

Balanced equi- n -squares

Saieed Akbari

Department of Mathematical Sciences
Sharif University of Technology, Tehran, Iran
`s_akbari@sharif.edu`

Trent G. Marbach Rebecca J. Stones Zhuanhao Wu

Nankai-Baidu Joint Laboratory, College of Computer Science
Nankai University, Tianjin, China

`trent.marbach@gmail.com`, `rebecca.stones82@gmail.com`, `zhwu95@gmail.com`

Submitted: Nov 6, 2019; Accepted: Oct 1, 2020; Published: Oct 16, 2020

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Abstract

We define a d -balanced equi- n -square $L = (l_{ij})$, for some divisor d of n , as an $n \times n$ matrix containing symbols from \mathbb{Z}_n in which any symbol that occurs in a row or column, occurs exactly d times in that row or column. We show how to construct a d -balanced equi- n -square from a partition of a Latin square of order n into $d \times (n/d)$ subrectangles. In design theory, L is equivalent to a decomposition of $K_{n,n}$ into d -regular spanning subgraphs of $K_{n/d, n/d}$. We also study when L is diagonally cyclic, defined as when $l_{(i+1)(j+1)} = l_{ij} + 1$ for all $i, j \in \mathbb{Z}_n$, which corresponds to cyclic such decompositions of $K_{n,n}$ (and thus α -labellings).

We identify necessary conditions for the existence of (a) d -balanced equi- n -squares, (b) diagonally cyclic d -balanced equi- n -squares, and (c) Latin squares of order n which partition into $d \times (n/d)$ subrectangles. We prove the necessary conditions are sufficient for arbitrary fixed $d \geq 1$ when n is sufficiently large, and we resolve the existence problem completely when $d \in \{1, 2, 3\}$.

Along the way, we identify a bijection between α -labellings of d -regular bipartite graphs and what we call d -starters: matrices with exactly one filled cell in each top-left-to-bottom-right unbroken diagonal, and either d or 0 filled cells in each row and column. We use d -starters to construct diagonally cyclic d -balanced equi- n -squares, but this also gives new constructions of α -labellings.

Mathematics Subject Classifications: 05B15, 05C51, 05C78

1 Introduction

An $n \times n$ matrix containing exactly n copies of each symbol from a set of size n is called an *equi- n -square* (or *gerechte framework* [9, 11], or *gerechte skeleton* [33]). We typically use \mathbb{Z}_n as the symbol set. We call an equi- n -square *d -balanced* if any symbol that occurs in a row or column, occurs exactly d times in that row or column. The question we focus on in this paper is:

For what parameters d and n do there exist d -balanced equi- n -squares?

Examples of balanced equi- n -squares are given in Figure 1. Balanced equi- n -squares are a generalization of Latin squares: Latin squares are precisely 1-balanced equi- n -squares.

0	0	1	1	2	2
0	2	0	3	3	2
3	0	0	1	3	1
3	4	4	3	5	5
4	2	4	5	2	5
4	4	1	5	5	1

0	0	0	3	3	3	6	6	6
7	1	1	1	4	4	4	7	7
8	8	2	2	2	5	5	5	8
0	0	0	3	3	3	6	6	6
7	1	1	1	4	4	4	7	7
8	8	2	2	2	5	5	5	8
0	0	0	3	3	3	6	6	6
7	1	1	1	4	4	4	7	7
8	8	2	2	2	5	5	5	8

Figure 1: An example of a 2-balanced equi-6-square and a 3-balanced equi-9-square.

A square matrix $L = (l_{ij})$ on the symbol set \mathbb{Z}_n , with rows and columns indexed by \mathbb{Z}_n , which satisfies the property

$$l_{(i+1)(j+1)} = l_{ij} + 1 \quad (1)$$

is called *diagonally cyclic*. The 3-balanced equi-9-square in Figure 1 is a diagonally cyclic matrix. Diagonally cyclic equi- n -squares are a generalization of diagonally cyclic Latin squares [34] (which are diagonally cyclic equi-1-squares).

We observe that: (a) diagonally cyclic $n \times n$ matrices on the symbol set \mathbb{Z}_n are equi- n -squares (since each of the n broken diagonals contains each symbol exactly once), and (b) diagonally cyclic matrices $L = (l_{ij})$ are uniquely determined by their first row.

A *subrectangle* (resp. *subsquare*) of a Latin square is a rectangular (resp. square) submatrix in which the number of distinct symbols equals the number of columns. The following lemma gives necessary conditions for the existence of d -balanced equi- n -squares, and describes a relationship between d -balanced equi- n -squares and Latin squares of order n which decompose into $d \times (n/d)$ subrectangles.

Lemma 1 (Necessary conditions).

1. For a d -balanced equi- n -square to exist, d must be a divisor of n , and $d^2 \leq n$.
2. If d is even, for a diagonally cyclic d -balanced equi- n -square to exist, d must be a divisor of $n/2$.
3. For a diagonally cyclic 1-balanced equi- n -square (i.e., a diagonally cyclic Latin square) to exist, n must be odd.
4. For $d \geq 1$, a Latin square of order n that decomposes into $d \times (n/d)$ subrectangles exists only if a d -balanced equi- n -square exists.

Proof. The symbols in the first row of a d -balanced matrix induce a partition of the n column indices into parts of size d , so d must divide n . A d -balanced equi- n -square $L = (l_{ij})$ has $\geq d$ rows in which the symbol 0 occurs, and in each of those rows the symbol 0 occurs d times. Thus, since the symbol 0 occurs n times, we must have $d^2 \leq n$. This proves the first claim.

Now suppose L is diagonally cyclic and d is even. The sum of the symbols in any row or column of L must be divisible by d (as each symbol which occurs, occurs exactly d times). In particular, the first column of L contains the multiset of symbols $\{l_{0j} - j\}_{j=0}^{n-1}$, from which we obtain

$$\underbrace{\sum_{j=0}^{n-1} (l_{0j} - j)}_{\substack{\text{sum of first column} \\ \Rightarrow \text{divisible by } d}} = \underbrace{\sum_{j=0}^{n-1} l_{0j}}_{\substack{\text{sum of first row} \\ \Rightarrow \text{divisible by } d}} + \frac{n(n-1)}{2}.$$

If $d = 1$ the first row and first column have the same sum, implying $n(n-1)/2 \equiv 0 \pmod{n}$ which is satisfied only if n is odd, which implies the third claim; now assume $d \geq 2$. As d divides n and $d \geq 2$, we know d is coprime to $n-1$, so the above equation implies that d divides $n/2$, proving the second claim.

We defer the proof of the fourth claim to Lemma 8. □

This paper proves, for $d \in \{1, 2, 3\}$ the necessary conditions in Lemma 1 are sufficient (with some small exceptions), and for all $d \geq 1$ the necessary conditions in Lemma 1 are sufficient except for finitely many n . Specifically, we prove the following two theorems.

Theorem 1. For $n \geq 1$,

1.
 - a 1-balanced equi- n -square exists for all n ,
 - a diagonally cyclic 1-balanced equi- n -square exists if and only if n is odd, and
 - there exists a Latin square of order n which decomposes into $1 \times n$ subrectangles for all n ;
2.
 - a 2-balanced equi- n -square exists if and only if n is even and $n \neq 2$,

- a diagonally cyclic 2-balanced equi- n -square exists if and only if 4 divides n , and
 - there exists a Latin square of order n which decomposes into $2 \times (n/2)$ subrectangles for all even $n \notin \{2, 6\}$; and
- 3.
- a 3-balanced equi- n -square exists if and only if 3 divides n and $n \geq 9$,
 - a diagonally cyclic 3-balanced equi- n -square exists if and only if 3 divides n and $n \geq 9$, and
 - there exists a Latin square of order n which decomposes into $3 \times (n/3)$ subrectangles if and only if 3 divides n and $n \geq 9$.

Theorem 2. For sufficiently large n ,

1. a d -balanced equi- n -square exists if and only if d divides n ,
2. a diagonally cyclic 1-balanced equi- n -square exists if and only if n is odd,
3. for odd $d \geq 3$, a diagonally cyclic d -balanced equi- n -square exists if and only if d divides n ,
4. for even $d \geq 2$, a diagonally cyclic d -balanced equi- n -square exists if and only if $2d$ divides n , and
5. there exists a Latin square of order n which partitions into $d \times (n/d)$ subrectangles if and only if d divides n .

When $d = 1$, we are working with Latin squares of order n , which exist for all $n \geq 1$ (e.g. the Cayley table of \mathbb{Z}_n); the rows of a Latin square of order n partition it into $1 \times n$ subrectangles. It is well known that diagonally cyclic Latin squares exist for odd $n \geq 1$ and do not exist for even n (an early proof was given by Euler [16]; the $d = 1$ case of Lemma 1 is essentially the same proof). This proves Theorem 1 for $d = 1$. The rest of this paper is primarily devoted to proving the remaining cases.

Except for 2-balanced equi-6-squares (which exist; see Figure 1), constructing the d -balanced equi- n -squares required to prove Theorems 1 and 2 is achieved through constructing Latin squares of order n that decompose into $d \times (n/d)$ subrectangles, then applying the construction in Section 3.

An equi- n -square that is orthogonal to a Latin square (i.e., like symbols in the equi- n -square correspond to n distinct symbols in the Latin square) are together called a gerechte design (attributed to [5] by [2]). When re-using a field after an agricultural experiment, a gerechte design can be used to balance the carry-over effects from the previous experiment [2]. Vaughan [33] showed the NP-completeness of deciding if an equi- n -square is orthogonal to some Latin square. A Latin square with a decomposition into $d \times (n/d)$ subrectangles differs from gerechte designs: in a d -balanced equi- n -square (a) each symbol in the equi- n -square corresponds to a set of n cells in the Latin square that contains exactly d copies of n/d distinct symbols, and (b) we insist on the subrectangles being $d \times (n/d)$ rectangular matrices.

2 Diagonally cyclic balanced equi- n -squares and α -labellings

A d -balanced equi- n -square is equivalent to a decomposition of $K_{n,n}$ into d -regular n -edge spanning subgraphs of $K_{n/d,n/d}$ (the k -th component has a biadjacency matrix corresponding to the occurrences of symbol k in the d -balanced equi- n -square). The decomposition equivalent to the 2-balanced equi-6-square in Figure 1 is given in Figure 2.

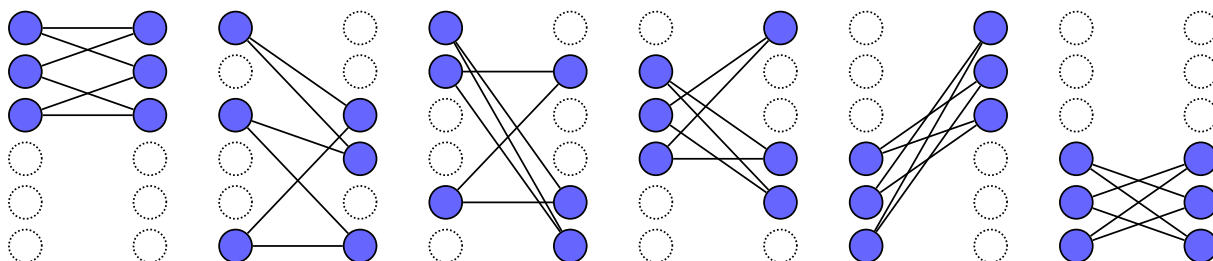


Figure 2: The decomposition of $K_{6,6}$ corresponding to the 2-balanced equi-6-square in Figure 1.

A decomposition \mathcal{G} of $K_{n,n}$ with vertex partition $\{1, \dots, n\} \cup \{n+1, \dots, 2n\}$ is *cyclic* if mapping each vertex to its next highest integer (except where n is mapped to 1 and $2n$ is mapped to $n+1$) is a permutation of the decomposition \mathcal{G} . Diagonally cyclic d -balanced equi- n -squares $L = (l_{ij})$ are equivalent to cyclic decompositions of $K_{n,n}$ into d -regular spanning subgraphs of $K_{n/d,n/d}$ as follows. We assume $K_{n,n}$ has the vertex bipartition $\{u_i\}_{i \in \mathbb{Z}_n} \cup \{v_i\}_{i \in \mathbb{Z}_n}$. We construct a d -regular spanning subgraph from L for each k by taking the set of edges $u_i v_j$ when $l_{ij} = k$. In particular, the *starter* is this d -regular spanning subgraph when $k = 0$. Note that by construction, each of the d -regular spanning subgraph can be obtained by cyclically rotating the starter.

We identify a particular type of starter (in matrix form) which is helpful in constructing diagonally cyclic d -balanced equi- n -squares. For $d \geq 1$, we call an $r \times s$ matrix A a *d-starter* if:

- every row either contains 0 or d filled cells,
- every column either contains 0 or d filled cells,
- every top-left to bottom-right (unbroken) diagonal contains exactly one filled cell, and
- the number of filled cells in A is $r + s - 1$.

Actually, the third condition above implies the fourth, but it is useful to make it explicit. Figure 3 depicts some 3-starters.

A d -starter A with n filled cells gives rise to a starter for $K_{n,n}$ with an edge $u_i v_j$ whenever cell (i, j) is filled. A starter for $K_{n,n}$ arising from a d -starter has three special properties: it has n edges, vertices have degree d or 0, and each value of $i - j \pmod n$ is used exactly once. (Actually, this last property holds for all starters.)

