_{1,3,1} - y_{2,3,1} + y_{3,4,1} + y_{3,4,2} = a_3 \end{array} \right\}.$$

From Corollary 11, we have

$$\begin{aligned} v_{A_3^+,m,\mathfrak{c}_{\text{nice}}}(a) &= \operatorname{Ires}_{x=0}\left(\frac{e^{a_1x_1+a_2x_2+a_3x_3}}{x_1^2x_2^2x_3^2(x_1-x_2)(x_1-x_3)(x_2-x_3)}\right) \\ &= \frac{1}{360}a_1^3(a_1^3 + 6a_1^2a_2 + 3a_1^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^3 + 30a_2^2a_3). \end{aligned}$$

3 Main theorems

In this section, we prove the main theorems of this paper. Let $\mathfrak{c}_{\text{nice}}$ be the nice chamber of A_r^+ and let $a = \sum_{i=1}^{r} a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}_{\text{nice}}}$.

Theorem 15. *For $a \in \overline{\mathfrak{c}_{\text{nice}}}$, let $P_{A_r^+,m}(a)$ be the flow polytope as in Definition 2 and let $v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ be the volume of $P_{A_r^+,m}(a)$. Then $v = v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$ satisfies the system of differential equations as follows:*

$$\begin{cases} \partial_r^{m_{r,r+1}} v = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v = 0 \\ \quad\vdots \\ (\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} v = 0, \end{cases} \tag{2}$$

where $\partial_i = \frac{\partial}{\partial a_i}$ *for* $i = 1, \ldots, r$.

Proof. We will prove the first two relations. Let $F = \frac{e^{a_1x_1+\cdots+a_rx_r}}{\prod_{i=1}^{r} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant r}(x_i-x_j)^{m_{i,j}}}$. It is easy to see that

$$P(\partial_1, \ldots, \partial_r)(\mathrm{Ires}_{x=0}F) = \mathrm{Ires}_{x=0}(P(\partial_1, \ldots, \partial_r)F) = \mathrm{Ires}_{x=0}(P(x_1, \ldots, x_r)F), \quad (3)$$

where P is a polynomial. Since $\frac{e^{a_1x_1+\cdots+a_kx_k}}{\prod_{i=1}^{k-1} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant k}(x_i-x_j)^{m_{i,j}}}$ is holomorphic at $x_k = 0$,

$$\mathrm{Res}_{x_k=0}\left(\frac{e^{a_1x_1+\cdots+a_kx_k}}{\prod_{i=1}^{k-1} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant k}(x_i-x_j)^{m_{i,j}}}\right) = 0 \quad (4)$$

for $k = 1, \ldots, r$. Therefore, from Corollary 11, (3) and (4), we obtain

$$\begin{aligned}\partial_r^{m_{r,r+1}} v &= \partial_r^{m_{r,r+1}} \mathrm{Ires}_{x=0}F = \mathrm{Ires}_{x=0}\partial_r^{m_{r,r+1}} F \\ &= \mathrm{Ires}_{x=0}\left(\frac{e^{a_1x_1+\cdots+a_rx_r}}{\prod_{i=1}^{r-1} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant r}(x_i-x_j)^{m_{i,j}}}\right) = 0,\end{aligned}$$

