On fractional fragility rates of graph classes*

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Abstract

We consider, for every positive integer a, probability distributions on subsets of vertices of a graph with the property that every vertex belongs to the random set sampled from this distribution with probability at most $1/a$. Among other results, we prove that for every positive integer a and every planar graph G, there exists such a probability distribution with the additional property that for any set X in the support of the distribution, the graph $G - X$ has component-size at most $(\Delta(G) - 1)^{a+O(\sqrt{a})}$, or treedepth at most $O(a^3 \log_2(a))$. We also provide nearly-matching lower bounds.

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1 Introduction

Planar graphs "almost" have bounded treewidth, in the following sense: For every assignment of weights to vertices and for every positive integer a, it is possible to delete vertices of at most $1/a$ fraction of the total weight so that the resulting subgraph has treewidth at most $3a - 3$. Equivalently, there exists a probability distribution on subsets of vertices whose complement induces a subgraph of treewidth at most $3a - 3$, such that each vertex belongs to a set sampled from this distribution with probability at most $1/a$. This property is the key ingredient of a number of approximation algorithms for planar graphs [1]. To study this phenomenon more generally, Dvořák [5] introduced the notion of

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fractional fragility of a graph class with respect to a graph parameter. Let us give the definitions we need to speak about this notion.

For $\varepsilon > 0$, we say that a probability distribution on the subsets of vertices of a graph G is *ε-thin* if for each vertex v, the probability that v belongs to a set sampled from this distribution is at most ε. For example, we commonly use the following $(1/a)$-thin probability distribution. Suppose that sets $X_1, \ldots, X_a \subseteq V(G)$ are pairwise disjoint, and that t of these sets are empty. We give to each non-empty set X_i the probability $1/a$ and to the empty set the probability t/a. All other sets are given probability 0. We call this distribution the *uniform distribution on* $\{X_1, \ldots, X_a\}$.

Let f be a graph parameter, that is, a function assigning to every graph a non-negative real number such that isomorphic graphs are assigned the same value. We will generally consider parameters that are *monotone* (satisfying that $f(H) \leqslant f(G)$ whenever H is a subgraph of G), or at least *hereditary* (satisfying $f(H) \leqslant f(G)$ whenever H is an induced subgraph of G). For a real number b and a graph G, let $G_{f\leqslant b}$ be the set of all subsets $X \subseteq V(G)$ such that $f(G[X]) \leqslant b$, and let $G_{f\downarrow b}$ be the set of all subsets $Y \subseteq V(G)$ such that $f(G - Y) \leqslant b$; thus $Y \in G_{f\downarrow b}$ if and only if $V(G) \setminus Y \in G_{f\leqslant b}$. For example, if tw is the function that to every graph assigns its treewidth, then $G_{\mathrm{tw}\downarrow 3a-3}$ is the set of vertex sets whose complement induces a subgraph of treewidth at most $3a - 3$.

Let $r \colon \mathbb{N} \to \mathbb{R}_0^+$ be a non-decreasing function. A graph G is *fractionally f-fragile at rate r* if for every positive integer a, there exists a $(1/a)$-thin probability distribution on $G_{f\downarrow r(a)}$. Of course, every graph is fractionally f-fragile at rate given by the constant function $r(a) \coloneqq f(G)$; so the notion is more interesting for graph classes. We say that a class of graphs is *fractionally f-fragile at rate r* if each graph from the class is, and we say that the class is *fractionally f-fragile* if it is fractionally f-fragile at some rate. Coming back to the introductory example, the class of planar graphs is known to be fractionally tw-fragile at rate $r(a) \coloneqq 3a - 3$; see Corollary 11 below for details.

Graphs from fractionally f-fragile classes can be viewed as being close to graphs for which the parameter f is bounded, and this proximity can be useful when reasoning about their structural and quantitative properties. There are also natural links to the theory classes of bounded expansion [5]. Furthermore, as we already mentioned in the introduction, the notion has algorithmic applications, especially in the design of approximation algorithms. Let us give an example of such an application, which was already presented before [5] along with other applications. Consider a property $\pi(G, X)$ of a graph G and a subset X of its vertices. We say the property is *downward-hereditary* if $\pi(G, X)$ being true implies that $\pi(H, V(H) \cap X)$ is true for every induced subgraph H of G, and *upward-hereditary* if $\pi(G, X)$ being true implies that $\pi(G', X)$ is true for every graph G' such that G is an induced subgraph of G'. As an example, the property "X is an independent set in G" is both downward- and upward-hereditary. Let $\alpha_\pi(G)$ be the largest size of a set $X \subseteq V(G)$ such that $\pi(G, X)$ is true.

Observation 1. *Let f be a graph parameter and let $r, t \colon \mathbb{N} \to \mathbb{R}_0^+$ be non-decreasing functions. Suppose that a class $\mathcal{G}$ of graphs is fractionally f-fragile at rate r and, moreover, that there exists an algorithm that for a graph $G \in \mathcal{G}$ and a positive integer a returns a set sampled from a $(1/a)$-thin probability distribution on $G_{f\downarrow r(a)}$ in polynomial time. Let π be a* *downward- and upward-hereditary property, and suppose that $\alpha_\pi(H)$ can be determined in time $t(f(H)) \cdot \mathrm{poly}(|H|)$ for any graph H. Then there exists a randomized algorithm that for a graph $G \in \mathcal{G}$ and a positive integer a returns in time $t(r(a)) \cdot \mathrm{poly}(|G|)$ an integer $\mathsf{b} \leqslant \alpha_\pi(G)$ such that $\mathbf{E}[\mathsf{b}] \geqslant (1-1/a)\alpha_\pi(G)$.*

Proof. Sample a set X from a $(1/a)$-thin probability distribution on $G_{f\downarrow r(a)}$, so that $f(G - \mathsf{X}) \leqslant r(a)$, and let $\mathsf{b} = \alpha_\pi(G - \mathsf{X})$. Since π is upward-hereditary, we have $\mathsf{b} \leqslant \alpha_\pi(G)$.

Moreover, consider a set $Y \subseteq V(G)$ such that $\pi(G, Y)$ is true and $|Y| = \alpha_\pi(G)$. Since X is sampled from a $(1/a)$-thin probability distribution, we have $\mathbf{E}[|Y \setminus \mathsf{X}|] \geqslant (1-1/a)|Y|$. Since π is downward-hereditary, $\pi(G - \mathsf{X}, Y \setminus \mathsf{X})$ is true, and thus

$$\mathbf{E}[\mathsf{b}] = \mathbf{E}[\alpha_\pi(G - \mathsf{X})] \geqslant \mathbf{E}[|Y \setminus \mathsf{X}|] \geqslant (1-1/a)|Y| = (1-1/a)\alpha_\pi(G). \qquad \square$$

For example, the independence number is hard to approximate within a polynomial factor [2] in general, but can be determined in time $3^{\mathrm{tw}(H)} \cdot \mathrm{poly}(|H|)$ for any graph H. Consequently, for any class of graphs that is fractionally tw-fragile at rate r (efficiently in the sense of Observation 1), the independence number can be approximated in an n-vertex graph from the class up to the factor of $(1-1/a)$ in time $3^{r(a)} \cdot \mathrm{poly}(n)$. Let us remark that in essentially all known cases of fractionally fragile classes, it is possible to find a probability distribution as described in Observation 1 with support of polynomial size, and thus the algorithm can be derandomized by trying all sets from the support rather than sampling one of them.

Fractional fragility is also related to generalizations of the (fractional) chromatic number arising from the following scheme introduced by Wood [16, Section 10]. An (f, b)-*coloring* of a graph G is an assignment φ of colors to the vertices such that $f(G[\varphi^{-1}(c)]) \leqslant b$ for every color c, that is, such that each color class belongs to $G_{f\leqslant b}$. We can now define $\chi_{f,b}(G)$ as the least number of colors in an (f, b)-coloring of G. For a class of graphs $\mathcal{G}$, we naturally define $\chi_f(\mathcal{G})$ as the smallest integer s such that for some positive integer b, all graphs $G \in \mathcal{G}$ satisfy $\chi_{f,b}(G) \leqslant s$. For example, let $\star(G)$ be the maximum of the orders of the components of the graph G. Then $\chi_{\star,1}(G)$ is just the ordinary chromatic number of G, while in general, the parameter $\chi_{\star,b}(G)$ has been studied as the *clustered chromatic number* [16].

Similarly to the way the fractional chromatic number is derived from the ordinary chromatic number [15], we can also derive the fractional variant of this generalization. A *fractional* (f, b)-*coloring* of a graph G is a function $\kappa\colon G_{f\leqslant b} \to [0, 1]$ such that for each vertex $v \in V(G)$,

$$\sum_{Y \in G_{f\leqslant b}, v \in Y} \kappa(Y) \geqslant 1;$$

the *number of colors* $|\kappa|$ used by this coloring is $\sum_{Y \in G_{f\leqslant b}} \kappa(Y)$. We define $\chi^{\mathrm{frac}}_{f,b}(G)$ to be the infimum of $|\kappa|$ over all fractional (f, b)-colorings κ of G. For a class $\mathcal{G}$ of graphs, we define $\chi^{\mathrm{frac}}_f(\mathcal{G})$ as the infimum of the real numbers s such that for some positive integer b, all graphs $G \in \mathcal{G}$ satisfy $\chi^{\mathrm{frac}}_{f,b}(G) \leqslant s$.

Note that unlike the ordinary fractional chromatic number case, this can indeed be a proper infimum: as b increases, the fractional (f, b)-coloring may need fewer colors, converging to but never reaching $\chi_f^{\text{frac}}(\mathcal{G})$. This motivates the following definition that captures the rate of the convergence. For a real number c and a function $r\colon \mathbb{N} \to \mathbb{R}_0^+$, we say that a class of graphs $\mathcal{G}$ is *fractionally f-colorable by c colors at rate r* if for every integer $a \geqslant 1$, every graph $G \in \mathcal{G}$ satisfies $\chi_{f,r(a)}^{\text{frac}}(G) \leqslant c + 1/a$. As we will see below (Lemma 2), fractional f-fragility is equivalent to fractional f-colorability by 1 color, at a matching rate.

1.1 Main results

The previous treatment of fractional fragility [5] was mostly qualitative. In this paper, we focus on the quantitative aspect: the rate of fractional fragility for various parameters and graph classes. Note that the rate is important in the applications, as it determines, *e.g.*, the multiplicative constant in the complexity of the approximation algorithm from Observation 1.

In Section 3, we consider the parameter $\star$, the maximum component size. By Lemma 5, only classes of graphs with bounded maximum degree can be fractionally $\star$-fragile.

- In Theorem 16, we prove that the fractional $\star$-fragility rate r of any class of graphs containing all trees of maximum degree $\Delta \geqslant 3$ is exponential, more precisely $r(a) \geqslant (\Delta - 1)^{a-3}$.
- Conversely, we show that graphs of bounded treewidth (Corollary 22) and planar graphs (Theorem 23) of maximum degree Δ are fractionally $\star$-fragile at a nearly matching rate $r(a) = (\Delta - 1)^{a+O(\sqrt{a})}$.

In Section 4, we turn our attention to another graph parameter, *treedepth*. This parameter naturally generalizes the component size, but fractional td-fragility does not require bounded maximum degree.