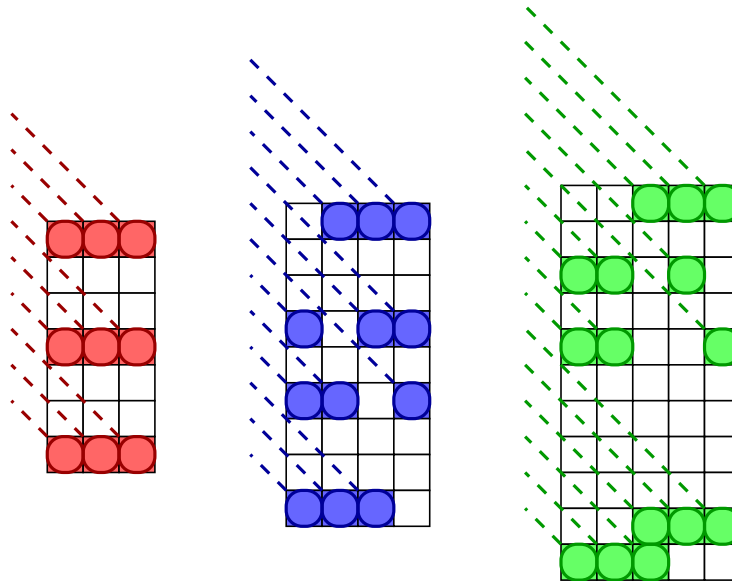


Figure 3: Examples of 3-starters.

A d -starter with n filled cells also describes the placement of zeroes in a diagonally cyclic d -balanced equi- n -square, with an example depicted in Figure 4.

Lemma 2. *If a d -starter with n filled cells exists, then a diagonally cyclic d -balanced equi- n -square exists.*

Proof. We embed the d -starter in an $n \times n$ diagonally cyclic matrix L over \mathbb{Z}_n . The filled cells in the d -starter form the 0s in L . The requirement that each unbroken diagonal in a d -starter has exactly one filled cell implies this process indeed generates a diagonally cyclic matrix. The diagonally cyclic property ensures L is an equi- n -square.

Suppose L is not d -balanced, and some row (resp. column) i contains $x \notin \{0, d\}$ copies of some symbol k . The diagonally cyclic property implies row (resp. column) $i - k$ contains x zeroes, contradicting the d -starter property. \square

Not all diagonally cyclic d -balanced equi- n -squares arise from d -starters; Figure 4 (right) gives a non-trivial example of a diagonally cyclic 2-balanced equi-12-squares which does not arise from a 2-starter. There are also 1-balanced equi- n -squares for all odd n , but not 1-starters as we note in the following lemma. However, 1-starters are a special case.

Lemma 3. *The only 1-starters have dimensions 1×1 .*

Lemma 3 can be proved by considering entries closest to the top-right corner; a full proof is neither challenging nor enlightening, so we omit it. Lemma 3 excludes some possibilities in applying the following lemma (Lemma 4).

9	0	0	0	4	4	4	7	7	9	7	9
10	10	1	1	1	5	5	5	8	8	10	8
9	11	11	2	2	2	6	6	6	9	9	11
0	10	0	0	3	3	3	7	7	7	10	10
11	1	11	1	1	4	4	4	8	8	8	11
0	0	2	0	2	2	5	5	5	9	9	9
10	1	1	3	1	3	3	6	6	6	10	10
11	11	2	2	4	2	4	4	7	7	7	11
0	0	0	3	3	5	3	5	5	8	8	8
9	1	1	1	4	4	6	4	6	6	9	9
10	10	2	2	2	5	5	7	5	7	7	10
11	11	11	3	3	6	6	8	6	8	8	8

0	0	7	7	9	9	4	4	6	6	10	10
11	1	1	8	8	10	10	5	5	7	7	11
0	0	2	2	9	9	11	11	6	6	8	8
9	1	1	3	3	10	10	0	0	7	7	9
10	10	2	2	4	4	11	11	1	1	8	8
9	11	11	3	3	5	5	0	0	2	2	9
10	10	0	0	4	4	6	6	1	1	3	3
4	11	11	1	1	5	5	7	7	2	2	4
5	5	0	0	2	2	6	6	8	8	3	3
4	6	6	1	1	3	3	7	7	9	9	4
5	5	7	7	2	2	4	4	8	8	10	10
11	6	6	8	8	3	3	5	5	9	9	11

Figure 4: Depicting how d -starters (blue cells, which contain symbol 0) gives rise to diagonally cyclic d -balanced equi- n -squares (which are equivalent to cyclic decompositions of $K_{n,n}$). Right: A diagonally cyclic 2-balanced equi-12-square that does not come from a 2-starter.

We make use of d -starters because of their simplicity, and because they admit the following direct product construction. We illustrate how this construction works in Figure 5. Also, we feel d -starters are interesting combinatorial matrices in their own right.

Lemma 4. *Given a d_1 -starter A with n_1 zeroes and a d_2 -starter B with n_2 zeroes, then there exists a d_1d_2 -starter $A \otimes B$ with n_1n_2 zeroes.*

Proof. We embed B in the top-right corner of an $n_2 \times n_2$ matrix B^* . There are n_2 unbroken diagonals of B^* which contain a cell in the top row, and each of them contains exactly one filled cell.

We blow up A , replacing each filled cell with a copy of B^* , and each empty cell with an $n_2 \times n_2$ all-empty matrix (this is the direct product of A and B^* , also known as the Kronecker product). We then delete any boundary rows and columns that do not contain a zero. The result is what we call $A \otimes B$.

Suppose A is an $r_1 \times s_1$ matrix, and B is an $r_2 \times s_2$ matrix. In this process, we delete the bottom $n_2 - r_2$ empty rows and left-most $n_2 - s_2$ empty columns. Thus we end up with an $(r_1n_2 - n_2 + r_2) \times (s_1n_2 - n_2 + s_2)$ matrix. The number of unbroken diagonals is

$$\begin{aligned} (r_1n_2 - n_2 + r_2) + (s_1n_2 - n_2 + s_2) - 1 &= (r_1 + s_1 - 1)n_2 - n_2 + r_2 + s_2 - 1 \\ &= n_1n_2 \end{aligned}$$

since $n_1 = r_1 + s_1 - 1$ and $n_2 = r_2 + s_2 - 1$. Since the number of filled cells is n_1n_2 , to prove that $A \otimes B$ is a d_1d_2 -starter, we check there are no empty unbroken diagonals.

Prior to deleting empty boundary rows and columns, the direct product is composed of $n_2 \times n_2$ blocks: some empty, and some containing the d_2 -starter. Shrinking the blocks to a single cell regains the d_1 -starter. Moreover, any unbroken diagonal of the direct product

(after deleting the empty boundary rows and columns) maps to an unbroken diagonal of the d_1 -starter corresponding to the $n_2 \times n_2$ blocks it intersects in their top rows. This unbroken diagonal of the d_1 -starter contains exactly one filled cell, which corresponds to the only non-empty $n_2 \times n_2$ block that the original unbroken diagonal of the direct product intersects in its top row. Thus, any unbroken diagonal in the direct product intersects the top row of some non-empty $n_2 \times n_2$ block. Since each non-empty $n_2 \times n_2$ block contains a copy of the d_2 -starter in the top-right corner, this unbroken diagonal has a filled cell.

It is clear from the construction that each row and each column contain either 0 or $d_1 d_2$ filled cells. \square

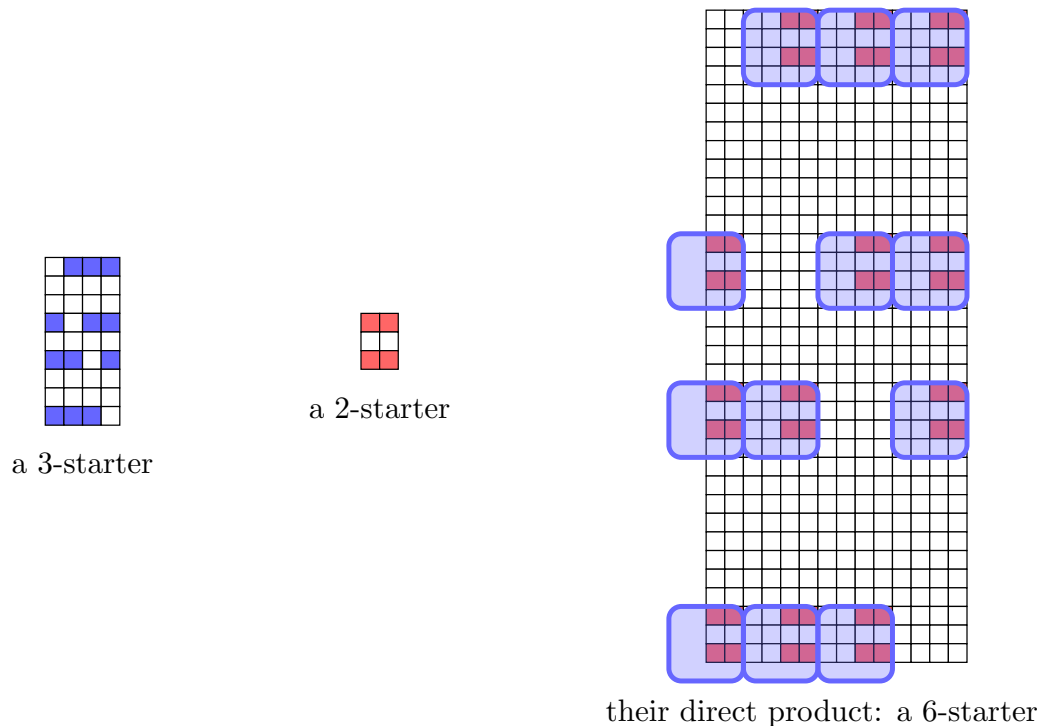


Figure 5: The direct product of a 3-starter with a 2-starter.

Lemma 1 implies that for a diagonally cyclic d -balanced equi- n -square to exist when $d \geq 2$, we must have either

- d is odd, and d divides n , or
- d is even, and $2d$ divides n .

Thus, for the purposes of proving Theorem 2 in the diagonally cyclic case, we need only consider prime d , then take direct products according to Lemma 4.

Since diagonally cyclic d -balanced equi- n -squares are equivalent to cyclic decompositions of $K_{n,n}$ into isomorphic copies of a d -regular graph with n edges (ignoring isolated vertices), we begin by looking at constructions in the graph-decomposition literature.

For the $d = 2$ case, we require $n \in \{4, 8, 12, \dots\}$ for a diagonally cyclic d -balanced equi- n -square to exist (by Lemma 1). Rosa [27] and Huang and Rosa [22] showed that when n is a multiple of 4, the complete bipartite graph $K_{n,n}$ can be cyclically decomposed into copies of C_n (see also [14]), which proves the $d = 2$ diagonally cyclic case of Theorem 1. The constructions of [22, 27] are effectively the 2-starters depicted in Figure 6, in the present paper's terminology.

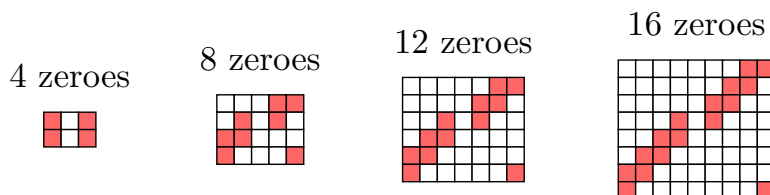


Figure 6: Small 2-starters which generate the 2-balanced equi- n -squares in [22, 27] (after permuting the rows).

For the $d = 3$ case, we require $n \in \{3, 6, 9, \dots\}$. Constructions of d -balanced equi- n -squares arise from α -labellings (attributed to Rosa [27] in Gallian's dynamic survey [19]). A *graceful labelling* of an n -edge graph G is an injection f from $V(G)$ to $\{0, 1, \dots, n\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. An α -labelling is a graceful labelling with the additional property that there exists an integer k so that for each edge xy either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. An example of an α -labelling is given in Figure 7.

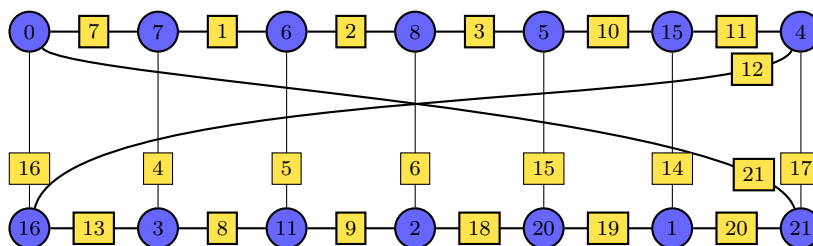


Figure 7: The α -labelling (blue) of the 14-vertex 21-edge Möbius ladder given in [24], with parameter $k = 6$.

El-Zanati and Vanden Eynden [12] showed that if an n -edge graph G admits an α -labelling, then $K_{n,n}$ cyclically decomposes into subgraphs isomorphic to G as follows. Suppose $K_{n,n}$ has the vertex bipartition $\{u_i\}_{i \in \mathbb{Z}_n} \cup \{v_i\}_{i \in \mathbb{Z}_n}$. Given an α -labelling f of G , we map edges as follows:

$$xy \mapsto \begin{cases} u_{f(x)}v_{f(y)} & \text{if } f(x) \leq k \\ v_{f(x)}u_{f(y)} & \text{otherwise.} \end{cases}$$

The distinct values of $|f(x) - f(y)|$ ensure edges are used exactly once when cyclically rotated. If G is a d -regular graph with n edges, then this decomposition is equivalent to

a diagonally cyclic d -balanced equi- n -square. Figure 8 depicts the biadjacency matrix of the Möbius ladder in Figure 7 arising from its given α -labelling.