and

$$\begin{aligned}&(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} v \\ &= \mathrm{Ires}_{x=0}(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} F \\ &= \mathrm{Ires}_{x=0}(\partial_{r-1} - \partial_r)^{m_{r-1,r}}\left(\frac{e^{a_1x_1+\cdots+a_rx_r}}{x_r^{m_{r,r+1}}\prod_{i=1}^{r-2} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant r}(x_i-x_j)^{m_{i,j}}}\right) \\ &= \mathrm{Ires}_{x=0}\left(\frac{e^{a_1x_1+\cdots+a_rx_r}}{x_r^{m_{r,r+1}}\prod_{i=1}^{r-2} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant r,(i,j)\neq(r-1,r)}(x_i-x_j)^{m_{i,j}}}\right) \\ &= \mathrm{Res}_{x_1=0}\cdots\left(\mathrm{Res}_{x_{r-1}=0}\left(\frac{e^{a_1x_1+\cdots+a_{r-1}x_{r-1}}}{\prod_{i=1}^{r-2} x_i^{m_{i,r+1}} \prod_{1\leqslant i<j\leqslant r-1}(x_i-x_j)^{m_{i,j}}}\right.\right. \\ &\quad \left.\left.\times \mathrm{Res}_{x_r=0}\left(\frac{e^{a_rx_r}}{x_r^{m_{r,r+1}}\prod_{i=1}^{r-2}(x_i-x_r)^{m_{i,r}}}\right)\right)\right) = 0.\end{aligned}$$

Similarly, we can verify the remaining expressions. □

Remark 16. In general, it is known that the volume function $v_{A_r^+,m}(a)$ of $P_{A_r^+,m}(a)$, as a distribution on V, satisfies the differential equation

$$Lv_{A_r^+,m}(a) = \delta(a),$$

where $L = \prod_{i<j}(\partial_i - \partial_j)^{m_{i,j}}$ and $\delta(a)$ is the Dirac delta function on V ([8, 11]). Note that ∂_{r+1} in the definition of L is supposed to be zero. Theorem 15 above, together with Proposition 17 and Theorem 18 as below, characterizes the function $v_{A_r^+,m,\mathfrak{c}_{\mathrm{nice}}}(a)$ on $\overline{\mathfrak{c}_{\mathrm{nice}}}$ more explicitly.

Let $M_\ell = \sum_{i=\ell+1}^{r+1} m_{\ell,i}$ for $\ell = 1, \ldots, r$. Then we have the following proposition.

Proposition 17. *The coefficient of $a_1^{M_1-1} a_2^{M_2-1} \cdots a_{r-1}^{M_{r-1}-1} a_r^{M_r-1}$ in the volume function $v_{A_r^+,m}(a)$ is given by*

$$\frac{1}{(M_1-1)!(M_2-1)!\cdots(M_{r-1}-1)!(M_r-1)!}.$$

Proof. From the Lidskii formula in Corollary 11, we have

$$v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a) = \sum_{|i|=\ell-r} \frac{a_1^{i_1}}{i_1!} \cdots \frac{a_r^{i_r}}{i_r!} \mathrm{Ires}_{x=0} \left(\frac{x_1^{i_1} \cdots x_r^{i_r}}{\prod_{i=1}^r x_i^{m_{i,r+1}} \prod_{1 \leqslant i<j \leqslant r} (x_i - x_j)^{m_{i,j}}} \right),$$

where $|i| = i_1 + \cdots + i_r$. When $i_\ell = M_\ell - 1$ for $\ell = 1, \ldots, r$,

$$\begin{aligned}
&\mathrm{Ires}_{x=0} \left(\frac{x_1^{M_1-1} \cdots x_r^{M_r-1}}{\prod_{i=1}^r x_i^{m_{i,r+1}} \prod_{1 \leqslant i<j \leqslant r} (x_i - x_j)^{m_{i,j}}} \right) \\
&= \mathrm{Res}_{x_1=0} \cdots \mathrm{Res}_{x_{r-1}=0} \mathrm{Res}_{x_r=0} \left(\frac{x_1^{(\sum_{i=2}^r m_{1,i})-1} \cdots x_{r-1}^{m_{r-1,r}-1}}{x_r \prod_{1 \leqslant i<j \leqslant r} (x_i - x_j)^{m_{i,j}}} \right) \\
&= \mathrm{Res}_{x_1=0} \cdots \mathrm{Res}_{x_{r-1}=0} \left(\frac{x_1^{(\sum_{i=2}^{r-1} m_{1,i})-1} \cdots x_{r-2}^{m_{r-2,r-1}-1}}{x_{r-1} \prod_{1 \leqslant i<j \leqslant r-1} (x_i - x_j)^{m_{i,j}}} \right) \\
&= \mathrm{Res}_{x_1=0} \frac{1}{x_1} = 1.
\end{aligned}$$