- In Theorem 31, we show that graphs of treewidth at most t are fractionally td-fragile at a polynomial rate $r(a) = O(a^t)$. We also give a matching lower bound. In particular, in Theorem 26, we prove that planar graphs of treewidth at most two have fractional td-fragility rate $r(a) = \Omega(a^2)$.
- For outerplanar graphs (an important subclass of graphs of treewidth at most two), we show that the fractional td-fragility rate is $r(a) = \Theta(a \log a)$, in Theorems 33 and 29.
- In Corollary 35, we show that planar graphs are fractionally td-fragile at rate $r(a) = O(a^3 \log a)$, in contrast to the lower bound $\Omega(a^2 \log a)$ for this class.

2 Preliminaries

In this section, we show some basic properties of the fractional f-fragility, and present several auxiliary results we need in the rest of the paper.

2.1 Basic properties of fractional fragility

The relationship between fractional f-colorability and fractional f-fragility is given by the following lemma.

Lemma 2. *Let $r\colon \mathbb{N} \to \mathbb{R}_0^+$ be a non-decreasing function, and let $r'\colon \mathbb{N} \to \mathbb{R}_0^+$ be defined by setting $r'(a) := r(a+1)$ for every $a \in \mathbb{N}$. Let f be a graph parameter whose value is at most $r(1)$ on the empty graph. A class $\mathcal{G}$ of graphs is fractionally f-fragile at rate r if and only if it is fractionally f-colorable by 1 color at rate r'.*

Proof. Suppose first that $\mathcal{G}$ is fractionally f-fragile at rate r. Hence, for any positive integer a and any graph $G \in \mathcal{G}$, there exists a $\frac{1}{a+1}$-thin probability distribution on $G_{f\downarrow r(a+1)}$. Recall that a subset of $V(G)$ belongs to $G_{f\downarrow r(a)}$ if and only if its complement belongs to $G_{f\leqslant r(a)}$. For $Y \in G_{f\leqslant r'(a)}$, let $\kappa(Y) := \frac{a+1}{a}\,\mathbf{Pr}(V(G)\setminus Y)$. For each vertex $v \in V(G)$ and a set X sampled from the probability distribution, we have

$$\begin{aligned}\sum_{Y\in G_{f\leqslant r'(a)},v\in Y}\kappa(Y) &= \sum_{Z\in G_{f\downarrow r'(a)},v\notin Z}\kappa(V(G)\setminus Z) = \frac{a+1}{a}\sum_{Z\in G_{f\downarrow r'(a)},v\notin Z}\mathbf{Pr}(Z)\\ &= \frac{a+1}{a}\,\mathbf{Pr}[v\notin \mathsf{X}] \geqslant \frac{a+1}{a}\Big(1-\frac{1}{a+1}\Big) = 1,\end{aligned}$$

and thus κ is a fractional $(f, r'(a))$-coloring of G using $|\kappa| = \frac{a+1}{a} = 1 + 1/a$ colors. Since this holds for every positive integer a and for all graphs in $\mathcal{G}$, the class $\mathcal{G}$ is f-colorable by 1 color at rate r'.

Conversely, suppose that $\mathcal{G}$ is f-colorable by 1 color at rate r'. Consider a positive integer a and a graph $G \in \mathcal{G}$. Note that setting $\mathbf{Pr}(V(G)) := 1$ and $\mathbf{Pr}(Z) := 0$ for all $Z \subsetneq V(G)$ gives a 1-thin probability distribution on $G_{f\downarrow r(1)}$, since $r(1) \geqslant f(G - V(G))$. Hence, we can assume that $a \geqslant 2$. Then there exists a fractional $(f, r'(a-1))$-coloring κ with $|\kappa| = 1 + \frac{1}{a-1}$, from which one can obtain a $(1/a)$-thin probability distribution on $G_{f\downarrow r(a)}$ by setting $\mathbf{Pr}(Z) := \frac{a-1}{a}\kappa(V(G)\setminus Z)$ for each $Z \in G_{f\downarrow r(a)}$. Indeed, for any vertex $v \in V(G)$ and a set X sampled from this distribution, we have

$$\begin{aligned}\mathbf{Pr}[v\in \mathsf{X}] = 1-\mathbf{Pr}[v\notin \mathsf{X}] &= 1-\sum_{Z\in G_{f\downarrow r(a)},v\notin Z}\mathbf{Pr}(Z)\\ &= 1-\frac{a-1}{a}\sum_{Z\in G_{f\downarrow r(a)},v\notin Z}\kappa(V(G)\setminus Z)\\ &= 1-\frac{a-1}{a}\sum_{Y\in G_{f\leqslant r'(a-1)},v\in Y}\kappa(Y) \leqslant 1-\frac{a-1}{a} = \frac{1}{a}.\end{aligned}$$

This shows that $\mathcal{G}$ is fractionally f-fragile at rate r. $\square$

Let us note the following necessary condition for fractional f-fragility. We say that a graph G is *f-breakable at rate r* if for every positive integer a, there exists a set $X \in G_{f\downarrow r(a)}$ of size at most $|V(G)|/a$. The next observation readily follows from the definitions by using the linearity of expectation.

Observation 3. *If a graph G is fractionally f-fragile at rate r, then it is also f-breakable at rate r.*

A seminal result on $\star$-breakability dates back to the work of Lipton and Tarjan [10]; they proved it in the special case of planar graphs, however, they proof directly generalizes to any class with sufficiently small balanced separators. A *separation* in a graph G is a pair (A, B) of subsets of vertices of G such that $V(G) = A \cup B$ and no edge of G has one end in $A \setminus B$ and the other end in $B \setminus A$; that is, $A \setminus B$ and $B \setminus A$ are unions of the vertex sets of the components of $G - (A \cap B)$. The *order* of the separation is $|A \cap B|$. The separation is *balanced* if $|A \setminus B| \leqslant \frac{2}{3}|V(G)|$ and $|B \setminus A| \leqslant \frac{2}{3}|V(G)|$. Let $s\colon \mathbb{N} \to \mathbb{R}_0^+$ be a non-decreasing function. A graph G has *balanced s-separators* if every induced subgraph H of G has a balanced separator of order at most $s(|V(H)|)$.

Theorem 4 (Lipton and Tarjan [10])**.** *Let β be a positive real number in $(0, 1]$. For every function $s(n) = O(n^{1-\beta})$, there exists a function $r(a) = O(a^{1/\beta})$ such that every graph with balanced s-separators is $\star$-breakable at rate r.*

We should also note the following property, already observed in an earlier work [5].

Lemma 5. *Suppose that f is a monotone graph parameter that is unbounded on stars. Then every fractionally f-fragile class of graphs has bounded maximum degree.*

Proof. Suppose that a class $\mathcal{G}$ of graphs is fractionally f-fragile at rate r. Since f is unbounded on stars, there exist an integer k such that $f(K_{1,k}) > r(3)$. We show that all graphs in $\mathcal{G}$ have maximum degree at most $3k - 3$. Suppose, on the contrary, that a graph $G \in \mathcal{G}$ contains a vertex v of degree at least $3k - 2$. Choose a set $\mathsf{X} \in G_{f\downarrow r(3)}$ at random from a $(1/3)$-thin probability distribution. Consider the random variable $\mathsf{R} := \deg(v) \cdot [v \in \mathsf{X}] + |N(v) \cap \mathsf{X}|$, where $[v \in \mathsf{X}]$ is 1 if $v \in \mathsf{X}$ and 0 otherwise. The linearity of expectation ensures that $\mathbf{E}[\mathsf{R}] \leqslant \frac{2}{3}\deg(v)$, and hence there exists a set $Z \in G_{f\downarrow r(3)}$ such that $\deg(v) \cdot [v \in Z] + |N(v) \cap Z| \leqslant \frac{2}{3}\deg(v)$. Consequently, $v \notin Z$ and $|N(v) \cap Z| \leqslant \frac{2}{3}\deg(v)$, and thus $\deg_{G-Z}(v) \geqslant \lceil \deg(v)/3 \rceil \geqslant k$. It follows that $K_{1,k}$ is a subgraph of $G - Z$. As f is monotone, we deduce that $f(G - Z) \geqslant f(K_{1,k}) > r(3)$, in contradiction to $Z \in G_{f\downarrow r(3)}$. $\square$

A linear programming dual formulation of fragility leads to the following observation. For an assignment $w\colon V(G) \to \mathbb{R}_0^+$ of weights to vertices and a set $Z \subseteq V(G)$, let $w(Z) := \sum_{v \in Z} w(v)$.

Lemma 6. *Let G be a graph that is fractionally f-fragile at rate r. Let a be a positive integer and $w\colon V(G) \to \mathbb{R}_0^+$ an assignment of weights to the vertices of G. Then there exists $Z \subseteq V(G)$ such that $w(Z) \leqslant w(V(G))/a$ and $f(G - Z) \leqslant r(a)$.*

Proof. Choose a set $\mathsf{X} \in G_{f\downarrow r(a)}$ at random from a $(1/a)$-thin probability distribution. By the linearity of expectation, $\mathbf{E}[w(\mathsf{X})] \leqslant w(V(G))/a$, and thus there exists a set $Z \in G_{f\downarrow r(a)}$ such that $w(Z) \leqslant w(V(G))/a$; *i.e.*, there exists $Z \subseteq V(G)$ such that $w(Z) \leqslant w(V(G))/a$ and $f(G - Z) \leqslant r(a)$. $\square$

2.2 Chordal graphs

Due to the following well-known observation, when considering graphs of bounded treewidth, it is often convenient to work in the setting of *chordal graphs*, that is, graphs not containing any induced cycles other than triangles.

Observation 7. *Every graph has a chordal supergraph with the same set of vertices and the same treewidth. Moreover, if G is chordal, then $tw(G) = \omega(G) - 1$.*

Indeed, this equivalent characterization of treewidth can be taken as its definition, and thus we do not provide its usual (somewhat technical) definition, which can be found, *e.g.*, in [14].

Each chordal graph G has an *elimination ordering*: an ordering of the vertices of G such that the neighbors of each vertex that precede it in the ordering induce a clique. By Observation 7, in an elimination ordering of G, each vertex is preceded by at most $\mathrm{tw}(G)$ of its neighbors. Moreover, for every induced path P in G, the last vertex of $V(P)$ according to the elimination ordering must be an end-vertex of P. In particular, this implies the following property.

Observation 8. *Let G be a connected chordal graph and let v be the first vertex in an elimination ordering L of G. For each vertex $u \in V(G) \setminus \{v\}$, the vertex preceding u on any shortest path from v to u also precedes u in L.*

The next observation is also based on this fact.

Lemma 9. *Let G be a connected chordal graph, let v be the first vertex in an elimination ordering of G, let i be a non-negative integer, and let H be a connected subgraph of G, all vertices of which are at distance greater than i from v. Let K be the set of vertices of G at distance exactly i from v that have a neighbor in $V(H)$. Then K induces a clique in G.*

Proof. Suppose for a contradiction that K does not induce a clique in G, and thus K contains distinct non-adjacent vertices x and y. In particular, this implies that $|K| \geqslant 2$, and thus $i > 0$, since only v is at distance 0 from v. Since both x and y are at distance $i > 0$ from v, we have $x \neq v \neq y$.

Since H is connected and every vertex in K has a neighbor in H, there exists a path between x and y in G with all internal vertices in H; let Q be a shortest such path. It follows that Q is an induced path. Consequently, the last vertex of Q in the elimination ordering is one of its end-points, say y. Let u be the neighbor of y in Q; since $xy \notin E(G)$, we have $u \in V(H)$. Since the distance from v to y is i and the distance to u is greater than i, there exists a shortest path P from v to u passing through y. But both v and u precede y in the elimination ordering, and thus the last vertex of P in the elimination ordering is neither of the ends of P. This is a contradiction, since P is an induced path. □

2.3 Planar graphs and treewidth

As we have mentioned in the introduction, planar graphs are fractionally tw-fragile. This is a well-known consequence of the fact that the treewidth of planar graphs is at most linear in their radius, which follows from ideas of Robertson and Seymour [13] and Baker [1]. The version we use, together with a short proof, can be found in a work by Eppstein [6, Lemma 4].