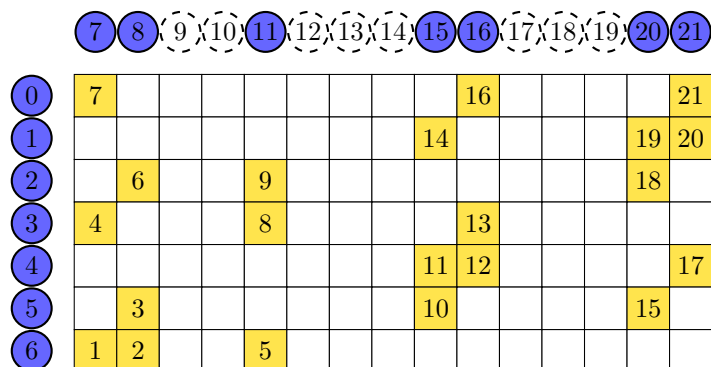


Figure 8: The biadjacency matrix of the Möbius ladder that cyclically decomposes $K_{21,21}$ derived from the α -labelling in Figure 7, with rows $\{0, 1, \dots, 6\}$ and columns $\{7, 8, \dots, 21\}$. The filled cells form a 3-starter. We number the filled cells from bottom-left to top-right using the numbers 1 to 21 sequentially; they correspond to the edge labels arising from the α -labelling in Figure 7.

A technicality for α -labellings is that we treat n and 0 distinctly in the α -labelling (since we require $|n - 0| = n$), but when constructing the bipartite graph or diagonally cyclic d -balanced equi- n -square, these represent the same index (since indices are in \mathbb{Z}_n).

Pasotti [24] described an α -labelling of the Möbius ladder on $2k$ vertices for odd $k \geq 3$ (and Figure 7 is one example). Frucht and Gallian [18] gave α -labellings for the prism $C_k \square K_2$ for even $k \geq 4$, and we give an example in Figure 9. In either case, the graphs have $3k$ edges, and thus give rise to diagonally cyclic 3-balanced equi- n -squares for all $n \in \{3k : \text{odd } k \geq 3\} \cup \{3k : \text{even } k \geq 4\} = \{9, 12, \dots\}$. Wannasit and El-Zanati [35] later showed that bipartite prisms, bipartite Möbius ladders, and connected bipartite graphs with at most 14 vertices admit “free” α -labellings (excluding C_4), which relates to Lemma 7.

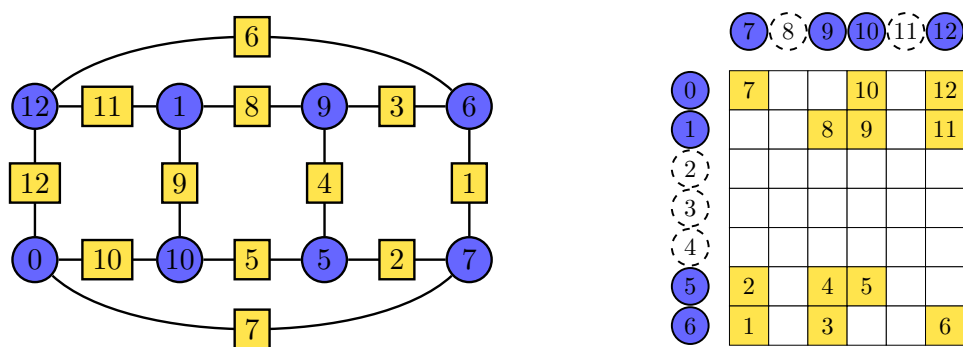


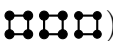
Figure 9: Left: The α -labelling of the 8-vertex 12-edge prism $C_4 \square K_2$ given in [18], with parameter $k = 6$. Right: its 3-starter with rows $\{0, 1, \dots, 6\}$ and columns $\{7, 8, \dots, 12\}$.

The following theorem establishes that d -starters are equivalent to α -labellings of d -regular bipartite graph.

Theorem 3. For $d \geq 1$, a d -starter A is equivalent to an α -labelling of the d -regular bipartite graph with biadjacency matrix corresponding to A . Thus, a d -starter exists if and only if some d -regular bipartite graph admits an α -labelling.

Proof. Suppose $A = (a_{ij})$ is an $r \times s$ d -starter. We construct a bipartite graph with biadjacency matrix $B = (b_{ij})$ with $b_{ij} = 1$ if and only if a_{ij} is filled. The rows of B correspond to vertices labeled $\{0, 1, \dots, r-1\}$ and the columns of B correspond to vertices labeled $\{r, r+1, \dots, r+s-1\}$. We delete vertices corresponding to an empty row or column. We observe the labelling is a graceful labelling: (a) if there is a filled cell in row i and column j , then by definition we have an edge from vertex j to vertex i labelled $j-i$, and this label is unique since A is a d -starter, and (b) the number of edges is $r+s-1$. It is an α -labelling with parameter $k = r-1$.

Now suppose we have an α -labelling f of a d -regular bipartite graph G , with parameter k and vertex labels belonging to $\{0, 1, \dots, h\}$ (where a vertex labelled h exists). We construct a $k \times (h-k+1)$ d -starter $A = (a_{ij})$, with rows $\{0, 1, \dots, k-1\}$ and columns $\{k, k+1, \dots, h\}$ with a filled cell a_{ij} whenever there is an edge between vertex i and vertex j . Since G is d -regular, we have either d or 0 filled cells per row and column. By definition of a graceful labelling, each value of $|f(x) - f(y)|$ occurs exactly once; this implies each unbroken diagonal in A contains exactly one filled cell. This verifies that A is a d -starter. \square

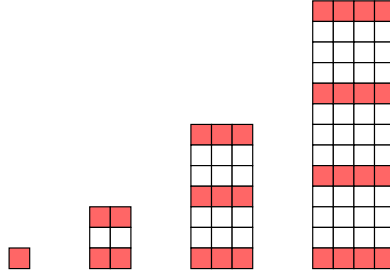
The zeroes in Figure 4 (right) form the biadjacency matrix of the graph $3C_4$ (i.e., ) which does not admit an α -labelling, so Theorem 3 implies it does not come from a 2-starter (even after cyclically permuting its rows and columns). Thus not all diagonally cyclic d -balanced equi- n -squares arise from d -starters, or equivalently from α -labellings.

In light of Theorem 3, Lemma 4 implies that if bipartite graphs with biadjacency matrices A and B both admit α -labellings, then so does the bipartite graph which has the biadjacency matrix $A \otimes B$, where \otimes denotes the Kronecker product of matrices. In general, this graph product is not the same as the Cartesian product for which there are many known constructions of α -labellings: it is essentially a “half-Kronecker product” of the graphs.

We turn our attention to proving the existence of d -starters with n filled cells when d and n satisfy the necessary conditions (Lemma 1) for sufficiently large n .

Lemma 5. For $d \geq 1$, there exists a d -starter with d^2 filled cells.

Proof. A d -starter is given by the $(d^2 - d + 1) \times d$ matrix where all the cells in rows $\{0, d, 2d, \dots, d^2 - d\}$ are filled, and all other cells are empty. These matrices are depicted below for $d \in \{1, 2, 3, 4\}$:



By design, each column contains exactly d filled cells, and each row contains either 0 or d filled cells. For all $i \in \{1, 2, \dots, d\}$ and $j \in \{0, 1, \dots, d-1\}$, the unbroken diagonal containing cell $(di - j, 0)$ contains exactly one filled cell, which is in row di . The diagonal containing cell $(0, i)$ contains exactly one filled cell, which is in row 0, for all $i \in \{0, 1, \dots, d-1\}$. Thus, all diagonals contain exactly one filled cell. \square

Theorem 3 implies Lemma 5 is equivalent to $K_{d,d}$ having an α -labelling, thus Lemma 5 also follows from work in [3, 27], which [19] claims contain α -labellings of $K_{d,d}$.

Lemma 6. For $d \geq 2$, there exists a d -starter with $2d^2 - 2d$ filled cells.

Proof. Define $r_i = (d-1)i$ and $\hat{r}_i = d^2 - d + (d-1)i$ for $i \in \{0, 1, \dots, d-2\}$. We construct a $(2d^2 - 4d + 3) \times (2d - 2)$ matrix with cells filled as tabulated below:

color	filled cell	whenever...
blue	(r_i, j)	$i \in \{0, 1, \dots, d-2\}$ and $j \in \{0, \dots, i\}$
green	(r_i, j)	$i \in \{0, 1, \dots, d-2\}$ and $j \in \{d+i-1, \dots, 2d-3\}$
red	(\hat{r}_i, j)	$i \in \{0, 1, \dots, d-2\}$ and $j \in \{d-2-i, \dots, 2d-3-i\}$

We color the filled cells blue, green, and red. This matrix is depicted for $d \in \{2, 3, 4, 5\}$ in Figure 10. We claim this is a d -starter. We first note that $(2d^2 - 4d + 3) + (2d - 2) - 1 = 2d^2 - 2d$, i.e., the required number of filled cells in a d -starter with these dimensions.

Check I: no monochromatic clashes. Cells in the same row cannot clash (i.e., belong to the same unbroken diagonal). If two cells (a, b) and (a', b') with $a > a'$ clash, then $a - a' = b - b'$. Thus, since indices of distinct non-empty rows differ by at least $d-1$, we must have $b - b' \geq d-1$. Thus, monochromatic clashes are not possible due to the restrictions on column indices j . For example, red-red clashes are excluded since if $a = \hat{r}_i$ then $b \leq 2d-3-i$ and $b' \geq d-2-i'$ for some $i' \leq i-1$, implying $b - b' \not\geq d-1$.

Check II: no blue-red and green-red clashes. For blue cells (a, b) , the maximum difference $a - b$ is $d^2 - 3d + 2$, and for green cells (a, b) , the maximum difference $a - b$ is $d^2 - 5d + 5$. For red cells (a, b) , the minimum difference $a - b$ is $d^2 - 3d + 3$. Then the difference $a - b$ for a red cell is always larger than the difference for a blue cell or a green cell. As each cell in an unbroken diagonal has the same difference between its row and column, this means that any unbroken diagonal containing a red cell cannot also contain a blue or green cell. Thus there are no blue-red and green-red clashes.

Check III: no blue-green clashes. Since indices of rows containing blue and green cells differ by a multiple of $d-1$, if blue cell (r_i, j) and green cell $(r_{i'}, j')$ clash, then $r_{i'} - r_i =$

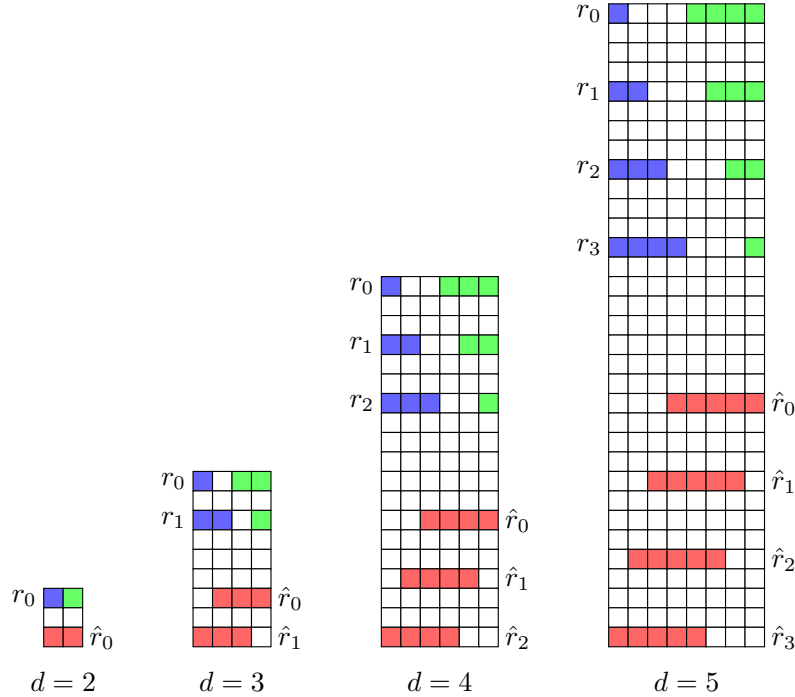


Figure 10: A d -starter in the proof of Lemma 6.

$j' - j \equiv 0 \pmod{d-1}$. Further $j < j' \leq 2d-3$, so we have $j' = j + d - 1$, which implies $r_{i'} = r_i + d - 1$, i.e., $i' = i + 1$. The blue cells in row r_i belong to columns $\{0, \dots, i\} \ni j$ and the green cells in row $r_{i'} = r_{i+1}$ belong to columns $\{d+i, \dots, 2d-3\} \ni j + d - 1$, which is impossible.