Thus we obtain the proposition. □

Theorem 18. *Let $\phi_r = \phi(a_1, \ldots, a_r)$ be a homogeneous polynomial of $a_1, \ldots, a_r$ with degree d and let $M = \sum_{1 \leqslant i<j \leqslant r+1} m_{i,j}$. Suppose ϕ_r satisfies the system of differential equations as follows:*

$$\begin{cases}
\partial_r^{m_{r,r+1}} \phi_r = 0 \\
(\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} \phi_r = 0 \\
\quad \vdots \\
(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r = 0.
\end{cases} \tag{5}$$

(i) *If $M - r < d$, then $\phi_r = 0$.*

(ii) *If $0 \leqslant d \leqslant M - r$, then there is a non trivial homogeneous polynomial ϕ_r satisfying* (4).

(iii) *If $d = M - r$ in particular, ϕ_r is equal to a constant multiple of $v = v_{A_r^+,m,\mathfrak{c}_{\text{nice}}}(a)$.*

Proof. We argue by induction on r. In the case that $r = 1$, we write

$$\phi_1 = \phi(a_1) = pa_1^d,$$

where p is a constant. If $m_{1,2} - 1 < d$ and ϕ_1 satisfies the differential equation $\partial_1^{m_{1,2}} \phi_1 = 0$, then $p = 0$ and hence $\phi_1 = 0$. If $0 \leqslant d \leqslant m_{1,2} - 1$, then for any $p \neq 0$, $\partial_1^{m_{1,2}} \phi_1 = 0$. Also, if $d = m_{1,2} - 1$, in particular, then $\phi_1 = pa_1^{m_{1,2}-1}$, while $v = \frac{1}{(m_{1,2}-1)!} a_1^{m_{1,2}-1}$ as in Example 12. Hence ϕ_1 is equal to a constant multiple of v.

We assume that the statement of this theorem holds for $r - 1$. We write ϕ_r as

$$\phi_r = \phi(a_1, \dots, a_r) = g_d(a_2, \dots, a_r) + a_1 g_{d-1}(a_2, \dots, a_r) + \cdots + a_1^d g_0(a_2, \dots, a_r),$$

where g_k is a homogeneous polynomial of $a_2, \dots, a_r$ with degree k for $k = 0, 1, \dots, d$. Then for $k = 0, 1, \dots, d$, g_k satisfies the differential equations as follows:

$$\begin{cases} \partial_r^{m_{r,r+1}} g_k = 0 \\ (\partial_{r-1} - \partial_r)^{m_{r-1,r}} \partial_{r-1}^{m_{r-1,r+1}} g_k = 0 \\ \quad \vdots \\ (\partial_2 - \partial_3)^{m_{2,3}} (\partial_2 - \partial_4)^{m_{2,4}} \cdots (\partial_2 - \partial_r)^{m_{2,r}} \partial_2^{m_{2,r+1}} g_k = 0. \end{cases} \tag{6}$$

We set $h = (\sum_{2 \leqslant i < j \leqslant r+1} m_{i,j}) - (r - 1)$. From the inductive assumption, if $0 \leqslant k \leqslant h$, then g_k is a homogeneous polynomial. On the other hand, if $h + 1 \leqslant k \leqslant d$, then $g_k = 0$, namely,

$$g_d(a_2, \dots, a_r) = g_{d-1}(a_2, \dots, a_r) = \cdots = g_{h+1}(a_2, \dots, a_r) = 0. \tag{7}$$