Theorem 10. *Every planar graph of radius at most d has treewidth at most $3d$.*

The fractional tw-fragility now follows by a standard layering argument [1, 7], which we restate in our notation.

Corollary 11. *The class of planar graphs is fractionally tw-fragile at rate $r(a) = 3a - 3$.*

Proof. Let G be a planar graph, without loss of generality connected, and let a be a positive integer, at least 2 since the statement is trivial for $a = 1$. Let v be an arbitrary vertex of G and for every non-negative integer i, let L_i be the set of vertices of G at distance exactly i from v. For $i \in \{0, \ldots, a-1\}$, set $X_i := L_i \cup L_{i+a} \cup L_{i+2a} \cup \cdots$ and consider any component C of the graph $G - X_i$. There is some integer j such that C contains only vertices at distance between $i+ja+1$ and $i+ja+a-1$ from v. Let G' be the graph obtained from G by deleting all vertices at distance at least $i + ja + a$ from v and by contracting all vertices at distance at most $\max(i + ja, 0)$ from v to a single vertex x. Clearly, G' is a minor of G, and thus G' is planar. Moreover, every vertex of G' is at distance at most $a - 1$ from x and $C \subseteq G'$. Consequently, $\mathrm{tw}(C) \leqslant \mathrm{tw}(G') \leqslant 3a - 3$ by Theorem 10. Since this is the case for every component of $G - X_i$, we have $\mathrm{tw}(G - X_i) \leqslant 3a - 3$, and thus $X_i \in G_{\mathrm{tw}\downarrow 3a-3}$. Since the uniform distribution on $\{X_0, \ldots, X_{a-1}\}$ is $(1/a)$-thin, planar graphs are fractionally tw-fragile at rate $r(a) = 3a - 3$. $\square$

Pilipczuk and Siebertz [12] demonstrated another relationship between planar graphs and graphs of bounded treewidth. Given a partition $\mathcal{P}$ of vertices of a graph G, let $G/\mathcal{P}$ be the graph obtained from G by contracting each part of $\mathcal{P}$ to a single vertex and suppressing the arising loops and parallel edges. A path P in a graph G is *geodesic* if for every $x, y \in V(P)$, the distance between x and y in G is the same as their distance in P. Pilipczuk and Siebertz [12] proved that every planar graph G admits a partition $\mathcal{P}$ of its vertices such that $G/\mathcal{P}$ has treewidth at most 8 and each part of $\mathcal{P}$ induces a geodesic path in G.

We need a variation which follows from a result proved by Dujmović *et al.*[4]. In a rooted tree T, a *vertical path* is an initial segment of a path from a vertex of T to the root; the *lower endpoint* of a vertical path is its vertex farthest from the root. Suppose T is a spanning tree of a graph G. Then a *tripod* is the union of the vertex sets of up to three pairwise disjoint vertical paths in T whose lower endpoints induce a clique in G.

Theorem 12 (Dujmović *et al.* [4, Theorem 16])**.** *For every plane triangulation G and every rooted spanning tree T of G, there exists a partition $\mathcal{P}$ of G into tripods such that $G/\mathcal{P}$ has treewidth at most 3.*

We can in fact say a bit more; note that the minor $G/\mathcal{P}$ of G is again a plane triangulation (after suppressing faces of length 1 and 2, but keeping non-facial loops and parallel edges), and thus by the following lemma, it is actually chordal.

Lemma 13. *Let H be a multigraph triangulating the plane. If H has treewidth at most 3, then the underlying simple graph of H is chordal.*

Proof. Suppose for a contradiction that C is an induced cycle in H of length at least four. By contracting all but four edges of C if necessary, we can assume C to have length exactly four. For each 2-cycle K in H, delete the vertices contained in the face of K disjoint from C and suppress the resulting 2-face, so that we can assume H does not have parallel edges. Since H is a triangulation, it does not have loops, either, and thus H is a simple graph. Consequently, $H - C$ has exactly two components (one drawn in each face of C), and contracting each of these components to a single vertex, we deduce from the fact that H triangulates the plane that H contains the octahedron graph as a minor. However, this graph has treewidth four, which is a contradiction. □

If T is a rooted breadth-first search spanning tree in some graph G, then all vertical paths in T are geodesic in G. Hence, applying Theorem 12 to such a spanning tree and using Lemma 13, we obtain the following result. We say that a partition $\mathcal{P}$ of the vertices of G is *trigeodesic* if every part of $\mathcal{P}$ induces in G a connected subgraph whose vertex set is covered by at most three geodesic paths of G.

Theorem 14. *For every plane triangulation G, there exists a trigeodesic partition $\mathcal{P}$ of $V(G)$ such that $G/\mathcal{P}$ is chordal and has treewidth at most* 3.

3 Maximum component size

Recall that $\star(G)$ is the maximum of the orders of the components of the graph G. The parameter $\chi_{\star,b}$ has been intensively studied under the name *clustered chromatic number* [16], and is among the most natural relaxations of the chromatic number. Clustered coloring specializes to the usual notion of vertex coloring, in the sense that $\chi_{\star,1}(G) = \chi(G)$.

In the special case of planar graphs, clustered chromatic number is in general no better than ordinary chromatic number: for every integer b, there exists a planar graph G_b such that $\chi_{\star,b}(G_b) = 4$. These graphs G_b necessarily have unbounded maximum degree: Esperet and Joret [8] proved that for every Δ, there exists b such that every planar graph G of maximum degree at most Δ satisfies $\chi_{\star,b}(G) \leqslant 3$. Moreover, the Hex lemma implies that this bound cannot be improved. The situation is different in the fractional setting due to Lemma 2, since planar graphs of bounded maximum degree are fractionally $\star$-fragile (the assumption of bounded maximum degree is necessary by Lemma 5). In fact, Dvořák [5] proved fractional $\star$-fragility in much greater generality, for all classes of bounded maximum degree with strongly sublinear separators.

Theorem 15 (Dvořák [5])**.** *Let β be a real number in $(0, 1]$. For every function $s(n) = O(n^{1-\beta})$ and every integer Δ, there exists a function r such that every graph with balanced s-separators and maximum degree at most Δ is fractionally $\star$-fragile at rate r.*

Let us remark that the argument used to prove Theorem 15 gives a very bad bound on the rate r, especially compared to the polynomial $\star$-breakability bound from Theorem 4. As shown by Lipton and Tarjan [9], planar graphs have balanced s-separators for $s(n) = 3\sqrt{n}$, and thus they are $\star$-breakable at rate $O(a^2)$. Considering Observation 3, it is natural to ask whether (subject to a bound on the maximum degree) planar graphs are also fractionally $\star$-fragile at quadratic rate $O(a^2)$. As our first result, we show that this is not the case, even for much more restricted graph classes.

Theorem 16. *Let $\Delta \geqslant 3$ be an integer and let $\mathcal{G}$ be a class of graphs that contains all trees of maximum degree at most Δ. If $\mathcal{G}$ is fractionally $\star$-fragile at rate r, then $r(a) \geqslant (\Delta-1)^{a-3}$ for every integer $a \geqslant 4$.*

Proof. Fix an integer $a \geqslant 4$. Let T be the complete rooted $(\Delta-1)$-ary tree of depth d (the root has depth 0 and the leaves have depth d, every non-leaf vertex has exactly $\Delta-1$ children), where $d \geqslant 3a-1$. We aim to use Lemma 6. For every vertex $v \in V(T)$ at depth k, let $w(v) := (\Delta-1)^{-k}$, so $w(V(T)) = d+1$. We prove that, for every set $X \subseteq V(T)$ with $w(X) \leqslant (d+1)/a$, the forest $T-X$ contains a component with at least $(\Delta-1)^{a/(1+a/(d+1))-3}$ vertices.

Consider any set $X \subseteq V(T)$ such that $w(X) \leqslant (d+1)/a$. Let X' consist of X and the root of T; we have $w(X') \leqslant 1 + (d+1)/a$. For a vertex $v \in X'$, let C_v be the set of all descendants of v in T (including v itself) that can be reached without passing through another vertex of X'. Then $\{C_v : v \in X'\}$ is a partition of $V(T)$. For $v \in X'$, set $\rho(v) := w(C_v)/w(v)$. We have

$$\begin{aligned}\frac{\sum_{v\in X'} w(v)\rho(v)}{w(X')} = \frac{\sum_{v\in X'} w(C_v)}{w(X')} &= \frac{w(V(T))}{w(X')}\\ &\geqslant \frac{d+1}{(d+1)/a+1} = \frac{1}{1+a/(d+1)}\cdot a.\end{aligned}$$

Let $a' := \frac{1}{1+a/(d+1)}\cdot a$, and note that $a' \geqslant \frac{3}{4}a \geqslant 3$ because $d \geqslant 3a-1$ and $a \geqslant 4$. Since the left side of the above inequality is a weighted average of the values $\rho(v)$ for $v \in X'$, there exists $v \in X'$ such that $\rho(v) \geqslant a'$, and thus $w(C_v) \geqslant a' w(v)$.

For each non-negative integer i, let n_i be the number of vertices in C_v whose depth is by i larger than the depth of v, so that $w(C_v) = w(v)\sum_{i\geqslant 0}(\Delta-1)^{-i}\cdot n_i$, and thus $a' \leqslant \sum_{i\geqslant 0}(\Delta-1)^{-i}\cdot n_i$. Subject to this inequality and to the constraints $n_i \leqslant (\Delta-1)^i$ for every i, the value $|C_v| = \sum_{i\geqslant 0} n_i$ is minimized when $n_i = (\Delta-1)^i$ for $i \in \{0,\ldots,m-1\}$ and $n_i = 0$ for $i \geqslant m+1$ where $m = \lfloor a' \rfloor \geqslant 3$ (as can be seen by a standard weight-shifting argument). It follows that

$$|C_v| \geqslant \sum_{i=0}^{m-1}(\Delta-1)^i = \frac{(\Delta-1)^m-1}{\Delta-2} \geqslant \frac{(\Delta-1)^{a'-1}-1}{\Delta-2} \geqslant (\Delta-1)^{a'-2}+1.$$

Consequently, $T[C_v]-v$ has a component with at least $(\Delta-1)^{a'-3}$ vertices (since v has $\Delta-1$ children in T), giving the same lower bound on $\star(T-X)$. By Lemma 6, we conclude that $r(a) \geqslant (\Delta-1)^{a'-3}$. Because this inequality holds for all $d \geqslant 3a-1$ and $\lim_{d\to\infty} a' = a$, the statement of the lemma follows. □

Conversely, many interesting graph classes, including planar graphs, nearly match the lower bound provided by Theorem 16. We start by an argument for graphs with bounded treewidth. We use the following well-known fact [14, (2.6)].

Observation 17. *Let k be an integer. If G is a graph of treewidth less than k and Z a subset of vertices of G, then G has a separation (D, B) of order at most k such that $|Z \setminus D| \leqslant \frac{2}{3}|Z|$ and $|Z \setminus B| \leqslant \frac{2}{3}|Z|$.*

Iterating this splitting procedure, we obtain the following generalization.