Check IV: non-empty rows and columns contain d filled cells. By design, each row contains either 0 or d filled cells. Also by design, each non-empty row is a cyclic shift of the other non-empty rows in the array, where all possible cyclic shifts occur in some row; this suffices to show each column contains d filled cells. \square

The graph G corresponding to the d -starter in Lemma 6 is the d -regular bipartite graph with vertex bipartition $\{x_i\}_{i \in \mathbb{Z}_n} \cup \{y_i\}_{i \in \mathbb{Z}_n}$, where $n = 2d - 2$, and edges between each x_i with each vertex in $\{y_i, y_{i+1}, \dots, y_{i+d-1}\}$. Theorem 3 therefore implies G admits an α -labelling.

The following lemma describes a way to “adjoin” special types of d -starters. It is equivalent to the d -regular case of [13, Th. 1] by El-Zanati and Vanden Eynden which adjoins “left-free” and “right-free” α -labellings of bipartite graphs.

Lemma 7. *If there exists a d -starter with n_1 filled cells with an empty second row and a d -starter with n_2 filled cells and an empty second-last row, then there exists a d -starter with $n_1 + n_2$ filled cells.*

The basic idea behind adjoining d -starters in Lemma 7 is depicted in Figure 11. We can use Lemma 7 recursively: if $\{A_i\}_{i \geq 1}$ is a set of d -starters, where A_i has n_i filled cells,

where each A_i has at least 4 rows, and both the second row and second-last row of each A_i is empty, then there exists a d -starter with $n_1 + \cdots + n_k$ filled cells, for all $k \geq 1$. We use this idea to prove the following theorem.

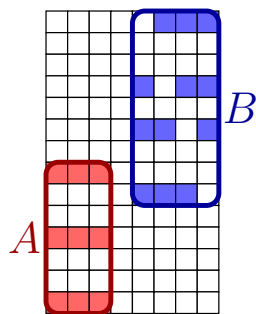


Figure 11: An example of adjoining d -starters A and B in Lemma 7.

Theorem 4. If $d > 1$ is odd, a d -starter with n filled cells exists for all sufficiently large n divisible by d ; it does not exist if d does not divide n .

If d is even, a d -starter with n filled cells exists for all sufficiently large n divisible by $2d$; it does not exist if $2d$ does not divide n .

Proof. When $d \geq 3$, Lemma 5 gives a d -starter with $n_1 := d^2$ filled cells, and Lemma 6 gives a d -starter with $n_2 := 2d^2 - 2d$ filled cells: these both have more than 4 rows and both have empty second rows and second-last rows. Thus, we recursively use Lemma 7 on these two d -starters.

Since

$$\gcd(n_1, n_2) = \begin{cases} d & \text{if } d \text{ is odd,} \\ 2d & \text{if } d \text{ is even,} \end{cases}$$

d -starters exist for all sufficiently large n that satisfy the necessary conditions in Lemma 1. \square

Theorem 4 implies diagonally cyclic d -balanced equi- n -squares exist for all sufficiently large admissible n .

3 Decompositions into subrectangles

The following lemma gives a method for using a Latin square of order n which has been partitioned into $d \times (n/d)$ subrectangles, for some divisor d of n , to obtain a d -balanced equi- n -square. Figure 12 illustrates Lemma 8 where $n = 10$ and $d = 2$.

Lemma 8. Suppose for some divisor d of n there exists a Latin square $L = (l_{ij})$ of order n whose entries can be partitioned into $d \times (n/d)$ subrectangles. If we index the subrectangles $0, 1, \dots, n/d - 1$, and define an $n \times n$ matrix M where cell (l_{ij}, j) contains the index of the subrectangle of L containing the cell (i, j) , then M is a d -balanced equi- n -square.

Proof. For each of the d copies of symbol k occurring in the t -th subrectangle of L , the symbol t occurs in row k of M . Hence any row of M containing t contains exactly d copies of t .

Similarly, for each of the entries in the j -th column of the t -th subrectangle of L , the symbol t occurs in column j of M . Hence any column of M containing t contains exactly d copies of t . \square

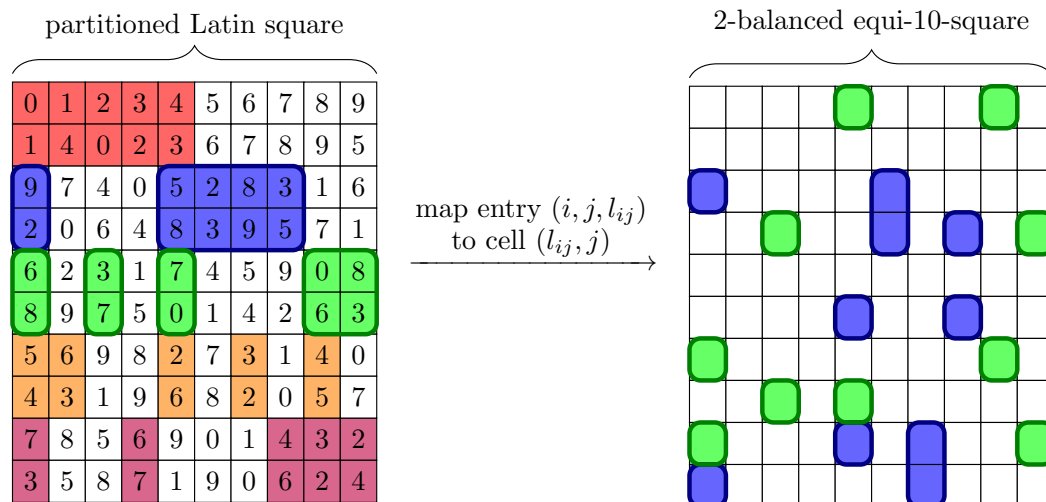


Figure 12: Illustrating how a Latin square of order 10 partitioned into 2×5 subrectangles gives rise to a 2-balanced equi-10-square as per Lemma 8. To avoid cluttering the figure, we only show two symbols from the 2-balanced equi-10-square.

Thus, in this section, we describe constructions of Latin squares of order n which decompose into $d \times (n/d)$ subrectangles. We streamline the proofs by using König's Theorem which states that regular spanning subgraphs of $K_{n,n}$ have 1-factorizations (although it appears in a variety of forms). The following lemma is how we use König's Theorem throughout the paper.

Lemma 9 (König's Theorem; variant). *Let M be an $n \times n$ matrix containing symbols in \mathbb{Z}_n and possibly some empty cells, in which any symbol $s \in \mathbb{Z}_n$ occurs exactly $a_s \in \{0, 1, \dots\}$ times in every row and every column. Choose a set $\{A_s\}_{s \in \mathbb{Z}_n}$ of disjoint subsets of \mathbb{Z}_n , where each $|A_s| = a_s$. Then there exists an $n \times n$ Latin square L whose cells contain a symbol in A_s if and only if the corresponding cell in M contains the symbol s .*

Furthermore, any $a_s \times a_s$ all- s submatrix in M corresponds with an $a_s \times a_s$ subsquare in L on the symbols A_s .

For any $a \geq 1$ and $b \geq a$, we define the $b \times b$ matrix $B_{a,b}(x)$ as having symbol x in cell (i, j) whenever $i + j \pmod{b} \in \{0, 1, \dots, a - 1\}$, and the remaining cells empty. Some examples are given in Figure 13.

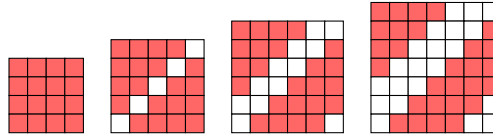


Figure 13: The matrix $B_{a,b}(x)$ when $a = 4$ and $b \in \{4, 5, 6, 7\}$. Cells colored red contain x .

We further define $A_{a,m}(x)$ as the $m \times m$ block-diagonal matrix with main-diagonal blocks

$$\overbrace{B_{a,a}(x), \dots, B_{a,a}(x), B_{a,a+r}(x)}^{\lfloor m/a \rfloor - 1 \text{ times}},$$

where $r = a \pmod{m}$. Some examples are given in Figure 14.

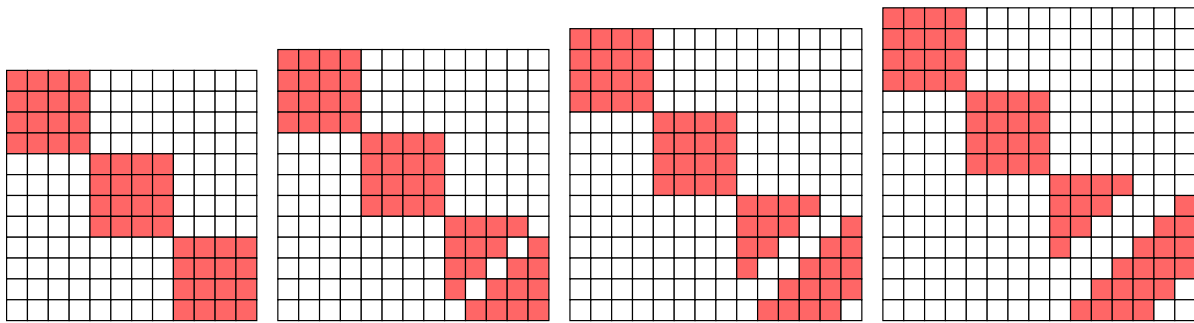


Figure 14: The matrix $A_{a,m}(x)$ when $a = 4$ and $m \in \{12, 13, 14, 15\}$. Cells colored red contain x .

For $t \in \mathbb{Z}_m$, let $A_{a,m}^{(t)}(x)$ denote $A_{a,m}(x)$ after mapping the contents of row $i \in \mathbb{Z}_m$ to row $i + t \in \mathbb{Z}_m$. Finally, we define $M_{a,m,k}$ as the union of the matrices

$$A_{a,m}(0), A_{a,m}^{(a+r)}(1), \dots, A_{a,m}^{(k-1)(a+r)}(k-1),$$

where $r = m \pmod{a}$, provided their non-empty cells do not overlap. By inspection, we see that they do not overlap if and only if $(k-1)(a+r) \leq m - (a+r)$, that is, $M_{a,m,k}$ is defined provided $m \geq k(a+r)$. Some examples of matrices $M_{4,m,3}$ are given in Figure 15 (we do not have an $m = 19$ example, since $m \geq k(a+r)$ is not satisfied in this case). In Figure 15, it is important to note that each row intersects an $a \times a$ all- i block in M , for some i .

The proofs in this section have a general theme:

1. We begin with the $m \times m$ matrix $M_{a,m,k}$ for $k \in \{2, 3\}$.
2. We use König's Theorem to turn it into an order- m Latin square.
3. We blow up this order- m Latin square using a direct product with a $d \times d$ Latin square.

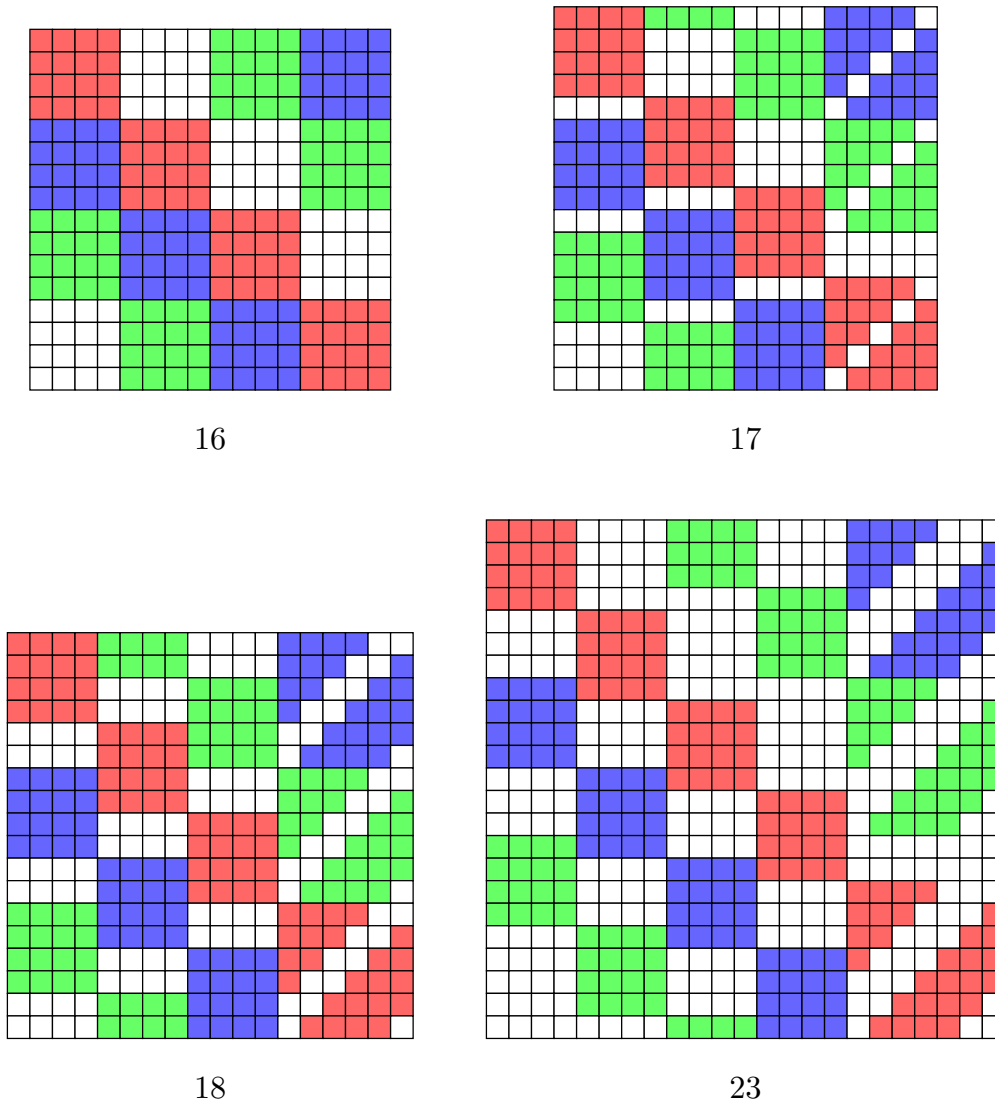


Figure 15: The matrix $M_{4,m,3}$ for $m \in \{16, 17, 18, 23\}$. Red cells contain 0, blue cells contain 1, and green cells contain 2.