(i) We consider the case of $M - r < d$. Let $M_1 = \sum_{i=2}^{r+1} m_{1,i}$. Now we compare the coefficients of $a_1^{d-h-M_1+n}$ in $(\partial_1 - \partial_2)^{m_{1,2}} (\partial_1 - \partial_3)^{m_{1,3}} \cdots (\partial_1 - \partial_r)^{m_{1,r}} \partial_1^{m_{1,r+1}} \phi_r$ for $n = 0, \dots, h$. For $q = 1, \dots, M_1 - m_{1,r+1}$, we define

$$\begin{aligned} D_q = &\sum_{2 \leqslant i_1 \leqslant r} \binom{m_{1,i_1}}{q} \partial_{i_1}^q + \cdots + \sum_{\substack{p_1 + \cdots + p_k = q \\ 2 \leqslant i_1 < \cdots < i_k \leqslant r}} \left(\prod_{1 \leqslant l \leqslant k} \binom{m_{1,i_l}}{p_l} \right) \partial_{i_1}^{p_1} \partial_{i_2}^{p_2} \cdots \partial_{i_k}^{p_k} \\ &+ \cdots + \sum_{2 \leqslant i_1 < \cdots < i_q \leqslant r} \left(\prod_{1 \leqslant l \leqslant q} \binom{m_{1,i_l}}{1} \right) \partial_{i_1} \partial_{i_2} \cdots \partial_{i_q}. \end{aligned}$$

Then we have the following equation:

$$\begin{aligned} &\frac{(d-h+n)!}{(d-h-M_1+n)!} g_{h-n}(a_2, \dots, a_r) - \frac{(d-h+n-1)!}{(d-h-M_1+n)!} D_1 g_{h-n+1}(a_2, \dots, a_r) \\ &+ \cdots + (-1)^j \frac{(d-h+n-j)!}{(d-h-M_1+n)!} D_j g_{h-n+j}(a_2, \dots, a_r) \\ &+ \cdots + (-1)^{M_1 - m_{1,r+1}} \frac{(d-h+n-(M_1 - m_{1,r+1}))!}{(d-h-M_1+n)!} D_{M_1 - m_{1,r+1}} g_{h-n+M_{1,r}}(a_2, \dots, a_r) \\ &= 0. \end{aligned} \tag{8}$$

When $n = 0$, from (7) and (8), we have

$$g_h(a_2, \ldots, a_r) = 0.$$

When $n = 1$, we have

$$\frac{(d-h+1)!}{(d-h-M_1+1)!} g_{h-1}(a_2, \ldots, a_r) - \frac{(d-h)!}{(d-h-M_1+1)!} D_1 g_h(a_2, \ldots, a_r) = 0.$$

Thus we have

$$g_{h-1}(a_2, \ldots, a_r) = 0.$$

Similarly, we have

$$g_{h-2}(a_2, \ldots, a_r) = g_{h-3}(a_2, \ldots, a_r) = \cdots = g_0(a_2, \ldots, a_r) = 0$$

and hence $\phi_r = 0$.

(ii) We consider the case of $0 \leqslant d \leqslant M - r$. By the inductive assumption, there is a non trivial homogeneous polynomial g_{h-n_1+i} satisfying (6) for $i = 1, \ldots, n_1$, where $n_1 = M - r - d + 1$. We can take

$$g_{h-n_1+i}(a_2, \ldots, a_r) \neq 0.$$

When $n = n_1$, from (7) and (8),

$$\begin{aligned} g_{h-n_1}(a_2, \ldots, a_r) = & \frac{(d-h+n_1-1)!}{(d-h+n_1)!} D_1 g_{h-n_1+1}(a_2, \ldots, a_r) \\ & - \frac{(d-h+n_1-2)!}{(d-h+n_1)!} D_2 g_{h-n_1+2}(a_2, \ldots, a_r) \\ & + \cdots + (-1)^{n_1-1} \frac{(d-h)!}{(d-h+n_1)!} D_{n_1} g_h(a_2, \ldots, a_r). \end{aligned}$$