Lemma 18. *Let k, s and p be positive integers such that $s \geqslant 12k$. If G is a graph of treewidth less than k and W a subset of vertices of G of order at most ps, then there exists a set $C \subseteq V(G)$ and non-empty sets $A_1, \dots, A_t \subseteq V(G)$ for some $t < 6p$ such that*

(i) $|C| < 6pk$;

(ii) $|A_i \cap (C \cup W)| \leqslant s$ *for each* $i \in \{1, \dots, t\}$;

(iii) $G = G[A_1] \cup \cdots \cup G[A_t]$; *and*

(iv) $A_i \cap A_j \subseteq C$ *if* $1 \leqslant i < j \leqslant t$.

Proof. We inductively define $\mathcal{A}_i$ and C_i for $i \in \{0, \dots, t-1\}$. Let $\mathcal{A}_0 \coloneqq \{V(G)\}$ and $C_0 \coloneqq \varnothing$. For $i \geqslant 0$, if there exists $X_i \in \mathcal{A}_i$ such that $|X_i \cap (C_i \cup W)| > s$, we apply Observation 17 to $G[X_i]$ with the subset $Z_i \coloneqq X_i \cap (C_i \cup W)$ of vertices, obtaining a separation (D_i, B_i) of $G[X_i]$ of order at most k; and we let $\mathcal{A}_{i+1} \coloneqq (\mathcal{A}_i \setminus \{X_i\}) \cup \{D_i, B_i\}$ and $C_{i+1} \coloneqq C_i \cup (D_i \cap B_i)$. If no such element X exists, the procedure stops and we set $t \coloneqq i + 1$, $C \coloneqq C_i$ and $\{A_1, \dots, A_t\} \coloneqq \mathcal{A}_i$. Assuming the construction stops, it is clear the conditions (ii), (iii) and (iv) hold. Since $|C_{i+1} \setminus C_i| \leqslant k$ for $i \in \{0, \dots, t-2\}$, it suffices to argue that the construction stops with $t < 6p$. Without loss of generality, we can assume that the construction does not stop in the first step, *i.e.*, that $|W| > s$.

If $0 \leqslant i \leqslant t-1$ and $X \subseteq V(G)$, we let $\partial_i X \coloneqq |X \cap (C_i \cup W)|$. Suppose that $i \leqslant t-2$. Note that if $X \in \mathcal{A}_i$ and $X \neq X_i$, then $X \cap C_{i+1} = X \cap C_i$, since $C_{i+1} \setminus C_i \subseteq X_i \setminus C_i$ is disjoint from X; hence, $\partial_{i+1} X = \partial_i X$. By the choice of the separation (D_i, B_i), we have

$$\partial_{i+1} D_i \geqslant |D_i \cap Z_i| = |Z_i| - |Z_i \setminus D_i| \geqslant |Z_i|/3 > s/3,$$

and symmetrically $\partial_{i+1} B_i > s/3$. We conclude that if $0 \leqslant i \leqslant t-1$, then

$$\sum_{X \in \mathcal{A}_i} \partial_i X > |\mathcal{A}_i| s/3 = (i+1)s/3.$$

On the other hand,

$$\partial_{i+1} D_i + \partial_{i+1} B_i \leqslant \partial_i X_i + 2|D_i \cap B_i| = \partial_i X_i + 2k,$$

and thus

$$\sum_{X \in \mathcal{A}_{i+1}} \partial_{i+1} X \leqslant 2k + \sum_{X \in \mathcal{A}_i} \partial_i X.$$

By induction, we conclude that for $i \in \{0, \ldots, t-1\}$, we have

$$\sum_{X \in \mathcal{A}_i} \partial_i X \leqslant |W| + 2ik \leqslant ps + 2ik \leqslant (p + i/6)s.$$

Combining the inequalities, we obtain $(i+1)/3 < (p + i/6)$, and hence $i < 6p - 2$. Consequently, the construction stops with $t < 6p$. □

Corollary 19. *Let k, s and p be positive integers such that $s \geqslant 12k$. If G is a graph of treewidth less than k and W a subset of vertices of G of order at most ps, then there exists a set $C \subseteq V(G) \setminus W$ and a partition $E_1, \ldots, E_t$ of $V(G) \setminus (C \cup W)$ for some $t < 6p$ such that*

(i) $|C| < 6pk$*;*

(ii) for each $i \in \{1, \ldots, t\}$, at most s vertices in $C \cup W$ have a neighbor in E_i;

(iii) for each non-isolated vertex v in C, either v has a neighbor in $C \cup W$, or in at least two of the sets $E_1, \ldots, E_t$;

(iv) $G = G[C \cup W \cup E_1] \cup \cdots \cup G[C \cup W \cup E_t]$.

Proof. Apply Lemma 18 and replace C by $C \setminus W$ if necessary, so that $C \cap W = \varnothing$. Let $E_i := A_i \setminus (C \cup W)$ for $i \in \{1, \ldots, t\}$, and remove from the list $E_1, \ldots, E_t$ the empty sets. Finally, if a vertex $v \in C$ has a neighbor in E_i and no neighbor in $\cup_{j \neq i} E_j$ for some $i \in \{1, \ldots, t\}$, then we can replace C by $C \setminus \{v\}$ and E_i by $E_i \cup \{v\}$. □

A *tree partition* (T, β) of a graph G consists of a tree T and a function β that to each vertex of T assigns a subset of vertices of G, such that

- the sets $\beta(v)$ for $v \in V(T)$ are pairwise disjoint and form a partition of $V(G)$; and
- if distinct vertices x and y of T are not adjacent, then G does not contain any edge with one end in $\beta(x)$ and the other in $\beta(y)$.

Equivalently, the graph obtained from G by contracting each set $\beta(x)$ for $x \in V(T)$ to a single vertex (and removing loops and multiple edges) is a subgraph of T. In a *rooted tree partition*, the tree T is additionally rooted. The *depth* of a vertex v of T is the length of the path in T from v to the root; in particular, the depth of the root is 0. The depth of T is the maximum of the depths of its vertices. For a subtree $S \subseteq T$, let $\beta(S) := \bigcup_{v \in V(S)} \beta(v)$. The subtree S is naturally rooted in the vertex of S nearest to the root of T. For every integer a, the *depth-a order* of the rooted tree partition is the maximum of $|\beta(S)|$ over all subtrees S of T of depth at most $a - 2$. We use the following simple observation.

Lemma 20. *Let $r \colon \mathbb{N} \to \mathbb{R}_0^+$ be a non-decreasing function. If for every positive integer a, the graph G admits a rooted tree partition (T_a, β) of depth-a order at most $r(a)$, then G is fractionally $\star$-fragile at rate r.*

Proof. Let a be a positive integer. For each $i \in \{0, \ldots, a-1\}$, let L_i be the set of vertices of T_a whose depth belongs to $\{i + ja : j \in \{0, 1, 2, \ldots\}\}$ and let $X_i := \bigcup_{v \in L_i} \beta(v)$. By the definition of a tree partition, the vertex set of each component of $G - X_i$ is contained in $\beta(S)$ for a component S of $T_a - L_i$. Each component of $T_a - L_i$ is a tree of depth at most $a-2$, and hence $|\beta(S)| \leqslant r(a)$. Consequently, $\star(G - X_i) \leqslant r(a)$, and thus $X_i \in G_{\star\downarrow r(a)}$. Note that $X_i \cap X_j = \varnothing$ if $i \neq j$. Considering the uniform distribution on $\{X_0, \ldots, X_{a-1}\}$, which is $(1/a)$-thin, we conclude that G is fractionally $\star$-fragile at rate r. □

For example, if $\Delta \geqslant 3$ and T is a tree of maximum degree at most Δ, then we can root T and define $\beta(v) := \{v\}$ for every $v \in V(T)$, thereby obtaining a rooted tree partition of T of depth-a order $O((\Delta-1)^{a-2})$. It thus follows from Lemma 20 that trees of maximum degree at most Δ are fractionally $\star$-fragile at rate $O((\Delta-1)^{a-2})$, essentially matching the bound from Theorem 16.

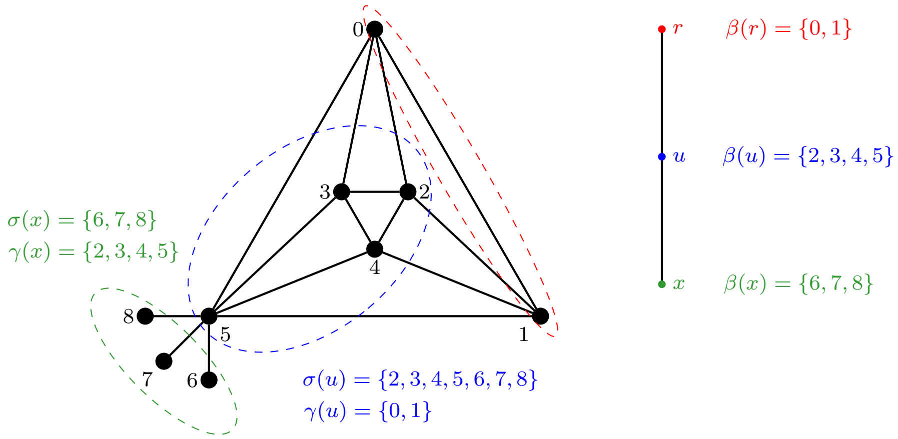

Figure 1: A tree decomposition (right) of the graph with 9 vertices on the left obtained by following the construction in the proof of Lemma 21, illustrating the case where $|W| \leqslant (\Delta-1)^{b-1}s$ (for some integer $b \geqslant 2$).

We now construct good tree partitions for graphs of bounded treewidth and maximum degree.

Lemma 21. *Let a, b, k and Δ be positive integers with $\Delta \geqslant 3$ and $a \geqslant b$. If G is a connected graph of treewidth less than k and maximum degree at most Δ, then G admits a rooted tree partition (T, β) of depth-a order at most $12k(\Delta-1)^{a+b-1}\left(1 + 6^{a/b}\right)$.*

Proof. Let $s := 12k$. We construct the tree partition starting from the root and adding children as described below. It might be helpful to look at Figures 1 and 2 while reading the construction. To every vertex v of T will be associated, in addition to $\beta(v)$, three sets, namely $\sigma(v)$, $\gamma(v)$ and $\kappa(v)$. When considering a vertex v with parent z in T, two of these will already have been defined in one of the previous steps: the set $\sigma(v) \subseteq V(G) \setminus \beta(z)$, which at the end of the construction will be equal to $\beta(S)$ for the subtree S of T consisting of v and all its descendants, and the set $\gamma(v) \subseteq \beta(z)$, which is of size at most $(\Delta-1)^{b-1}s$ and such that in G, all neighbors of vertices from $\sigma(v)$ are contained in $\sigma(v) \cup \gamma(v)$, and each vertex in $\gamma(v)$ has at most $\Delta - 1$ neighbors in $\sigma(v)$. The set $\kappa(v)$, which must be contained in $\beta(v)$, is defined when v is considered; its role becomes clear later.

Clearly, we can assume that G has at least three vertices. For the root r of T, we start the construction by letting $\beta(r)$ consist of two adjacent vertices of G and $\kappa(r) \coloneqq \varnothing$, adding a child u of r to T, and setting $\sigma(u) \coloneqq V(G) \setminus \beta(r)$ and $\gamma(u) \coloneqq \beta(r)$. Suppose now that the construction reaches a vertex v of T with parent z. Let W be the set of vertices in $\sigma(v)$ that have a neighbor (in G) in $\gamma(v)$.