4. We replace the $da \times da$ subsquares that arise from the all- i blocks in $M_{a,m,k}$ by $da \times da$ subsquares which decompose into $d \times (d + 1)$ subrectangles (or something similar). This gives an $md \times md$ Latin square in which consecutive sets of d rows $\{di, di + 1, \dots, di + d - 1\}$ decompose into a small number of $d \times (d + 1)$ subrectangles and a large number of $d \times d$ subsquares.
5. We identify a partition of dm into d partitions of m , such that each partition of m only has parts of sizes d and $d + 1$. We group together the $d \times (d + 1)$ subrectangles and $d \times d$ subsquares in rows $\{di, di + 1, \dots, di + d - 1\}$ accordingly in order to obtain a decomposition of those rows into $d \times m$ subrectangles. For example, if $d = 2$ and

$m = 11$ and the partition is

$$2 \times 11 = (3 + 3 + 3 + 2) + (3 + 2 + 2 + 2 + 2), \quad (2)$$

then one 2×11 subsquare is formed from grouping three 2×3 subrectangles with one 2×2 subsquare (i.e., an *intercalate*), and the other 2×11 subsquare is formed from grouping one 2×3 subrectangle with four intercalates.

We require some modifications to this general approach in individual cases. For brevity, we use standard partition notation such as

$$2 \times 11 = (3^3, 2) + (3, 2^4)$$

to represent (2).

3.1 Case $d = 2$

When $d = 2$, for a 2-balanced equi- n -square to exist (and for a Latin square of order n that can be partitioned into $2 \times (n/2)$ subrectangles to exist), Lemma 1 implies n must be even. When $n \equiv 0 \pmod{4}$ there is an easy construction: a direct product gives a Latin square of order n which decomposes into four $(n/2) \times (n/2)$ subsquares, each of which decomposes into $2 \times (n/2)$ subrectangles. So we consider $n \equiv 2 \pmod{4}$.

An exhaustive computer search reveals that no 6×6 Latin square decomposes into 2×3 subrectangles. Figure 12 gives an example of order 10 and Figure 16 gives an example of order 14. The next smallest case is 18.

We get close to decomposing a Latin square of order 6 into 2×3 subrectangles with the following:

0	1	2	3	4	5
1	2	0	4	5	3
3	4	5	0	1	2
4	5	3	1	2	0
2	0	1	5	3	4
5	3	4	2	0	1

3	4	5	0	1	2
4	5	3	1	2	0
2	0	1	5	3	4
5	3	4	2	0	1
0	1	2	3	4	5
1	2	0	4	5	3

2	0	1	5	3	4
5	3	4	2	0	1
0	1	2	3	4	5
1	2	0	4	5	3
3	4	5	0	1	2
4	5	3	1	2	0

(3)

Here, the unhighlighted pairs of rows decompose into three 2×2 subsquares.

We find larger Latin squares containing many 6×6 subsquares, and replace them by the Latin squares in (3) (after relabelling the symbols), to ensure pairs of rows in the larger Latin square contain odd-length cycles. This is the main idea of the following construction.

Theorem 5. Let $m \geq 9$ be odd. Suppose there exists an $m \times m$ matrix M containing symbols in $\{0, 1, 2\}$ and empty cells with the properties:

- every row and every column contains exactly 3 copies of each symbol,

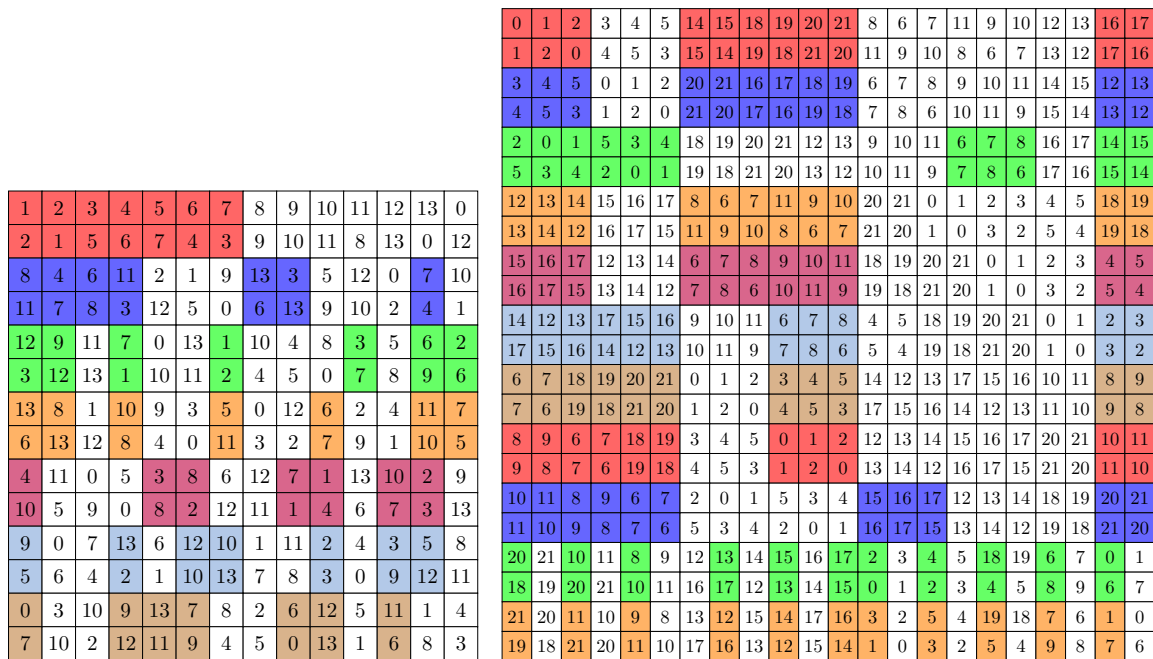


Figure 16: Latin squares of order $n \in \{14, 22\}$ partitioned into $2 \times (n/2)$ subrectangles.

- there exists a set $\{U_t\}$ of non-overlapping 3×3 submatrices in which
 - U_t is an all- i matrix for some $i \in \{0, 1, 2\}$, and
 - we can choose 1 or 2 rows from the submatrices U_t such that every row of M is represented exactly once.

Then there exists a $2m \times 2m$ Latin square that can be partitioned into $2 \times m$ subrectangles.

Proof. We prove the theorem with the aid of a running example for $m = 13$, beginning with the matrix in Figure 17 (left). We highlight how to select 1 or 2 rows from the 3×3 submatrices U_t .

Lemma 9 implies that we can construct an $m \times m$ Latin square L , with symbols in $\{0, 1, 2\}$ wherever the symbol 0 occurs in M , symbols in $\{3, 4, 5\}$ wherever the symbol 1 occurs in M , symbols in $\{6, 7, 8\}$ wherever the symbol 2 occurs in M , and symbols in $\{9, 10, \dots, m-1\}$ wherever an empty cell occurs in M . Moreover, the chosen rows of the 3×3 submatrices U_t in M map to rows of 3×3 subsquares of L . This is depicted in Figure 17 (right); we highlight the subsquares we subsequently discuss in this proof.

We next take a direct product of L with a 2×2 Latin square, giving a $2m \times 2m$ Latin square. Consequently, we also blow up the selected rows of the 3×3 subsquares of L to 2×6 subrectangles. Denote these subrectangles S_i for $i \in \{0, 1, \dots, m-1\}$.

We next replace each of the 6×6 subsquares arising from the submatrices U_t (and only those) with one of the Latin squares in (3). This requires relabelling the symbols so that each replacement subsquare has the same set of symbols as the subsquare it replaced.

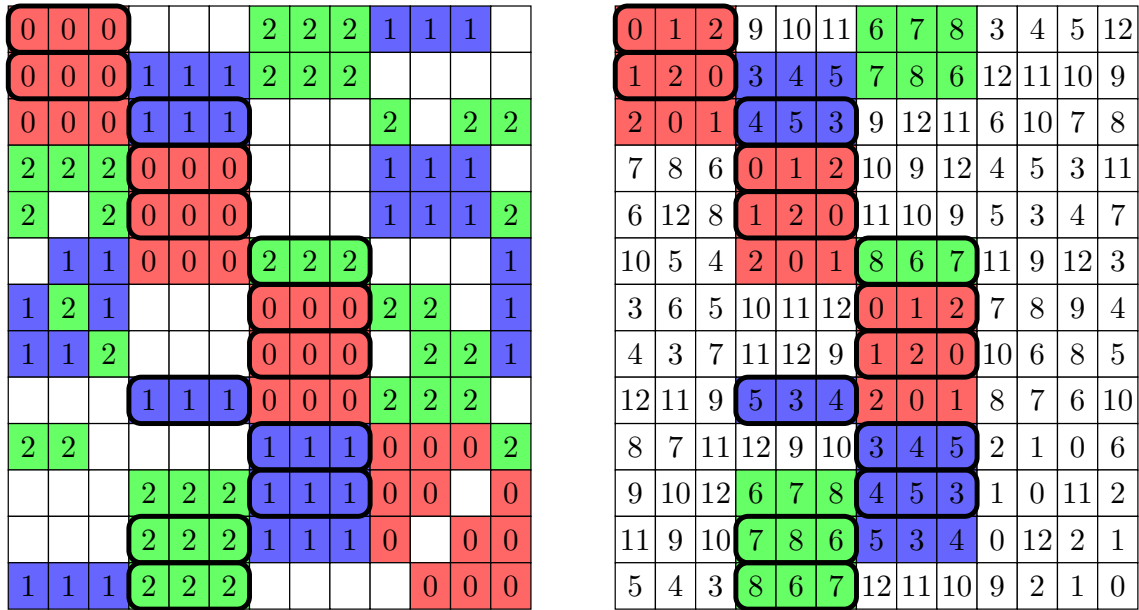


Figure 17: Left: A matrix satisfying the conditions of Theorem 5. Right: A Latin square derived from it via König's Theorem (Lemma 9).

We choose the 6×6 subsquares so that each submatrix S_i decomposes into two 2×3 subrectangles. This is possible since we choose only 1 or 2 rows from U_t . We also select one 2×3 subrectangle from each S_i after it has been replaced in this way.

By design, the $2m \times 2m$ Latin square has row pairs $\{2i, 2i + 1\}$ that decompose according to one of the following cases:

- *Case I*: Two 2×3 subrectangles and $m - 3$ intercalates. In this case, one $2 \times m$ subrectangle is formed by the union of one 2×3 subrectangle and $(m - 3)/2$ intercalates. The other $2 \times m$ subrectangle is formed from the remaining entries in those rows. We write this as the sum of two partitions of m :

$$2m = (3, 2^{(m-3)/2}) + (3, 2^{(m-3)/2})$$

(and similarly for the subsequent cases).