When $n = n_1 + 1$,

$$\begin{aligned} g_{h-(n_1+1)}(a_2, \ldots, a_r) = & \frac{(d-h+n_1)!}{(d-h+n_1+1)!} D_1 g_{h-n_1}(a_2, \ldots, a_r) \\ & - \frac{(d-h+n_1-1)!}{(d-h+n_1+1)!} D_2 g_{h-n_1+1}(a_2, \ldots, a_r) \\ & + \cdots + (-1)^{n_1} \frac{(d-h)!}{(d-h+n_1+1)!} D_{n_1+1} g_h(a_2, \ldots, a_r). \end{aligned}$$

Similarly, for $n = n_1 + 2, \ldots, h$, we can express $g_{h-j}(a_2, \ldots, a_r)$ $(j = n_1, n_1 + 1, \ldots, h)$ in terms of $g_{h-j+i}(a_2, \ldots, a_r)$ $(i = 1, \ldots, j)$ and their partial derivatives. Namely, we can express ϕ_r in terms of $g_{h-n_1+i}(a_2, \ldots, a_r)$ and their partial derivatives. It follows that $\phi_r \neq 0$ when $0 \leqslant d \leqslant M - r$.

(iii) If $d = M - r$ in particular, then $n_1 = 1$, and g_{h-j} $(j = 1, \ldots, h)$ becomes the linear combination of g_h and their partial derivatives. Therefore ϕ_r is uniquely determined by

g_h. Moreover, from the inductive assumption, $g_h = C \cdot v_{A^+_{r-1},m',\mathfrak{c}'_{\text{nice}}}$, where C is a constant, $m' = (m_{i,j})_{2\leqslant i<j\leqslant r+1}$, and $\mathfrak{c}'_{\text{nice}}$ is a nice chamber of A^+_{r-1}. Hence the solution of (5) is unique up to a constant multiple. On the other hand, by Theorem 15, $v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}$ satisfies the system of differential equations (5). Hence ϕ_r is equal to a constant multiple of $v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}$. □

Recall that in the proof of Theorem 18, we have defined the operator

$$D_q = \sum_{2\leqslant i_1\leqslant r} \binom{m_{1,i_1}}{q} \partial^q_{i_1} + \cdots + \sum_{\substack{p_1+\cdots+p_k=q \\ 2\leqslant i_1<\cdots<i_k\leqslant r}} \left(\prod_{1\leqslant l\leqslant k} \binom{m_{1,i_l}}{p_l} \right) \partial^{p_1}_{i_1}\partial^{p_2}_{i_2}\cdots\partial^{p_k}_{i_k}$$
$$+ \cdots + \sum_{2\leqslant i_1<\cdots<i_q\leqslant r} \left(\prod_{1\leqslant l\leqslant q} \binom{m_{1,i_l}}{1} \right) \partial_{i_1}\partial_{i_2}\cdots\partial_{i_q} \tag{9}$$

for $q = 1, \ldots, M_1 - m_{1,r+1}$.

Remark 19. Let $M_1 = \sum_{i=2}^{r+1} m_{1,i}$. When $d = M - r$, from the proof of Theorem 18 (iii), g_{h-j} $(j = 1, \ldots, h)$ is uniquely determined as follows:

$$\begin{cases} g_{h-1} = \frac{(M_1-1)!}{M_1!} D_1 g_h \\ g_{h-2} = \frac{(M_1-1)!}{(M_1+1)!}(D_1^2 - D_2) g_h \\ g_{h-3} = \frac{(M_1-1)!}{(M_1+2)!}(D_1^3 - 2D_1D_2 + D_3) g_h \\ \quad\vdots \\ g_0 = \frac{(M_1-1)!}{(M-r)!}(D_1^h - (h-1)D_1^{h-2}D_2 + \cdots + (-1)^{h-1}D_h) g_h. \end{cases}$$

Let $m' = (m_{i,j})_{2\leqslant i<j\leqslant r+1}$, $\mathfrak{c}'_{\text{nice}}$ a nice chamber of A^+_{r-1} and $a' = \sum_{i=2}^{r} a_i(e_i - e_{r+1}) \in \overline{\mathfrak{c}'_{\text{nice}}}$. From Proposition 17 and Remark 19, we obtain the following theorem.