If $|W| \leqslant (\Delta-1)^{b-1}s$, we let $\beta(v) \coloneqq W$ and $\kappa(v) \coloneqq \varnothing$; when $\sigma(v) = W$, then v is a leaf of T, otherwise, we add a child x to v and set $\sigma(x) \coloneqq \sigma(v) \setminus W$ and $\gamma(x) \coloneqq W$ (see Figure 1). Notice that if $y \in \sigma(x)$, then all neighbors of y in G are contained in $\sigma(v) \cup \gamma(v)$, since $\sigma(x) \subseteq \sigma(v)$. Moreover, because $y \notin W$ we know that y has no neighbor in $\gamma(v)$, and hence all neighbors of y are contained in $\sigma(v) = \sigma(x) \cup \gamma(x)$. Let us also point out that a vertex $w \in \gamma(x)$ has less than Δ neighbors in $\sigma(x)$, because w has a neighbor in $\gamma(v)$, which is disjoint from $\sigma(v)$.

Let us consider the case that $|W| > (\Delta-1)^{b-1}s$; in this case, we say that v is a *branching vertex*. Since $|\gamma(v)| \leqslant (\Delta-1)^{b-1}s$ and each vertex in $\gamma(v)$ has at most $\Delta - 1$ neighbors in $\sigma(v)$, we have $|W| \leqslant (\Delta-1)^b s$. Let $C, E_1, \dots, E_t \subseteq \sigma(v)$ be the sets obtained by applying Corollary 19 to $G[\sigma(v)]$ and W, with p being $(\Delta-1)^b$. We set $\beta(v) \coloneqq W \cup C$, $\kappa(v) \coloneqq C$, we add t children $u_1, \dots, u_t$ to v, and set $\sigma(u_i) \coloneqq E_i$ and let $\gamma(u_i)$ consist of all vertices in $W \cup C$ with a neighbor in E_i for $i \in \{1, \dots, t\}$ (see Figure 2). Let us point out that $|\gamma(u_i)| \leqslant s \leqslant (\Delta-1)^{b-1}s$ by property (ii) from Corollary 19, and that vertices in $\sigma(u_i)$ only have neighbors in $\sigma(u_i) \cup \gamma(u_i)$ by property (iv) from Corollary 19. Let us also remark that each vertex in $\gamma(u_i)$ has at most $\Delta - 1$ neighbors in $\sigma(u_i)$, since G has maximum degree at most Δ, each vertex in W has a neighbor in $\gamma(v)$, and due to the property (iii) from Corollary 19 for vertices in $C \cap \gamma(u_i)$.

Note that since $|\gamma(v)| \leqslant s$ when v is the child of a branching vertex, and $|\beta(v)| \leqslant (\Delta-1)|\gamma(v)|$ when v is not a branching vertex, if x and y are two distinct branching vertices and x is an ancestor of y, then the depth of x is by at least b larger than the depth of y. Note also that every branching vertex x has less than $6(\Delta-1)^b$ children and satisfies $|\kappa(x)| < 6(\Delta-1)^b k$.

The described construction clearly results in a rooted tree partition of G. Let us now consider any subtree S of T of depth at most $a-2$, with root w. The *level* of a branching vertex x of S is the number of branching vertices on the path from x to w, excluding x itself; hence, each branching vertex has level at most $\lfloor (a-2)/b \rfloor \leqslant \lfloor a/b \rfloor$. The number of branching vertices of S of level i is at most $\left(6(\Delta-1)^b\right)^i$. If w is the root of T, then let $B \coloneqq \beta(w)$, otherwise let B be the set of vertices in $\sigma(w)$ with a neighbor in $\gamma(w)$; in either case, we have $|B| \leqslant (\Delta-1)^b s$. Note that each vertex in $\beta(S)$ is either at distance at most $a-2$ from B, or at distance at most $a-2-b\cdot i$ from a vertex in $\kappa(x)$ for some branching vertex $x \in V(S)$ of level i. Therefore, we have

$$\begin{aligned}|\beta(S)| &\leqslant (\Delta-1)^{a-1}\left((\Delta-1)^b s + \sum_{i=0}^{\lfloor a/b \rfloor}\left(6(\Delta-1)^b\right)^{i+1}(\Delta-1)^{-bi}k\right)\\ &= 12k(\Delta-1)^{a-1}\left((\Delta-1)^b + \frac{(\Delta-1)^b}{2}\sum_{i=0}^{\lfloor a/b \rfloor}6^i\right)\\ &\leqslant 12k(\Delta-1)^{a+b-1}\left(1+6^{a/b}\right),\end{aligned}$$

as required. $\square$

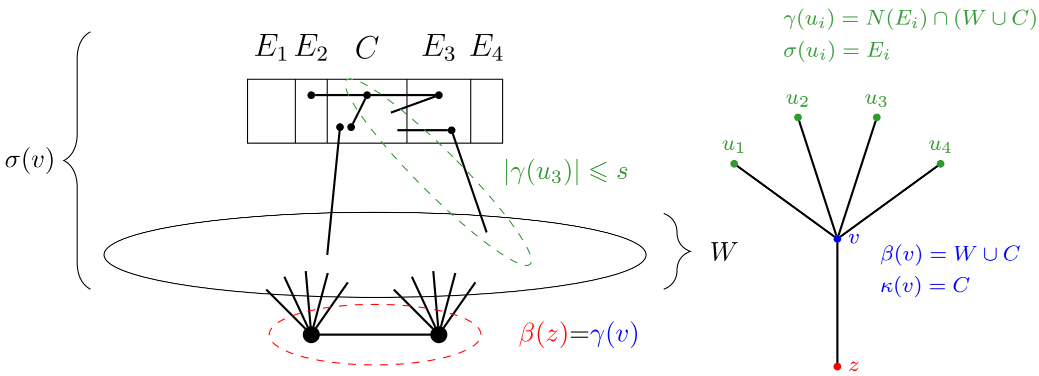

Figure 2: A schematic illustration of the construction of the tree decomposition in the proof of Lemma 21, in the case where $|W| > (\Delta-1)^{b-1}s$ and v is thus a branching vertex.

We now combine Lemmas 20 and 21, choosing $b = \Theta(\sqrt{a})$ in the latter.

Corollary 22. *Let k and Δ be positive integers with $\Delta \geqslant 3$. The class of graphs of treewidth less than k and maximum degree at most Δ is fractionally $\star$-fragile at rate $r(a) = k(\Delta-1)^{a+O(\sqrt{a})}$.*

The result can be extended to planar graphs using their fractional tw-fragility; that is, by combining Corollaries 11 and 22.

Theorem 23. *For every integer $\Delta \geqslant 3$, the class of planar graphs with maximum degree at most Δ is fractionally $\star$-fragile at rate $r(a) = (\Delta-1)^{a+O(\sqrt{a})}$.*

Proof. Let G be a planar graph of maximum degree at most Δ. Consider an integer $a > 256$, let $a' := \lceil 2^{\sqrt{a}} \rceil$, let $a'' := a+1$ and note that $1/a' + 1/a'' < 1/a$. Choose $\mathsf{X} \in G_{\mathrm{tw}\downarrow 3a'-3}$ at random from the $(1/a')$-thin probability distribution given by Corollary 11. Then $G - \mathsf{X}$ is a planar graph of treewidth less than $3a'-2$ and maximum degree at most Δ. Choose $\mathsf{Y} \in (G-\mathsf{X})_{\star\downarrow(3a'-2)(\Delta-1)^{a''+O(\sqrt{a''})}}$ from the $(1/a'')$-thin probability distribution given by Corollary 22, and let $\mathsf{Z} := \mathsf{X} \cup \mathsf{Y}$. Then $\star(G-\mathsf{Z}) = \star((G-\mathsf{X})-\mathsf{Y}) \leqslant$

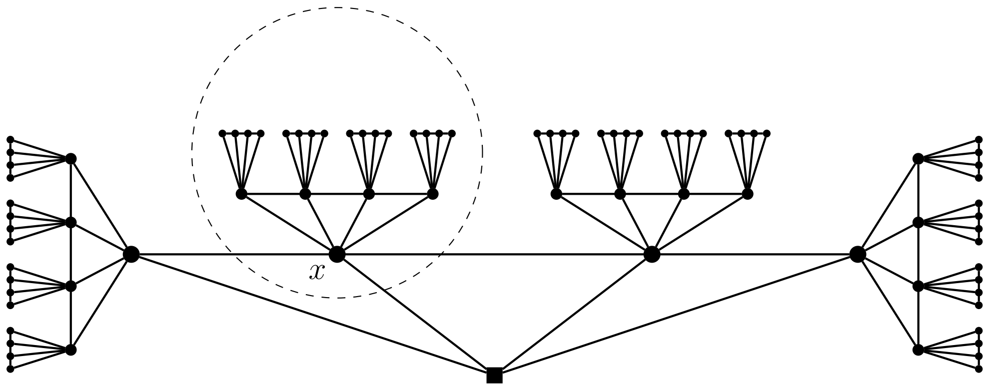

Figure 3: The graph $T_3(P_4)$: the handle is the vertex represented by a square, and the circled part is the jug of the vertex x.

$(3a'-2)(\Delta-1)^{a''+O(\sqrt{a''})} = (\Delta-1)^{a+O(\sqrt{a})}$. Consequently, choosing the random set Z in this way gives a probability distribution on $G_{\star\downarrow(\Delta-1)^{a+O(\sqrt{a})}}$, and $\mathbf{Pr}[v \in \mathsf{Z}] \leqslant \mathbf{Pr}[v \in \mathsf{X}] + \mathbf{Pr}[v \in \mathsf{Y}] \leqslant 1/a' + 1/a'' < 1/a$ for every $v \in V(G)$. We conclude that G is fractionally $\star$-fragile at rate $(\Delta-1)^{a+O(\sqrt{a})}$. $\square$

4 Treedepth

By Lemma 5, we cannot hope to extend the results on fractional $\star$-fragility to any class with unbounded maximum degree. The natural parameter to consider in graphs with unbounded maximum degree is the *treedepth* [11]: firstly, stars have treedepth at most 2, and secondly, a connected graph of maximum degree at most Δ and treedepth at most d has at most Δ^d vertices, thus giving us about as good a relationship to $\star$ as one may hope for in the case where the maximum degree is bounded from above. The *treedepth* $\mathrm{td}(G)$ of a graph G is the minimum integer d for which there exists a rooted tree T of depth at most $d-1$ with vertex set $V(G)$ such that every edge of G joins a vertex to one of its ancestors or descendants in T.

Given Corollary 22 and the relationship between $\star$ and td outlined above, one could perhaps hope that graphs of bounded treewidth are fractionally td-fragile at a linear rate. However, this is not the case. For the simplicity of presentation, we only give the counterargument for the case of graphs of treewidth two, but it can be naturally generalized to show that the class of all graphs of treewidth at most t cannot be fractionally td-fragile at rate better than $\Omega(a^t)$.