- *Case II*: Four 2×3 subrectangles and $m - 6$ intercalates. We have the partition

$$2m = (3^3, 2^{(m-9)/2}) + (3, 2^{(m-3)/2}).$$

- *Case III*: Six 2×3 subrectangles and $m - 9$ intercalates. We have the partition

$$2m = (3^3, 2^{(m-9)/2}) + (3^3, 2^{(m-9)/2}).$$

The hypothesis $m \geq 9$ and m is odd imply $(m - 9)/2$ and $(m - 3)/2$ are non-negative integers, and so these partitions (and the corresponding $2 \times m$ subrectangles) exist. \square

0	1	2	3	4	5	18	19	20	21	22	23	15	16	17	12	13	14	6	7	8	9	10	11	24	25
1	2	0	4	5	3	19	18	21	20	23	22	16	17	15	13	14	12	7	6	9	8	11	10	25	24
3	4	5	0	1	2	8	6	7	11	9	10	14	12	13	17	15	16	24	25	22	23	20	21	18	19
4	5	3	1	2	0	11	9	10	8	6	7	17	15	16	14	12	13	25	24	23	22	21	20	19	18
2	0	1	5	3	4	6	7	8	9	10	11	18	19	24	25	22	23	12	13	20	21	14	15	16	17
5	3	4	2	0	1	7	8	6	10	11	9	19	18	25	24	23	22	13	12	21	20	15	14	17	16
14	15	16	17	12	13	0	1	2	3	4	5	20	21	18	19	24	25	8	9	10	11	6	7	22	23
15	14	17	16	13	12	1	2	0	4	5	3	21	20	19	18	25	24	9	8	11	10	7	6	23	22
12	13	24	25	16	17	3	4	5	0	1	2	22	23	20	21	18	19	10	11	6	7	8	9	14	15
13	12	25	24	17	16	4	5	3	1	2	0	23	22	21	20	19	18	11	10	7	6	9	8	15	14
20	21	10	11	8	9	2	0	1	5	3	4	12	13	14	15	16	17	22	23	18	19	24	25	6	7
21	20	11	10	9	8	5	3	4	2	0	1	13	14	12	16	17	15	23	22	19	18	25	24	7	6
6	7	12	13	10	11	20	21	22	23	24	25	0	1	2	3	4	5	14	15	16	17	18	19	8	9
7	6	13	12	11	10	21	20	23	22	25	24	1	2	0	4	5	3	15	14	17	16	19	18	9	8
8	9	6	7	14	15	22	23	24	25	18	19	3	4	5	0	1	2	20	21	12	13	16	17	10	11
9	8	7	6	15	14	23	22	25	24	19	18	4	5	3	1	2	0	21	20	13	12	17	16	11	10
24	25	22	23	18	19	9	10	11	6	7	8	2	0	1	5	3	4	16	17	14	15	12	13	20	21
25	24	23	22	19	18	10	11	9	7	8	6	5	3	4	2	0	1	17	16	15	14	13	12	21	20
16	17	14	15	22	23	24	25	18	19	20	21	6	7	8	9	10	11	4	5	2	3	0	1	12	13
17	16	15	14	23	22	25	24	19	18	21	20	7	8	6	10	11	9	5	4	3	2	1	0	13	12
18	19	20	21	24	25	14	12	13	17	15	16	9	10	11	6	7	8	2	3	0	1	22	23	4	5
19	18	21	20	25	24	17	15	16	14	12	13	10	11	9	7	8	6	3	2	1	0	23	22	5	4
22	23	18	19	20	21	12	13	14	15	16	17	8	6	7	11	9	10	0	1	24	25	4	5	2	3
23	22	19	18	21	20	13	14	12	16	17	15	11	9	10	8	6	7	1	0	25	24	5	4	3	2
10	11	8	9	6	7	15	16	17	12	13	14	24	25	22	23	20	21	18	19	4	5	2	3	0	1
11	10	9	8	7	6	16	17	15	13	14	12	25	24	23	22	21	20	19	18	5	4	3	2	1	0

Figure 18: The Latin square formed from taking a direct product of the Latin square in Figure 17 with a 2×2 Latin square, then replacing its chosen 6×6 subsquares using (3). We highlight a 2×3 subrectangle in each pair of consecutive rows $\{2i, 2i+1\}$. This enables us to decompose each pair of consecutive rows $\{2i, 2i+1\}$ into 2×13 subrectangles.

Browning, Vojtěchovský, and Wanless [10, Lemma 4] classified when there exists a Latin square containing two non-overlapping subsquares. In the case of two 6×6 subsquares with 2 rows in common in a Latin square of order $2m$, we must have $2m \geq 18$, implying the technique in the proof of Theorem 5 could not work in smaller cases.

The matrix $M_{3,m,3}$ exists when $m \in \{9\} \cup \{13, 15, \dots\}$. When $m = 9$ we choose the block rows according to Figure 19 to satisfy the conditions of Theorem 5. Figure 16 contains an example of a Latin square of order 22 which decomposes into 2×11 , thereby resolving the $m = 11$ case.

We next check for $m \geq 13$ that it is possible to choose block rows of $M = M_{3,m,3}$ to satisfy the conditions of Theorem 5. We use $r = m \pmod 3$, so $3\lfloor m/3 \rfloor = m - r$. The all- i blocks of $M_{3,m,3}$ are denoted $J_{i,\sigma}$ where $\sigma \in \{0, 1, \dots, \lfloor m/3 \rfloor - 2\}$ (ignoring the

0	0	0	1	1	1	2	2	2
0	0	0	1	1	1	2	2	2
0	0	0	1	1	1	2	2	2
1	1	1	2	2	2	0	0	0
1	1	1	2	2	2	0	0	0
1	1	1	2	2	2	0	0	0
2	2	2	0	0	0	1	1	1
2	2	2	0	0	0	1	1	1
2	2	2	0	0	0	1	1	1

Figure 19: A matrix and a selection of rows which satisfies the conditions of Theorem 5.

additional all- i block when $r = 0$).

We find $J_{i,\sigma}$ intersects rows $\{i(3+r) + 3\sigma, i(3+r) + 3\sigma + 1, i(3+r) + 3\sigma + 2\}$ for all $i \in \{0, 1, 2\}$ and $\sigma \in \{0, 1, \dots, \lfloor m/3 \rfloor - 2\}$.

- We choose the rows 3σ and $3\sigma + 1$ from $J_{0,\sigma}$ for all $\sigma \in \{0, 1, \dots, \lfloor m/3 \rfloor - 2\}$.
- We choose row $3\sigma + 5$ from $J_{1,\sigma}$ for all $\sigma \in \{0, 1, \dots, \lfloor m/3 \rfloor - 3\}$.

This selection covers all but a few boundary rows. We still need to choose rows for indices $\{2\} \cup \{m - r - 3, m - r - 2, \dots, m - 1\}$.

- We choose row 2 from $J_{2,\lfloor m/3 \rfloor - 2}$.
- If $r \in \{1, 2\}$, we choose row $m - 4$ from $J_{2,\lfloor m/3 \rfloor - 4}$.
- If $r = 2$, we choose row $m - 5$ from $J_{1,\lfloor m/3 \rfloor - 3}$. (Since $r = 2$, the row we have already chosen from this block is $3\sigma + 5 = m - 6$, so this second choice is different.)
- We choose rows $m - 2$ and $m - 3$ from $J_{1,\lfloor m/3 \rfloor - 2}$.
- We choose row $m - 1$ from $J_{2,\lfloor m/3 \rfloor - 3}$.

Examples of this construction are given in Figure 20. We conclude that this construction satisfies the conditions of Theorem 5. Finally, combining the results in this section (i.e., a direct product for $m \equiv 0 \pmod{2}$; explicit examples for $m \in \{5, 7, 11\}$; a special case of Theorem 5 for $m = 9$; and this construction for odd $m \geq 13$), we completely resolve the existence problem for Latin squares of order n which decompose into $2 \times (n/2)$ subrectangles.

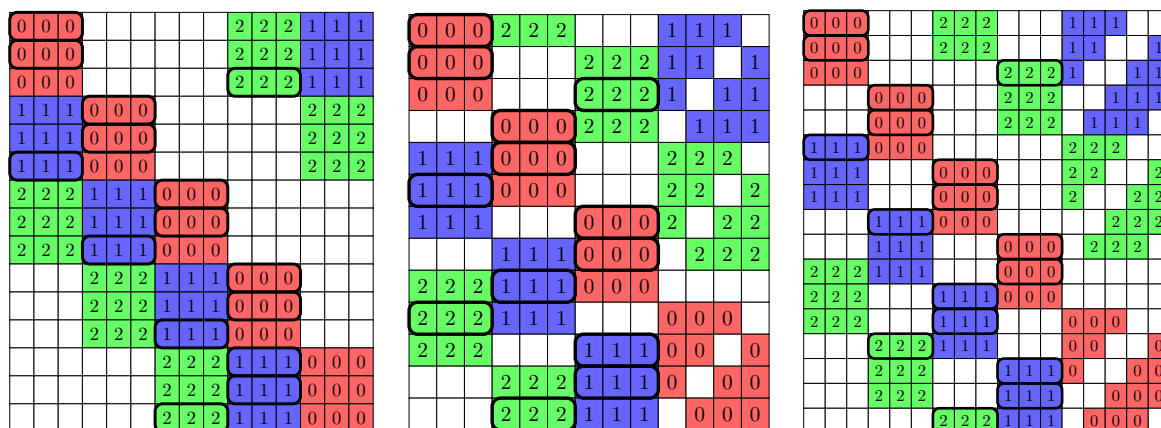


Figure 20: Examples of $m \times m$ matrices which satisfy the conditions of Theorem 5 for $m \in \{15, 13, 17\}$.

3.2 Case $d = 3$

The construction we give for decomposing Latin squares of order n into $3 \times (n/3)$ subrectangles is similar to the $2 \times (n/2)$ construction, so we only explain the differences. In fact, the construction is made easier in this case because there exist Latin squares of order 12 which decompose into 3×4 subrectangles (as opposed to Latin squares of order 6 which do not decompose into 2×3 rectangles); we include one example in Figure 21 (left) found by computer search.

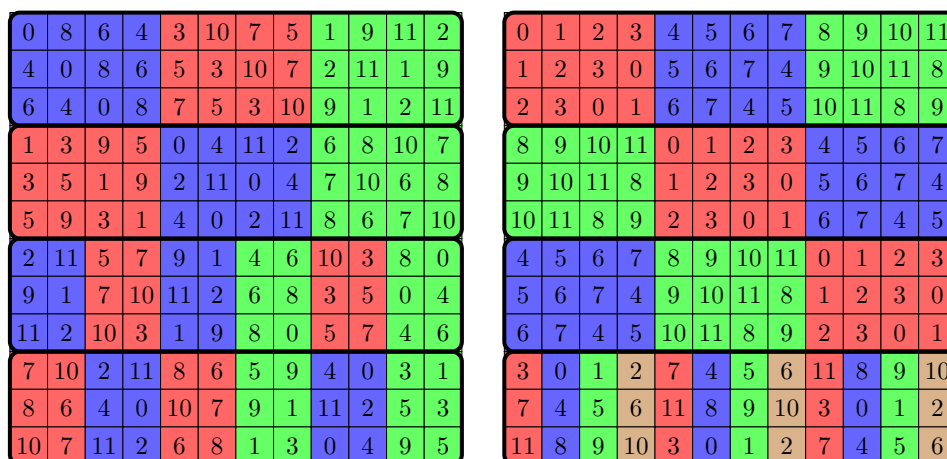


Figure 21: Left: A Latin square of order 12 where consecutive sets of 3 rows decompose into three 3×4 subrectangles. Right: A Latin square of order 12 where consecutive sets of 3 rows decompose into three 3×4 subrectangles, except for the last three rows, which decomposes into four 3×3 subsquares.

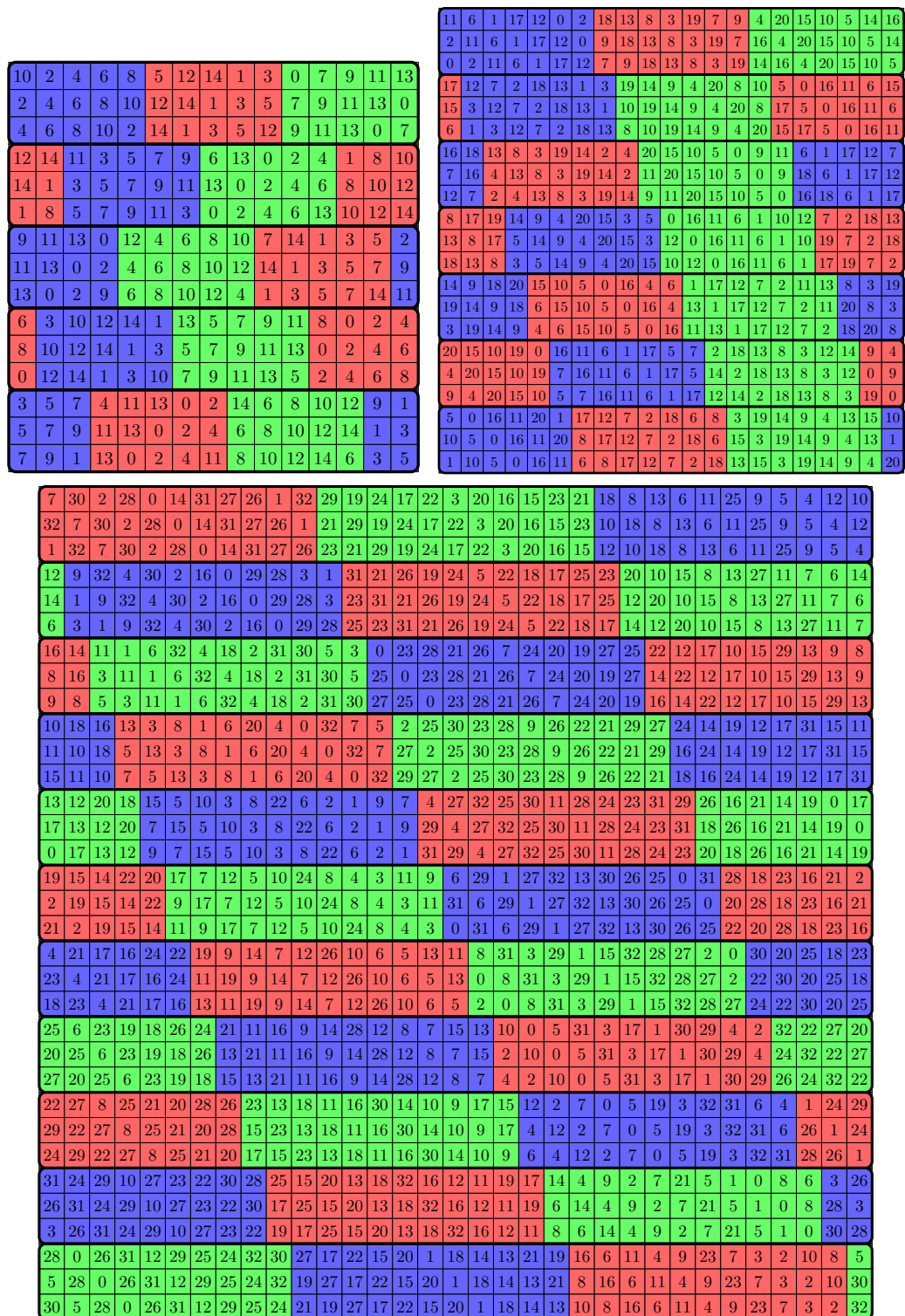


Figure 22: Latin squares of orders $n \in \{15, 21, 33\}$ partitioned into $3 \times (n/3)$ subrectangles.