Theorem 20. *Let $h = (\sum_{2\leqslant i<j\leqslant r+1} m_{i,j}) - (r-1)$ and let D_q $(q = 1, \ldots, h)$ be as in (9). Then $v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}(a)$ is written by the linear combination of $v_{A^+_{r-1},m',\mathfrak{c}'_{\text{nice}}}(a')$ and its partial derivatives as follows:*

$$\begin{aligned} v_{A^+_r,m,\mathfrak{c}_{\text{nice}}}(a) = \Bigg\{ & \frac{a_1^{M_1-1}}{(M_1-1)!} + \frac{a_1^{M_1}}{M_1!} D_1 + \frac{a_1^{M_1+1}}{(M_1+1)!}(D_1^2 - D_2) \\ & + \frac{a_1^{M_1+2}}{(M_1+2)!}(D_1^3 - 2D_1D_2 + D_3) + \cdots \\ & + \frac{a_1^{M-r}}{(M-r)!}(D_1^h - (h-1)D_1^{h-2}D_2 + \cdots + (-1)^{h-1}D_h) \Bigg\} v_{A^+_{r-1},m',\mathfrak{c}'_{\text{nice}}}(a'). \end{aligned} \tag{10}$$

Example 21. Let $r = 3$, let $a = \sum_{i=1}^{3} a_i(e_i - e_4) \in \overline{\mathfrak{c}_{\text{nice}}}$ and let $a' = \sum_{i=2}^{3} a_i(e_i - e_4) \in \overline{\mathfrak{c}'_{\text{nice}}}$. We set $m_{1,2} = 1, m_{1,3} = 1, m_{1,4} = 2, m_{2,3} = 1, m_{2,4} = 2$ and $m_{3,4} = 2$ as in Example 14. Then we have

$$v_{A^+_3,m,\mathfrak{c}_{\text{nice}}}(a) = \frac{1}{360} a_1^3(a_1^3 + 6a_1^2a_2 + 3a_1^2a_3 + 15a_1a_2^2 + 15a_1a_2a_3 + 10a_2^3 + 30a_2^2a_3).$$

We can check that $v = v_{A_3^+, m, \mathfrak{c}_{\text{nice}}}(a)$ satisfies the system of differential equations as follows:

$$\begin{cases} \partial_3^2 v = 0 \\ (\partial_2 - \partial_3)\partial_2^2 v = 0 \\ (\partial_1 - \partial_2)(\partial_1 - \partial_3)\partial_1^2 v = 0. \end{cases}$$

Also, from Proposition 17, the coefficient of the term $a_1^3 a_2^2 a_3$ is $\frac{1}{3!2!1!} = \frac{1}{12}$. When $r = 2$,

$$v_{A_2^+, m', \mathfrak{c}'_{\text{nice}}}(a') = \frac{1}{6} a_2^2 (a_2 + 3a_3).$$

Therefore, we have

$$\left\{ \frac{a_1^3}{6} + \frac{a_1^4}{24} D_1 + \frac{a_1^5}{120}(D_1^2 - D_2) + \frac{a_1^6}{720}(D_1^3 - 2D_1 D_2 + D_3) \right\} v_{A_2^+, m', \mathfrak{c}'_{\text{nice}}}(a')$$
$$= \frac{a_1^3 a_2^3}{36} + \frac{a_1^3 a_2^2 a_3}{12} + \frac{a_1^4 a_2^2}{24} + \frac{a_1^4 a_2 a_3}{24} + \frac{a_1^5 a_2}{60} + \frac{a_1^5 a_3}{120} + \frac{a_1^6}{360} = v_{A_3^+, m, \mathfrak{c}_{\text{nice}}}(a)$$

as in (10).

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