For a graph H and a non-negative integer d, let $T_d(H)$ be the graph inductively defined as follows: $T_0(H)$ is the graph consisting of a single vertex v, which we call the *handle* of $T_0(H)$. For $d \geqslant 1$, let $T_d(H)$ be the graph obtained from H by adding, for each $x \in V(H)$, a copy of $T_{d-1}(H)$ and identifying its handle with x, and finally adding a vertex v adjacent to all vertices of H; the vertex v is the *handle* of $T_d(H)$. Figure 3 gives a representation of $T_3(H)$ when H is the 4-vertex path P_4. Note that for each vertex x of $T_d(H)$, there is a unique index $i \in \{0, \dots, d\}$ such that x is the handle of a copy of $T_i(H)$; let us call this copy the *jug* of x. Given a non-identically-zero function $w\colon V(H) \to \mathbb{R}_0^+$ assigning weights to vertices of H, let $w_d\colon V(T_d(H)) \to \mathbb{R}_0^+$ be defined as follows. For the handle v, we set $w_d(v) \coloneqq 1$, and when $d \geqslant 1$, for each vertex x in the copy of H we set $w_d(x) \coloneqq w(x)/w(V(H))$, and in the copy of $T_{d-1}(H)$ attached at x, we set the weights according to $w_d(x) \cdot w_{d-1}$. It may help to follow the sequel to notice that $w_d(V(T_d(H))) = d + 1$ for every non-negative integer d and every graph H. Further, if the jug J of x in $T_d(H)$ is a copy of $T_i(H)$ with $i \geqslant 1$, then $\sum_{v \in N_J} w_d(v) = w_d(x)$ where N_J is the set of neighbors of x in $T_d(H)$ that belong to J.

Let B_d be the complete binary tree of depth d (let us remark that $B_d = T_d(2K_1)$, where $2K_1$ is the graph with two vertices and no edge), and let $t_d\colon V(B_d) \to \mathbb{R}_0^+$ be the weight function assigning to each vertex of depth i the weight 2^{-i} (so $t_d = w_d$ for the weight function w assigning to both vertices of $2K_1$ the same weight). Let us start with an observation on complete binary subtrees in heavy subsets of B_d. For a graph H with a handle h, we say that a graph G with a vertex s *contains H as a minor rooted in s* if there exists an assignment μ of pairwise disjoint non-empty sets of vertices of G to the vertices of H, such that

- $s \in \mu(h)$;
- for each vertex $v \in V(H)$, the graph $G[\mu(v)]$ is connected;
- for each edge $uv \in E(H)$, there exists an edge of G with one end in $\mu(u)$ and the other end in $\mu(v)$.

The sets $\mu(v)$ are called the *bags* of the minor.

Lemma 24. *Let d and p be non-negative integers and let S be a subtree of B_d with root s such that $t_d(V(S)) \geqslant (2p + 1)t_d(s)$. Then S contains B_p as a minor rooted in s.*

Proof. We prove the statement by induction on the non-negative integer p. The case $p = 0$ being trivial, suppose that $p \geqslant 1$. For $x \in V(S)$, let S_x be the subtree of S induced by x and all its descendants. We can assume that $t_d(V(S_x)) < (2p+1)t_d(x)$ for every $x \in V(S) \setminus \{s\}$, as otherwise we can consider S_x instead of S and combine the obtained minor with the path from x to s in S. In particular, for a child x_1 of s in S we have $t_d(S_{x_1}) < (2p+1)t_d(x_1) = (p+1/2)t_d(s)$, and thus $t_d(V(S) \setminus (V(S_{x_1}) \cup \{s\})) > (p-1/2)t_d(s)$. Consequently, s has another child x_2 in S and $t_d(S_{x_2}) = t_d(V(S) \setminus (V(S_{x_1}) \cup \{s\})) > (p - 1/2)t_d(s) = (2p - 1)t_d(x_2)$. Symmetrically, $t_d(S_{x_1}) > (2p - 1)t_d(x_1)$. By the induction hypothesis, each of S_{x_1} and S_{x_2} contains B_{p-1} as a minor rooted in x_1 and x_2, respectively, which combine with s to form B_p as a minor rooted in s. $\square$

Next, let us lift this result to $T_d(B_d)$. For a non-negative integer p, let us define $q(p) \coloneqq 10\sqrt{p + 1}$. Let us remark that the function q is chosen so that $q(p)-2 \geqslant q(p-\lfloor (q(p)-4)/8 \rfloor)$ holds for $p \geqslant 1$.

Lemma 25. *Let d and p be non-negative integers such that $d \geqslant q(p)-1$, let $G := T_d(B_d)$ and $w := (t_d)_d$, let s be a vertex of G and let S be a connected induced subgraph of G contained in the jug of s and containing s. If $w(V(S)) \geqslant q(p)w(s)$, then S contains B_p as a minor rooted in s.*

Proof. We prove the statement by induction on the non-negative integer p. The case $p = 0$ being trivial, suppose that $p \geqslant 1$. For $x \in V(S)$, let S_x be the intersection of S with the jug of x. We can assume that $w(V(S_x)) < q(p)w(x)$, as otherwise we can consider S_x instead of S, find the required minor in S_x, and combine it with a path from s to x.

Let T be the subgraph of G induced by the neighbors of s in the jug of s (note that T is a copy of B_d), and let N be the set of neighbors of s in G that belong to S. Notice that $N \subseteq V(T)$ since S is contained in the jug of s by assumptions. We have

$$q(p)w(s) \leqslant w(V(S)) = w(s) + \sum_{x \in N} w(V(S_x)) < w(s) + q(p)w(N),$$

and thus

$$w(N) > \Big(1 - \frac{1}{q(p)}\Big)w(s) = \Big(1 - \frac{1}{q(p)}\Big)w(V(T)). \tag{1}$$

Let B consist of the vertices x in N such that $w(V(S_x)) < (q(p)-2)w(x)$. Then

$$\begin{aligned} q(p)w(s) &\leqslant w(s) + \sum_{x \in N} w(V(S_x)) \\ &= w(s) + \sum_{x \in N \setminus B} w(V(S_x)) + \sum_{x \in B} w(V(S_x)) \\ &< w(s) + q(p)w(N) - 2w(B) \\ &\leqslant w(s) + q(p)w(s) - 2w(B), \end{aligned}$$

and hence

$$w(B) < \frac{w(s)}{2} = \frac{w(V(T))}{2}. \tag{2}$$

Let r be the root of T and set $X := (V(T) \setminus N) \cup \{r\}$. By (1) and the assumption $d \geqslant q(p) - 1$,

$$w(X) < \frac{w(V(T))}{q(p)} + w(r) = \Big(\frac{1}{q(p)} + \frac{1}{d+1}\Big)w(V(T)) \leqslant \frac{2}{q(p)}w(V(T)). \tag{3}$$

For $x \in X$, let T_x be the subtree of the forest $T[N \cup \{x\}]$ induced by x and its descendants, and set $a(x) := (w(V(T_x)) - w(V(T_x) \cap B))/w(x)$. By (2) and (3), we have

$$\frac{\sum_{x \in X} w(x)a(x)}{w(X)} = \frac{w(V(T)) - w(B)}{w(X)} > \frac{q(p)}{4}. \tag{4}$$

Since the left side of (4) is a weighted average of the values $a(x)$ for $x \in X$, there exists $x \in X$ such that $a(x) > q(p)/4$. Let T'_x be the smallest subtree of T_x containing x and all vertices in $V(T_x) \setminus B$. Note that $w(V(T'_x)) \geqslant a(x)w(x) > \frac{q(p)}{4}w(x)$ and no leaf of T'_x belongs to B. Set $p' \coloneqq \lfloor (q(p)-4)/8 \rfloor$. Lemma 24 ensures that T'_x contains $T_{p'}$ as a minor μ, which can be extended so that for every leaf u, the bag $\mu(u)$ contains a leaf y of T'_x; this leaf in particular does not belong to B. Hence, $w(V(S_y)) \geqslant (q(p)-2)w(y)$, which is at least $q(p-p')w(y)$ by the definition of q, since $p \geqslant 1$. Consequently, the induction hypothesis implies that S_y contains $T_{p-p'}$ as a minor rooted in y. Adding these minors of $T_{p-p'}$ for each leaf of $T_{p'}$, and replacing x by s in the root bag, we obtain T_p in S as a minor rooted in s, as required. $\square$

We now use Lemma 6 to give the desired lower bound.

Theorem 26. *Let $r\colon \mathbb{N} \to \mathbb{R}_0^+$ be a non-decreasing function. If all planar graphs of treewidth at most two are fractionally td-fragile at rate r, then $r(a) = \Omega(a^2)$.*

Proof. Consider two integers a and d such that $d \geqslant a \geqslant 20$. Let $G \coloneqq T_d(B_d)$ and $w \coloneqq (t_d)_d$. Note that G is planar and has treewidth at most two. Suppose that X is a subset of $V(G)$ such that $w(X) \leqslant w(V(G))/a$. Let r be the handle of G and let $X' = X \cup \{r\}$; we have $w(X') \leqslant w(X) + 1 = w(X) + \frac{w(V(G))}{d+1} \leqslant \frac{2}{a} w(V(G))$. For $x \in X'$, let J_x be the jug of x and let S_x be the component of $J_x - (X' \setminus \{x\})$ containing x. We have

$$\frac{a}{2} w(X') \leqslant w(V(G)) = \sum_{x \in X'} w(V(S_x)),$$

and thus there exists $x \in X'$ such that $w(V(S_x)) \geqslant \frac{a}{2} w(x)$. Set $p \coloneqq \lfloor a^2/400 - 1 \rfloor$, so p is a non-negative integer. Because $10\sqrt{p+1} \leqslant \frac{a}{2}$, we deduce from Lemma 25 that S_x contains B_p as a minor.

Note that B_p has treedepth $p+1$, that deleting a vertex decreases the treedepth by at most one, and that treedepth is minor-monotone [11]. Since $S_x - x \subseteq G - X$, we have

$$\mathrm{td}(G - X) \geqslant \mathrm{td}(S_x - x) \geqslant \mathrm{td}(S_x) - 1 \geqslant \mathrm{td}(B_p) - 1 \geqslant p.$$

Since this holds for every set X with $w(X) \leqslant w(V(G))/a$, Lemma 6 implies that $r(a) \geqslant p = \Omega(a^2)$. $\square$

Outerplanar graphs are planar and have treewidth two; however, the graphs $T_d(B_d)$ are not outerplanar if $d \geqslant 2$. As we will see below, outerplanar graphs are actually fractionally td-fragile at a subquadratic rate. Nevertheless, even for outerplanar graphs the rate is not linear, as we now show. Let us start by showing that $T_d(P_n)$ has substantial treedepth, where P_n is the n-vertex path.

Lemma 27. *Let $d \geqslant 0$, $a \geqslant 1$ and $n \geqslant 2^a$ be integers. The graph $T_d(P_n)$ has treedepth at least $ad + 1$.*

Proof. We prove the statement by induction on the non-negative integer d. The case $d = 0$ is trivial, and we thus assume that $d \geqslant 1$. Set $G \coloneqq T_d(P_n)$, let v be the handle of G, and let Q be the n-vertex path induced by the neighbors of v. For a subpath Q' of Q, we define $J_{Q'}$ to be the union of the vertex sets of every jug the handle of which is contained in Q'. Suppose that R is a rooted tree witnessing the treedepth of $T_d(P_n)$. By finite induction we build a path $u_0 \dots u_a$ in R starting at the root u_0 of R and a decreasing sequence $Q_0 \supset Q_1 \supset \cdots \supset Q_a$ of subpaths of Q such that the following invariants hold for each $i \in \{0, \dots, a\}$.

(i) $|V(Q_i)| = 2^{a-i}$;

(ii) $u_0, \dots, u_{i-1} \notin J_{Q_i}$; and

(iii) the subtree of R rooted at u_i contains all vertices of J_{Q_i}.