Theorem 6. Let $m \in \{12, 15, 16\} \cup \{18, 19, \dots\}$. Suppose there exists an $m \times m$ matrix M containing symbols in $\{0, 1, 2\}$ and empty cells with the two properties:

- every row and every column contains exactly 4 copies of each symbol, and
- every row intersects at least two 4×4 all- i matrices, where $i \in \{0, 1, 2\}$.

Then there exists a $3m \times 3m$ Latin square that can be partitioned into $3 \times m$ subrectangles.

Proof. Lemma 9 (König's Theorem) implies that there exists an $m \times m$ Latin square L , with symbols in $\{0, 1, 2, 3\}$ wherever the symbol 0 occurs in M , symbols in $\{4, 5, 6, 7\}$ wherever the symbol 1 occurs in M , symbols in $\{8, 9, 10, 11\}$ wherever the symbol 2 occurs in M , and symbols in $\{12, 13, \dots, n-1\}$ wherever an empty cell occurs in M . We use U_t to denote the t -th 4×4 subsquare that arises from the all- i blocks in M for $i \in \{0, 1, 2\}$.

We take a direct product of L with a 3×3 Latin square, giving a $3m \times 3m$ Latin square. The subsquares U_t blow up to 12×12 subsquares, and we replace them with the 12×12 Latin square in Figure 21 (left), after adjusting the symbol set appropriately. This results in consecutive row triples $\{3i, 3i+1, 3i+2\}$ of L having subrectangles as in one of the following cases. (Note that the theorem assumes $m \in \{12, 15, 16\} \cup \{18, 19, \dots\}$.)

- *Case I:* Six 3×4 subrectangles and $(3m-24)/3$ subsquares of order 3. In this case, we have the partitions of $3m$ into three partitions of m as follows: when $m \equiv 0 \pmod{3}$,

$$3m = (4^3, 3^{(m-12)/3}) + (4^3, 3^{(m-12)/3}) + (3^{m/3});$$

when $m \equiv 1 \pmod{3}$,

$$3m = (4^4, 3^{(m-16)/3}) + (4, 3^{(m-4)/3}) + (4, 3^{(m-4)/3});$$

and when $m \equiv 2 \pmod{3}$,

$$3m = (4^2, 3^{(m-8)/3}) + (4^2, 3^{(m-8)/3}) + (4^2, 3^{(m-8)/3}).$$

- *Case II:* Nine 3×4 subrectangles and $(3m-36)/3$ subsquares of order 3. In this case, we have the partitions of $3m$ into three partitions of m as follows: when $m \equiv 0 \pmod{3}$,

$$3m = (4^3, 3^{(m-12)/3}) + (4^3, 3^{(m-12)/3}) + (4^3, 3^{(m-12)/3});$$

when $m \equiv 1 \pmod{3}$,

$$3m = (4^4, 3^{(m-16)/3}) + (4^4, 3^{(m-16)/3}) + (4, 3^{(m-4)/3});$$

and when $m \equiv 2 \pmod{3}$,

$$3m = (4^5, 3^{(m-20)/3}) + (4^2, 3^{(m-8)/3}) + (4^2, 3^{(m-8)/3}).$$

Thus, since $m \in \{12, 15, 16\} \cup \{18, 19, \dots\}$, it is possible to partition the rows in $\{3i, 3i+1, 3i+2\}$ into $3 \times m$ subrectangles by grouping together 3×4 subrectangles and 3×3 subsquares. \square

We have designed $M_{4,m,3}$ (which exists when $m \in \{12, 16, 17, 18\}$ and $m \geq 20$) so that it satisfies the two matrix conditions in Theorem 6. (We omit the details of a formal check of these properties; they are straightforward to prove, and are apparent from Figure 15). Thus the conditions of Theorem 6 are satisfied when $m \in \{12, 16, 18\} \cup \{20, 21, \dots\}$, in which case there exists a $3m \times 3m$ Latin square that can be partitioned into $3 \times m$ subrectangles.

We still need to resolve the $m \in \{3, 4, \dots, 11\} \cup \{13, 14, 15, 17, 19\}$ cases. If m is divisible by 3, we take a direct product of a Latin square of order 3 and a Latin square of order m to obtain a Latin square of order $3m$ which can be partitioned into 3×3 subsquares, which we group together to give a decomposition into $3 \times m$ subrectangles. When $m \in \{4, 5, 7, 11\}$ the existence problem is settled by examples in Figure 21 and Figure 22. By taking a direct product of a Latin square in Figure 21 or Figure 22 with a Latin square of order 2, we obtain 24×24 , 30×30 , and 42×42 Latin squares which decompose into 3×8 , 3×10 , and 3×14 subrectangles, respectively, thereby resolving the $m \in \{8, 10, 14\}$ cases. This leaves the $m \in \{13, 17, 19\}$ cases.

Case $m = 19$. We use $M_{4,m,2}$, in which each row only intersects either one or two 4×4 all- i matrices for $i \in \{0, 1\}$. We blow it up (taking a direct product with a 3×3 Latin square), and replace the resulting 12×12 subsquares by one that decomposes into 3×4 subrectangles. We have the partitions

$$3 \times 19 = (4, 3^5) + (4, 3^5) + (4, 3^5)$$

and

$$3 \times 19 = (4^4, 3) + (4, 3^5) + (4, 3^5).$$

of $3m$ into three partitions of m . Thus regardless of whether the three consecutive rows $\{3i, 3i+1, 3i+2\}$ decomposes into three 3×4 subrectangles and 15 subsquares of order 3, or six 3×4 subrectangles and 11 subsquares of order 3, it is possible to group them into 3×19 subrectangles.

Case $m = 13$. We begin with the matrix on the left in Figure 23. We apply Lemma 9 (König's Theorem) to obtain a Latin square of order 13 with subsquares marked A , B , C , and D . We take a direct product with a 3×3 Latin square, to obtain a Latin square of order 39 with four subsquares of order 12 arising from A , B , C , and D . We make the replacements tabulated below:

after blowing up...	we replace it with...
A, C	Figure 21 (right) rotated 180°
B	Figure 21 (right)
D	Figure 21 (left)
...after adjusting the symbols	

We have designed this construction so that in the resulting Latin square of order 39, sets of 3 consecutive rows decompose into exactly three 3×4 subrectangles, and exactly nine 3×3 subsquares. Since

$$3 \times 13 = (4, 3^3) + (4, 3^3) + (4, 3^3)$$

is a partition of 39 into three partitions of 13, it is possible to group these subrectangles and subsquares together to obtain three 3×13 subrectangles.

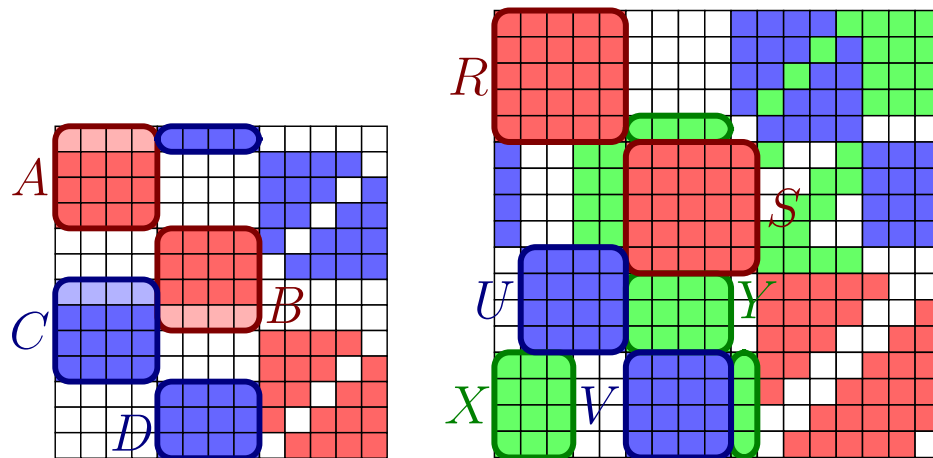


Figure 23: Left: Used to construct a Latin square of order 39 which decomposes into 3×13 subrectangles. Right: Used to construct a Latin square of order 51 which decomposes into 3×17 subrectangles.

Case $m = 17$. This is similar to the $m = 13$ case, so we just highlight the differences. We blow up the matrix on the right in Figure 23. We make the replacements tabulated below:

after blowing up...	we replace it with...
R, S	Figure 22 (top left)
U, V, X, Y	Figure 21 (left)
...after adjusting the symbols	

In the resulting Latin square of order 51, sets of 3 consecutive rows decompose into one of the following:

- exactly three 3×5 subrectangles, and exactly twelve 3×3 subsquares;
- exactly three 3×5 subrectangles, exactly three 3×4 subrectangles, and exactly eight 3×3 subsquares; or
- exactly six 3×4 subrectangles, and exactly nine 3×3 subsquares.

Since

$$\begin{aligned}
 3 \times 17 &= (5, 3^4) + (5, 3^4) + (5, 3^4) \\
 &= (5^2, 4, 3) + (5, 3^4) + (4^2, 3^3) \\
 &= (4^2, 3^3) + (4^2, 3^3) + (4^2, 3^3)
 \end{aligned}$$

describes partitions of 51 into three partitions of 17, in every case, it is possible to group these subrectangles and subsquares together to obtain three 3×17 subrectangles.

3.3 Sufficiently large n

We begin with the following Latin square where sets of d consecutive rows can be partitioned into one $d \times x$ subrectangle and one $d \times y$ subrectangle, where $x \equiv 1 \pmod{d}$ and $y \equiv -1 \pmod{d}$.

Theorem 7. For $d \geq 3$, there exists a Latin square of order $n := d(d+1)$ where, for any $k \in \{0, 1, \dots, d\}$, the d rows $\{kd, kd+1, \dots, kd+d-1\}$ partition into one $d \times (d+1)$ subrectangle and one $d \times (d^2-1)$ subrectangle.

Proof. We define an $n \times n$ matrix containing symbols 0 and 1 (and some empty cells) in which every row and every column has exactly $d+1$ occurrences of symbol 0 and exactly $d+1$ occurrences of symbol 1. The construction is illustrated in Figure 24 for $d \in \{3, 4\}$. We add all- i blocks according to the table below (all other cells are empty; we “mark” the submatrices to match Figure 24):

dimensions	top-left corner	symbol	mark
$(d+1) \times (d+1)$	$((d+1)\sigma + d + 2, (d+1)\sigma)$ for all $\sigma \in \{0, 1, \dots, d-1\}$ (Note: indices are in \mathbb{Z}_n)	1	A
$d \times (d+1)$	$(d\sigma, (d+1)\sigma)$ for all $\sigma \in \{0, 1, \dots, d-3\}$	0	B
$1 \times (d+1)$	$(n-1-\sigma, (d+1)\sigma)$ for all $\sigma \in \{0, 1, \dots, d-3\}$	0	C
$(d+1) \times (d+1)$	$(n-3d, n-2(d+1))$	0	D
$(d+1) \times (d+1)$	$(n-2d+1, n-(d+1))$	0	E

The $d \times (d+1)$ and $1 \times (d+1)$ all-1 blocks combine to form $(d+1) \times (d+1)$ all-1 submatrices.

Applying Lemma 9 (König’s Theorem) to this matrix gives a Latin square L , which has $(d+1) \times (d+1)$ subsquares wherever an all- i submatrix occurs.

From these, we select contiguous $d \times (d+1)$ subrectangles (also depicted in Figure 24): we choose subrectangles with top-left corner $(d\sigma, (d+1)\sigma)$ for all $\sigma \in \{0, 1, \dots, d-2\}$ (in the blocks marked **B** and **D**) and top-left corner $(d(d-1), (d+1)(d-3))$ and $(d^2, (d+1)(d-2))$ (in two of the blocks marked **A**). This divides the d rows $\{d\sigma, \dots, d\sigma+d-1\}$, for any $\sigma \in \{0, 1, \dots, d\}$, into a $d \times (d+1)$ subrectangle and a $d \times (n-d-1)$ subrectangle. \square

By taking a direct product of the Latin square of order $d(d+1)$ in Theorem 7 with a Latin square of order $(d-1)^2$, we let S_d be a Latin square of order $d(d+1)(d-1)^2$ in which sets of d consecutive rows $\{di, di+1, \dots, di+d-1\}$ contain $\geq (d-1)^2$ distinct $d \times (d+1)$ subrectangles.