We proceed by finite induction on $i \in \{0, \dots, a\}$. The path Q_0 is chosen arbitrarily among the subpaths of Q with 2^a vertices and u_0 is the root of R. For $i \in \{1, \dots, a\}$, the path Q_i is selected as one of the halves of Q_{i-1} so that J_{Q_i} does not contain u_{i-1}. It follows that Q_i satisfies (i) and (ii). By (iii), we know that the subtree of R rooted at u_{i-1} contains all vertices of $J_{Q_{i-1}}$, and thus also all vertices of J_{Q_i}. Since $G[J_{Q_i}]$ is connected and $u_{i-1} \notin J_{Q_i}$, we can choose u_i as the unique child of u_{i-1} in R such that the subtree of R rooted at u_i contains all vertices of J_{Q_i}, so that (iii) is satisfied. This concludes the construction.

Now let $x \in V(Q_a)$, let J_x be the jug of x and let R_x be the subtree of R rooted at u_a. We know by (iii) that $V(J_x) \subseteq V(R_x)$. Because J_x is isomorphic to $T_{d-1}(P_n)$, the induction hypothesis implies that R_x has depth at least $a(d-1)$. Since u_a has depth a in R, it follows that R has depth at least ad. Consequently, the treedepth of $T_d(P_n)$ is at least $ad+1$. □

We now give an argument analogous to that of Lemma 25. For the d-vertex path P_d, let $p_d\colon V(P_d) \to \mathbb{R}_0^+$ be the mapping that assigns 1 to each vertex of P_d.

Lemma 28. *Let d, p and b be non-negative integers such that $d \geqslant 4b+2p+2$, let $G \coloneqq T_d(P_d)$ and $w \coloneqq (p_d)_d$. Let s be a vertex of G and let S be a connected induced subgraph of G contained in the jug of s and containing s. If $w(V(S)) \geqslant (4b+2p+2)w(s)$, then S contains $T_p(P_b)$ as a minor rooted in s.*

Proof. We prove the statement by induction on the non-negative integer p. The case $p = 0$ being trivial, we suppose that $p \geqslant 1$. For $x \in V(S)$, let S_x be the intersection of S with the jug of x. We can assume that $w(V(S_x)) < (4b+2p+2)w(x)$, as otherwise we can consider S_x instead of S, find the required minor in S_x, and combine it with a path from s to x.

Let P be the subgraph of G induced by the neighbors of s in the jug of s. Note that P is a copy $v_1 \dots v_d$ of P_d, and let N be the set of neighbors of s in G that are contained in S. Notice that $N \subseteq V(P)$ by hypothesis. We have

$$\begin{aligned}(4b+2p+2)w(s) \leqslant w(V(S)) &= w(s) + \sum_{x \in N} w(V(S_x)) \\ &< w(s) + (4b+2p+2)w(N),\end{aligned}$$

and thus

$$w(N) > \left(1 - \frac{1}{4b+2p+2}\right) w(s) = \left(1 - \frac{1}{4b+2p+2}\right) w(V(P)). \tag{5}$$

Let B consist of the vertices x in N such that $w(V(S_x)) < (4b+2p)w(x)$. Since

$$\begin{aligned}(4b+2p+2)w(s) &\leqslant w(V(S)) = w(s) + \sum_{x\in N} w(V(S_x)) \\ &< w(s) + (4b+2p+2)w(N) - 2w(B) \\ &\leqslant w(s) + (4b+2p+2)w(s) - 2w(B),\end{aligned}$$

we have

$$w(B) < \frac{w(s)}{2} = \frac{w(V(P))}{2}. \tag{6}$$

Set $X \coloneqq (V(P) \setminus N) \cup \{v_1\}$. By (5), we have

$$\begin{aligned}w(X) &< \frac{w(V(P))}{4b+2p+2} + w(v_1) \\ &= \left(\frac{1}{4b+2p+2} + \frac{1}{d}\right) w(V(P)) \\ &\leqslant \frac{1}{2b+p+1} w(V(P)). \end{aligned} \tag{7}$$

Given $v_i, v_j \in V(P)$, the vertex v_j is *to the right* of v_i if $j > i$. For $x \in X$, let P_x be the subpath of $P[N \cup \{x\}]$ induced by x and the vertices to the right of x. Observe that $(V(P_x))_{x\in X}$ is a partition of $V(P)$. Consequently, setting $a(x) \coloneqq (w(V(P_x)) - w(V(P_x) \cap B))/w(x)$, we deduce from (6) and (7) that

$$\frac{\sum_{x\in X} w(x)a(x)}{w(X)} = \frac{w(V(P)) - w(B)}{w(X)} > \frac{2b+p+1}{2}. \tag{8}$$

Since the left side of (8) is a weighted average of the values $a(x)$ for $x \in X$, there exists $x \in X$ such that $a(x) > (2b+p+1)/2$. Since all vertices of P have the same weight, we deduce that $P_x - x$ contains at least $(2b+p+1)/2 - 1 \geqslant b$ vertices not belonging to B. For each such vertex y, we have $w(V(S_y)) \geqslant (4b+2p)w(y)$, and by the induction hypothesis, S_y contains $T_{p-1}(P_b)$ as a minor rooted in y. Since $P_x - x$ is a subpath of $P[N]$ and s is adjacent to every vertex in N, these minors along with s combine to form $T_p(P_b)$ as a minor rooted in s and contained in S, as required. □

Now are ready to give a lower bound for outerplanar graphs.

Theorem 29. *Let $r\colon \mathbb{N} \to \mathbb{R}_0^+$ be a non-decreasing function. If all outerplanar graphs are fractionally td-fragile at rate r, then $r(a) = \Omega(a \log a)$.*

Proof. Consider two integer a and d such that $d \geqslant a \geqslant 28$. Set $G \coloneqq T_d(P_d)$ and $w \coloneqq (p_d)_d$. Note that G is outerplanar. Suppose that X is a subset of $V(G)$ such that $w(X) \leqslant w(V(G))/a$. Let r be the handle of G and let $X' \coloneqq X \cup \{r\}$; we have $w(X') \leqslant w(X)+1 = w(X) + \frac{w(V(G))}{d+1} \leqslant \frac{2}{a}w(V(G))$. For $x \in X'$, let J_x be the jug of x and let S_x be the component of $J_x - (X' \setminus \{x\})$ containing x. We have

$$\frac{a}{2}w(X') \leqslant w(V(G)) = \sum_{x \in X'} w(V(S_x)),$$

and thus there exists $x \in X'$ such that $w(V(S_x)) \geqslant \frac{a}{2}w(X')$. Set $c \coloneqq \lfloor (a-4)/12 \rfloor$. We deduce from Lemma 28 that S_x contains $T_c(P_c)$ as a minor. Since $S_x - x \subseteq G - X$, Lemma 27 implies that

$$\mathrm{td}(G-X) \geqslant \mathrm{td}(S_x) - 1 \geqslant \mathrm{td}(T_c(P_c)) - 1 \geqslant c\lfloor \log_2 c \rfloor.$$

As this holds for every set X with $w(X) \leqslant w(V(G))/a$, Lemma 6 implies that $r(a) \geqslant c\lfloor \log_2 c \rfloor = \Omega(a \log a)$. ☐

Next, we will give a general upper bound for graphs with bounded treewidth. To this end, we need the following property of treedepth.

Lemma 30. *Let $H, H_1, \ldots, H_t$ be induced subgraphs of a graph G such that*

- *$G = H \cup H_1 \cup \ldots \cup H_t$;*
- *$H_i \cap H_j \subseteq H$ whenever $1 \leqslant i < j \leqslant t$; and*
- *$H_i \cap H$ is a clique for $i \in \{1, \ldots, t\}$.*

Then

$$td(G) \leqslant td(H) + \max\{td(H_i - V(H)) \,:\, 1 \leqslant i \leqslant t\}.$$

Proof. Let $T, T_1, \ldots, T_t$ be rooted trees respectively witnessing the treedepths of $H, H_1 - V(H), \ldots, H_t - V(H)$. For $i \in \{1, \ldots, t\}$, since $H \cap H_i$ is a clique, all its vertices are contained in a root-leaf path of T; let ℓ_i be the leaf of such a path. Taking $T \cup T_1 \cup \cdots \cup T_t$ and, for $i \in \{1, \ldots, t\}$, adding an edge from the root of T_i to ℓ_i, we obtain a tree witnessing that the treedepth of G is at most $\mathrm{td}(H) + \max\{\mathrm{td}(H_i - V(H)) \,:\, 1 \leqslant i \leqslant t\}$. ☐

We are now ready to give the following upper bound on the rate of td-fragility for graphs with bounded treewidth.

Theorem 31. *For every non-negative integer t, the class of graphs with treewidth at most t is fractionally td-fragile at rate $r(a) = 2^{t(t+1)/2+1}a^t$.*

Proof. We proceed by induction on the non-negative integer t. Graphs of treewidth 0 have no edges, and thus they have treedepth 1. Hence, suppose that $t \geqslant 1$. Let a be a positive integer and let G be a graph of treewidth at most t, which we can assume to be connected and chordal by Observation 7 without loss of generality. Let us fix an elimination ordering of G, and let v be the first vertex in this ordering. For a non-negative integer d, let L_d be the set of vertices of G at distance exactly d from v. For $i \in \{0, \dots, 2a-1\}$, let $X_i \coloneqq \bigcup_{s \geqslant 0} L_{i+s\cdot 2a}$, and choose X randomly from the uniform distribution on $\{X_0, \dots, X_{2a-1}\}$.

Consider a non-negative index j, and note that $G[L_j]$ has treewidth at most $t-1$: for $j = 0$ it is obvious, while for $j \geqslant 1$ it follows from the fact that each vertex of L_j has a neighbor in L_{j-1} preceding it in the elimination ordering by Observation 8, and hence in the restriction of the elimination ordering to $G[L_j]$, each vertex is preceded by at most $t-1$ of its neighbors. The induction hypothesis thus implies that for each j, we can choose a set $\mathsf{Y}_j \in G[L_j]_{\mathrm{td}\downarrow 2^{(t-1)t/2+1}(2a)^{t-1}}$ at random such that $\mathbf{Pr}[v \in \mathsf{Y}_j] \leqslant \frac{1}{2a}$ for each $v \in L_j$. Set $\mathsf{Z} \coloneqq \mathsf{X} \cup \mathsf{Y}_0 \cup \mathsf{Y}_1 \cup \cdots$. If $v \in V(G)$, then there exists a unique index j such that $v \in L_j$, and hence

$$\mathbf{Pr}[v \in \mathsf{Z}] \leqslant \mathbf{Pr}[v \in \mathsf{X}] + \mathbf{Pr}[v \in \mathsf{Y}_j] \leqslant 1/a.$$

Consequently, it suffices to show that $\mathsf{Z} \in G_{\mathrm{td}\downarrow 2^{t(t+1)/2+1}a^t}$. As $G[L_j \setminus \mathsf{Y}_j]$ has treedepth at most $2^{(t-1)t/2+1}(2a)^{t-1} = 2^{t(t+1)/2}a^{t-1}$ for each j, the conclusion follows by repeatedly applying Lemmas 9 and 30. Indeed, let $i \in \{0, \dots, 2a-1\}$ be the index such that $\mathsf{X} = X_i$. It suffices to bound the treedepth of the subgraph of G induced by $(L_{i+1} \setminus \mathsf{Y}_{i+1}) \cup \cdots \cup (L_{i+2a-1} \setminus \mathsf{Y}_{i+2a-1})$, and of that induced by $\cup_{j=0}^{i-1} L_j \setminus \mathsf{Y}_j$ in the border case — which is omitted as similar to what follows only with different index boundaries yielding fewer applications of the lemmas. To this end, for any $j \in \{1, \dots, 2a-2\}$ define H to be $G[\cup_{s=i+1}^{i+j} L_s \setminus \mathsf{Y}_s]$ and, for each component C of $G[L_{i+j+1} \setminus \mathsf{Y}_{i+j+1}]$, define H_C to be the subgraph of G induced by the union of $V(C)$ and the subset of vertices of H with a neighbor (in G) that belongs to $V(C)$. Lemma 9 ensures that $H \cap H_C$ is a clique, and Lemma 30 that the treedepth of $H \cup \bigcup_C H_C$ is at most $\mathrm{td}(H) + 2^{t(t+1)/2}a^{t-1}$. Therefore, the conclusion follows by finite induction on $j \in \{1, \dots, 2a-1\}$. □

Let us remark that Theorem 31 implies that every fractionally tw-fragile class of graphs is also fractionally td-fragile; for example, this includes all proper minor-closed classes [3]. More precisely, if a graph G is fractionally tw-fragile at rate t, then it is also fractionally td-fragile at rate $r(b) = 2^{t(2b)(t(2b)+1)/2+1}(2b)^{t(2b)}$, as seen by first sampling a set X from a $\frac{1}{2b}$-thin probability distribution on $G_{\mathrm{tw}\downarrow t(2b)}$ and then applying Theorem 31 with $a = 2b$ to $G - \mathsf{X}$.

As we mentioned before, the bound provided by Theorem 31 can be improved for the special case of outerplanar graphs. Firstly, we note that the following holds.

Observation 32. *Suppose that G is an outerplanar graph, that $K \subseteq V(G)$ induces a connected subgraph of G, and let H be a connected subgraph of $G - K$ such that each vertex of H has a neighbor in K. Then H is a path.*

Proof. If H is not a path, it either is a cycle or contains a vertex of degree at least three. Contracting K to a single vertex, and considering it along with H, we obtain either K_4 or $K_{2,3}$ as a minor of G, contradicting the assumption that G is outerplanar. □

We can now modify the argument used to demonstrate Theorem 31. We use the fact that a path with n vertices has treedepth $\lceil \log_2(n+1) \rceil$, see [11].

Theorem 33. *The class of outerplanar graphs is fractionally td-fragile at rate $r(a) = 2a(1 + \lceil \log_2 a \rceil)$.*

Proof. Let a be a positive integer. Let G be an outerplanar graph, which without loss of generality can be assumed to be connected. By triangulating the inner faces, we can assume that G is chordal. Let $L_0, L_1, \ldots$ and $X_0, \ldots, X_{2a-1}$, and X be defined in the same way as in the proof of Theorem 31. From Lemma 9 and Observation 32, we infer that $G[L_j]$ is a disjoint union of paths, for each non-negative integer j. Repeating the same layering argument in $G[L_j]$, for each j, we can choose a set $\mathsf{Y}_j \subseteq L_j$ at random so that $\mathbf{Pr}[v \in Y_j] \leqslant \frac{1}{2a}$ for every $v \in V(L_j)$ and $G[L_j \setminus \mathsf{Y}_j]$ is a disjoint union of paths with at most $2a-1$ vertices. Consequently, $\mathrm{td}(G[L_j \setminus \mathsf{Y}_j]) \leqslant \lceil \log_2(2a) \rceil$ for every j, and letting $\mathsf{Z} \coloneqq \mathsf{X} \cup \mathsf{Y}_0 \cup \mathsf{Y}_1 \cup \cdots$, we apply Lemmas 9 and 30 similarly as in the proof of Theorem 31 to infer that $\mathsf{Z} \in G_{\mathrm{td}\downarrow 2a(1+\lceil \log_2 a \rceil)}$, while $\mathbf{Pr}[v \in \mathsf{Z}] \leqslant 1/a$ for each $v \in V(G)$. □

Combining Theorem 31 with Corollary 11 yields that planar graphs are fractionally td-fragile at rate $a^{O(a)}$. A much better bound can be obtained using Theorem 14. To this end, let us introduce another variation on Theorem 31.

Theorem 34. *The class of planar chordal graphs is fractionally td-fragile at rate $r(a) = 8a^2(2 + \lceil \log_2 a \rceil)$.*

Proof. Let a be a positive integer. Let G be a planar chordal graph, which without loss of generality can be assumed to be connected. Let $L_0, L_1 \ldots$ and $X_0, \ldots, X_{2a-1}$, and X be defined in the same way as in the proof of Theorem 31. For every $j \geqslant 1$, Lemma 9 implies the neighborhood of every component of $G[L_j]$ in L_{j-1} induces a connected subgraph of G, and since G is planar, $G[L_j]$ is outerplanar. By Theorem 33, we can for each j choose a set $\mathsf{Y}_j \subseteq L_j$ at random so that $\mathbf{Pr}[v \in \mathsf{Y}_j] \leqslant \frac{1}{2a}$ for each $v \in V(L_j)$ and $G[L_j \setminus \mathsf{Y}_j]$ has treedepth at most $4a(2 + \lceil \log_2 a \rceil)$. Consequently, letting $\mathsf{Z} \coloneqq \mathsf{X} \cup \mathsf{Y}_0 \cup \mathsf{Y}_1 \cup \cdots$, Lemmas 9 and 30 imply that $\mathsf{Z} \in G_{\mathrm{td}\downarrow 8a^2(2+\lceil \log_2 a \rceil)}$ similarly as before, while $\mathbf{Pr}[v \in \mathsf{Z}] \leqslant 1/a$ for each $v \in V(G)$. □

Let us point out that the rate from Theorem 34 cannot be substantially improved: the graphs $T_d(T_d(P_d))$ are planar, chordal, and combining the ideas of Theorems 26 and 29, one can show that they cannot be fractionally td-fragile at rate better than $\Omega(a^2 \log a)$. We can now compose the results to obtain a bound for planar graphs.

Corollary 35. *The class of planar graphs is fractionally td-fragile at rate $r(a) = 384a^3(3 + \lceil \log_2 a \rceil)$.*

Proof. Let a be a positive integer. Let G be a planar graph, which without loss of generality can be assumed to be connected. Let v be a vertex of G, and for $d \geqslant 0$, let L_d be the set of vertices of G at distance exactly d from v in G. For $i \in \{0, \ldots, 2a-1\}$, let $X_i \coloneqq L_i \cup L_{i+2a} \cup \cdots$, and choose X from the uniform distribution on $\{X_0, \ldots, X_{2a-1}\}$.

Consider any component H of $G - \mathsf{X}$. This component contains only vertices at distance from v between $i + 2aj + 1$ and $i + 2aj + 2a - 1$ for some integer j. Let G' be the graph obtained from G by deleting all vertices at distance at least $i + 2aj + 2a$ from v and by contracting all vertices at distance at most $\max(i + 2aj, 0)$ from v to a single vertex x. Clearly, G' is a minor of G, and thus G' is planar. Moreover, every vertex of G' is at distance at most $2a - 1$ from x and $H \subseteq G'$. Let G'' be a triangulation of G', and let $\mathcal{P}$ be a trigeodesic partition of $V(G'')$ such that $G''/\mathcal{P}$ is chordal, which exists by Theorem 14. By Theorem 34, we can choose a set $\mathsf{Y}'_H \subseteq V(G''/\mathcal{P})$ at random such that $\operatorname{td}(G''/\mathcal{P} - \mathsf{Y}'_H) \leqslant 32a^2(3 + \log_2 a)$ and $\mathbf{Pr}[z \in \mathsf{Y}'_H] \leqslant \frac{1}{2a}$ for every $z \in V(G''/\mathcal{P})$. We can naturally view Y'_H as a subset of $\mathcal{P}$; with this in mind, let $\mathsf{Y}_H \coloneqq V(H) \cap \bigcup_{P \in \mathsf{Y}'_H} P$. Clearly, for every $u \in V(H)$, we have $\mathbf{Pr}[u \in \mathsf{Y}_H] \leqslant \frac{1}{2a}$. Furthermore, note that since G'' has radius less than $2a$, every geodesic path in G'' has less than $4a$ vertices, and since $\mathcal{P}$ is trigeodesic, if follows that $|P| < 12a$ for every $P \in \mathcal{P}$. Consequently, we can turn the tree T witnessing the treedepth of $G''/\mathcal{P} - \mathsf{Y}'_H$ into one for $H - \mathsf{Y}_H$ by replacing each vertex $P \in V(G''/\mathcal{P} - \mathsf{Y}'_H)$ in T by a path consisting of the vertices contained in $P \cap V(H)$. Therefore $\operatorname{td}(H - \mathsf{Y}_H) < 12a \operatorname{td}(G''/\mathcal{P} - \mathsf{Y}'_H) \leqslant 384a^3(3 + \lceil \log_2 a \rceil)$.

Letting Z be the union of X and the sets Y_H for each component H of $G - \mathsf{X}$, we conclude that $\operatorname{td}(G - \mathsf{Z}) \leqslant 384a^3(3 + \lceil \log_2 a \rceil)$ and $\mathbf{Pr}[u \in \mathsf{Z}] \leqslant 1/a$ for each $u \in V(G)$. □

As we mentioned before, the planar graphs $T_d(T_d(P_d))$ cannot be fractionally td-fragile at rate better than $\Omega(a^2 \log a)$. We leave open the problem of determining the correct rate for planar graphs (between the bounds of $\Omega(a^2 \log a)$ and $O(a^3 \log a)$ we obtained).

5 Algorithmic remarks

In this paper, we mostly focused on the structural aspects of fractional fragility. Nevertheless, all our upper bound results can be straightforwardly turned into polynomial-time algorithms to sample from the corresponding probability distribution. Moreover, an inspection of the proofs shows that the distributions have supports of bounded size. More precisely:

- For a graph G of bounded treewidth and maximum degree, the support of the $(1/a)$-thin distribution on $G_{\star\downarrow r(a)}$ obtained in Corollary 22 has size at most a.
- For a planar graph G of bounded maximum degree, the support of the $(1/a)$-thin distribution on $G_{\star\downarrow r(a)}$ obtained in Theorem 23 has size at most $(a+1)\lceil 2^{\sqrt{a}} \rceil$.
- For a graph G of treewidth at most t, the support of the $(1/a)$-thin distribution on $G_{\operatorname{td}\downarrow r(a)}$ obtained in Theorem 31 has size at most $f_t(a)$, where $f_0(a) = 1$ and $f_t(a) = 2af_{t-1}(2a)$. Note that to obtain this bound, we use the fact that the choices of the sets Y_j do not have to be independent; consequently, we can use the same random choices for each j, obtaining only $f_{t-1}(2a)$ choices for the random set $\mathsf{Y}_0 \cup \mathsf{Y}_1 \cup \cdots$, rather than $\left(f_{t-1}(2a)\right)^{\text{the number of layers}}$ a naive analysis would suggest.

- For an outerplanar graph G, the support of the $(1/a)$-thin distribution on $G_{\mathrm{td}\downarrow r(a)}$ obtained in Theorem 33 has size at most $4a^2$.
- For a planar graph G, the support of the $(1/a)$-thin distribution on $G_{\mathrm{td}\downarrow r(a)}$ obtained in Corollary 35 has size at most $512a^4$.

As noted after Observation 1, this implies that in the algorithmic applications, we obtain deterministic rather than randomized algorithms.

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