We define the $m \times m$ matrix $P = M_{p,m,2}$ where $p := (d+1)(d-1)^2$ and observe that every row of P intersects a $p \times p$ all-0 block and/or a $p \times p$ all-1 block. We use Lemma 9 (König’s Theorem) to show that there exists an $m \times m$ Latin square L with $p \times p$ subsquares occurring whenever a $p \times p$ all-1 block occurs in P . We take a direct product of L with a Latin square of order d and replace the $dp \times dp$ subsquares which arise due to all- i blocks in P with copies of S_d after relabelling the symbols appropriately. This gives an $n \times n$

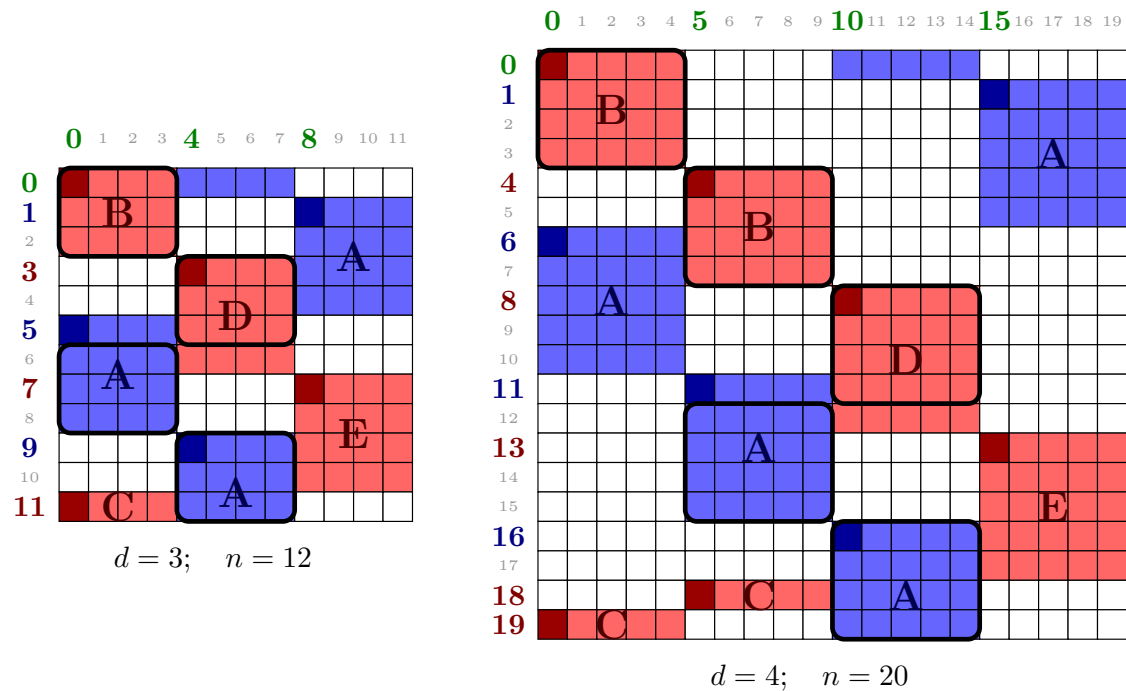


Figure 24: The construction in Theorem 7 for $d \in \{3, 4\}$. Red cells contain 0, and blue cells contain 1.

Latin square (where $n := dm$) which can be partitioned into sets of d consecutive rows containing at least $(d-1)^2$ distinct $d \times (d+1)$ subrectangles. Moreover, the remainder of every set of d consecutive rows decomposes into $d \times d$ subsquares, implying there are $m - O(1)$ such subsquares (as $m \rightarrow \infty$).

For each set of d consecutive rows, we identify $d-1$ disjoint $d \times m$ subrectangles by combining exactly $r := m \pmod{d}$ distinct $d \times (d+1)$ subrectangles with exactly $(m-r)/d - r$ distinct $d \times d$ subsquares. (The remaining $d \times m$ subrectangle is formed from what is outside these $d-1$ subrectangles.) This requires $m(d-1)/d + O(1)$ distinct $d \times d$ subsquares in those rows, and we have $m - O(1)$, which is enough when m is large. This gives a construction for Latin squares of order n which decompose into $d \times (n/d)$ subrectangles for all sufficiently large n divisible by d .

We note that “sufficiently large n ” in this construction is at least asymptotically d^4 , which likely far exceeds the actual minimum n .

4 Concluding remarks

We tabulate the answers to the existence problems for small n in Table 1. Some values are determined as follows: if a Latin square of order n decomposes into $2 \times (n/2)$ subrectangles, then taking a direct product with a Latin square of order 2 yields a Latin square of order $2n$ which decomposes into $4 \times n$ subrectangles.

The smallest unresolved case is the existence of a 4-balanced equi-20-square, and a 20×20 Latin square that decomposes into 4×5 subrectangles. An exhaustive computer search rules out the existence of a 5-starter with 30 filled cells (and thus $K_{6,6} - I$, where I is a 1-factor, does not admit an α -labelling).

n	some Latin square of order n can be partitioned into $d \times (n/d)$ subrectangles (and $d \leq n/d$)	a diagonally cyclic d -balanced equi- n -square exists	a d -balanced equi- n -square exists
4	2	2	2
6	2	2	2
8	2	2	2
9	3	3	3
10	2	2	2
12	2, 3	2, 3	2, 3
14	2	2	2
15	3	3	3
16	2, 4	2, 4	2, 4
18	2, 3	2 , 3	2, 3
20	2 [4]	2, 2	2 [4]
21	3	3	3
22	2	2	2
24	2, 3, 4	2, 3, 4	2, 3, 4
25	5	5	5
26	2	2	2
27	3	3	3
28	2 [4]	2, 4	2 [4]
30	2, 3 [5]	2 , 3 [5]	2, 3 [5]

Table 1: For a given n , the divisors $d \neq 1$ with $d^2 \leq n$ for which there exists (a) a Latin square that can be partitioned into $d \times (n/d)$ subrectangles (second column), (b) a diagonally cyclic d -balanced equi- n -square (third column), and (b) a d -balanced equi- n -square (fourth column). Numbers that are canceled are d values where a construction is impossible. Numbers in square brackets indicate d values which are unresolved.

The proof of Theorem 3 also works for α -labellings of bipartite graphs in general (after dropping the condition that there are “ d filled cells in each row and column”), which we illustrate in Figure 25. While we do not need this generality in this paper, it would be interesting to explore α -labellings from this perspective in future research. Moreover, the proof of the product construction in Lemma 4 holds for this generalization of d -starters, which implies the following lemma.

Lemma 10. *If two bipartite graphs with the biadjacency matrices A and B admit α -labellings, then the graph with biadjacency matrix $A \otimes B$, where \otimes is the matrix Kronecker*

product, also admits an α -labelling.

Brankovic and Wanless [8] found a relation between α -labellings of a path and partial transversals of the Cayley table of the cyclic group, which has a similar style to this construction. Graceful labelings of paths give rise to cyclic oriented triangular embeddings of complete graphs [20].

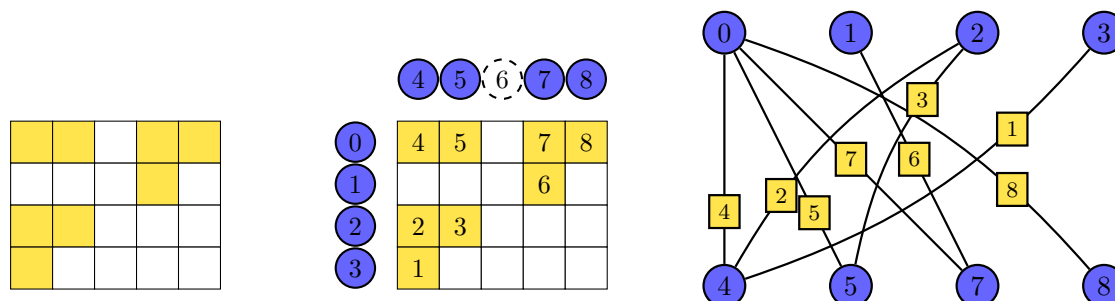


Figure 25: Left: A matrix with exactly one filled cell in each unbroken (top-left to bottom-right) diagonal. Middle: Illustrating how we convert to (and from) an α -labelling. Right: The α -labelling (parameter $k = 3$) and the bipartite graph determined from the matrix on the left.

It is straightforward to construct α -labellings (and hence graceful labelings) of all caterpillars via these matrices: if the main path has degree sequence $(d_i)_{i=1}^n$, we start at the bottom right, fill $d_1 = 1$ cells vertically, then d_2 cells horizontally, then d_3 cells vertically, and so on. This is depicted in Figure 26. Graceful labellings of caterpillars in particular are studied in connection with multi-protocol label switching in IP networks [1, 4].

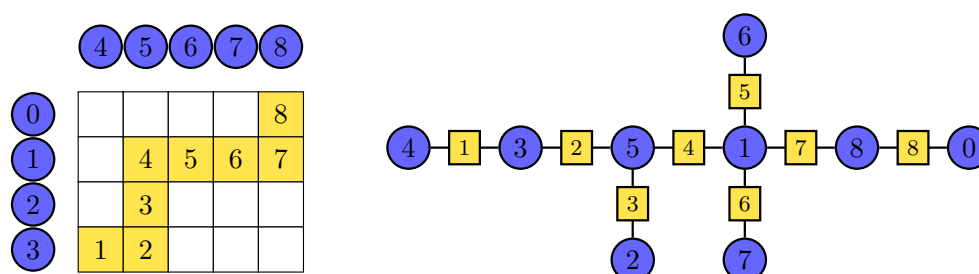


Figure 26: An α -labelling of a caterpillar.

We observe that if we take a d -starter with n zeroes, rotate it by 90° , then superimpose it on the multiplication table of \mathbb{Z}_n , we obtain a kind of generalized transversal. In this way, 1-starters correspond to transversals of the multiplication table of \mathbb{Z}_n . For example, a 24-element 4-starter embeds in the multiplication table of \mathbb{Z}_{24} as in Figure 27. This selection of entries has a unique copy of each symbol in \mathbb{Z}_n , and each row and each column is either unrepresented, or is represented exactly d times.

A long-standing problem is if all equi- n -squares have a near-transversal [29], i.e., a selection of $n-1$ entries which do not have a common row, column, nor symbol. Pokrovskiy

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1
3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5
7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5	6
8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5	6	7
9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	0	1	2	3	4	5	6	7	8

Figure 27: A 4-starter embedded in the multiplication table of \mathbb{Z}_{24} . Not all rows are shown; the omitted rows do not involve the 4-starter.

and Sudakov [25] recently obtained a breakthrough on this problem: a construction of (non-Latin) equi- n -squares without near-transversals. The same problem for Latin squares [15] is yet to be resolved. The current best answer to “how close we can get” to a transversal in a Latin square is by Keevash, Pokrovskiy, Sudakov and Yepremyan [23], improving older results by Hatami and Shor [21, 28] (see also [6]); similar questions arise for the aforementioned generalized transversals, which is a possible future research direction.

In this paper, we also describe decomposing a Latin square into subrectangles. This leads to a range of interesting problems, where many problems related to graph decomposition have analogues with Latin squares. The subrectangles we consider are not necessarily structurally equivalent (isotopic), so we have not resolved the problem of decomposing Latin squares into isotopic copies of a subrectangle. In fact, we might consider dropping the constraint that the subrectangles be horizontally aligned. Moreover, we need not limit ourselves to rectangular submatrices: it is interesting to ask when there exists a Latin square that decomposes into isotopic copies of a given partial Latin square.

Along these lines, the existence of a Latin square that decomposes into $d \times (n/d)$ submatrices containing all n symbols was resolved in [11]: it is possible for all divisors d of n . In [6], the authors observe that all Latin squares of order n have an $O(n^{1/2+\epsilon}) \times O(n^{1/2+\epsilon})$ submatrix containing $n - O(n^{1/2+\epsilon})$ distinct symbols, which raised the existence problem for Latin squares of order n^2 which cannot decompose into $n \times n$ submatrices which contain all n symbols.

Diagonally cyclic equi- n -squares are equivalent to equi- n -squares that admit an n -cycle automorphism. Latin squares that admit automorphisms are used in secret sharing [32], and the Latin square could easily be replaced by an equi- n -square, and interpreted as a graph decomposition [7] (see [30] for a broad and detailed treatment). In abstract algebra, diagonally cyclic equi- n -squares correspond to n -element magmas (sometimes called groupoids) that admit n -cycle automorphisms.

Diagonally cyclic equitable rectangles [17] are another kind of generalization of diago-

nally cyclic Latin squares. Equitable rectangles are studied in connection with generalized mix functions [26,31]. Like diagonally cyclic d -balanced equi- n -squares, they admit a compact description, where we need only store one entry from each diagonal.

Acknowledgements

The authors thank Daniel Horsley for directing us to relevant references, Amir Hossein Ghodrati for some useful discussions, and Ian Wanless for his advice. Stones acknowledges the use of the online mathematics forums math.stackexchange.com and mathoverflow.net to discuss topics relating to this research.

The research of the first author was partly funded by Iran National Science Foundation (INSF) under the contract No. 96004167. Also, the research of the first author was supported by grant number (G981202) from the Sharif University of Technology. Marbach and Stones were partially supported by NSF of China (61602266, 61872201), the Science and Technology Development Plan of Tianjin (17JCYBJC15300, 16JCYBJC41900), and the Fundamental Research Funds for the Central Universities and SAFEA: Overseas Young Talents in Cultural and Educational Sector. Stones was supported by her NSFC Research Fellowship for International Young Scientists (grant numbers: 11450110409, 11550110491), and the Thousand Youth Talents Plan in Tianjin.

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