On the size of $(K_t, \mathcal{T}_k)$-co-critical graphs

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Abstract

Given an integer $r \geqslant 1$ and graphs $G, H_1, \ldots, H_r$, we write $G \to (H_1, \ldots, H_r)$ if every r-coloring of the edges of G contains a monochromatic copy of H_i in color i for some $i \in \{1, \ldots, r\}$. A non-complete graph G is $(H_1, \ldots, H_r)$-*co-critical* if $G \nrightarrow (H_1, \ldots, H_r)$, but $G + e \to (H_1, \ldots, H_r)$ for every edge e in $\overline{G}$. In this paper, motivated by Hanson and Toft's conjecture [Edge-colored saturated graphs, J Graph Theory 11(1987), 191–196], we study the minimum number of edges over all $(K_t, \mathcal{T}_k)$-co-critical graphs on n vertices, where $\mathcal{T}_k$ denotes the family of all trees on k vertices and $G \to (K_t, \mathcal{T}_k)$ if for every 2-coloring $\tau : E(G) \to \{\text{red, blue}\}$, G has a red K_t or a blue tree in $\mathcal{T}_k$. Following Day [Saturated graphs of prescribed minimum degree, Combin. Probab. Comput. 26 (2017), 201–207], we apply graph bootstrap percolation on a not necessarily K_t-saturated graph to prove that for all $t \geqslant 4$ and $k \geqslant \max\{6, t\}$, there exists a constant $c(t, k)$ such that, for all $n \geqslant (t-1)(k-1)+1$, if G is a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, then

$$e(G) \geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(t, k).$$

Furthermore, this linear bound is asymptotically best possible when $t \in \{4, 5\}$ and $k \geqslant 6$. The method we develop in this paper may shed some light on attacking Hanson and Toft's conjecture.

Mathematics Subject Classifications: 05C55, 05C35

1 Introduction

All graphs considered in this paper are finite, and without loops or multiple edges. For a graph G, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number

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of vertices, $e(G)$ the number of edges, $N_G(x)$ the neighborhood of vertex x in G, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree, and $\overline{G}$ the complement of G. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete to* B if every vertex in A is adjacent to every vertex in B; and A is *anti-complete to* B if no vertex in A is adjacent to a vertex in B. The subgraph of G induced by A, denoted $G[A]$, is the graph with vertex set A and edge set $\{xy \in E(G) : x, y \in A\}$. We denote by $B \setminus A$ the set $B - A$, $e_G(A, B)$ the number of edges between A and B in G, $e_G(A)$ the number of edges of $G[A]$, and $G \setminus A$ the subgraph of G induced on $V(G) \setminus A$, respectively. If $A = \{a\}$, we simply write $B \setminus a$, $e_G(a, B)$, and $G \setminus a$, respectively. For any edge e in $\overline{G}$, we use $G + e$ to denote the graph obtained from G by adding the new edge e. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. For an integer $t \geqslant 1$ and a graph H, we define tH to be the union of t disjoint copies of H. We use K_n and T_n to denote the complete graph and a tree on n vertices, respectively. For any positive integer r, we write $[r]$ for the set $\{1, 2, \ldots, r\}$. We use the convention "$A :=$" to mean that A is defined to be the right-hand side of the relation.

Given an integer $r \geqslant 1$ and graphs G, $H_1, \ldots, H_r$, we write $G \to (H_1, \ldots, H_r)$ if every r-coloring of $E(G)$ contains a monochromatic H_i in color i for some $i \in [r]$. The classical *Ramsey number* $R(H_1, \ldots, H_r)$ is the minimum positive integer n such that $K_n \to (H_1, \ldots, H_r)$. Following Nešetřil [13], and Galluccio, Simonovits and Simonyi [10],

Definition 1. A non-complete graph G is $(H_1, \ldots, H_r)$*-co-critical* if $G \nrightarrow (H_1, \ldots, H_r)$, but $G + e \to (H_1, \ldots, H_r)$ for every edge e in $\overline{G}$.

It is simple to check that K_6^- is (K_3, K_3)-co-critical, where K_6^- denotes the graph obtained from K_6 by deleting exactly one edge. It is worth noting that every $(H_1, \ldots, H_r)$-co-critical graph has at least $R(H_1, \ldots, H_r)$ vertices.

Remark. Following Galluccio, Simonovits and Simonyi [10], we exclude the complete graphs in the definition of $(H_1, \ldots, H_r)$-co-critical graphs, else every complete graph on fewer than $R(H_1, \ldots, H_r)$ vertices is $(H_1, \ldots, H_r)$-co-critical.

The notation of co-critical graphs was initiated by Nešetřil [13] in 1986 when he asked the following question regarding (K_3, K_3)-co-critical graphs:

> Are there infinitely many *minimal* co-critical graphs, i.e., co-critical graphs which lose this property when any vertex is deleted? Is K_6^- the only one?

This was answered in the positive by Galluccio, Simonovits and Simonyi [10]. They constructed infinitely many minimal (K_3, K_3)-co-critical graphs without K_5 as a subgraph. Szabó [15] then constructed infinitely many nearly regular (K_3, K_3)-co-critical graphs with low maximum degree. It remains unknown whether there are infinitely many *strongly* minimal co-critical graphs, where an $(H_1, \ldots, H_r)$-co-critical graph is *strongly minimal co-critical* if it contains no proper subgraph which is also $(H_1, \ldots, H_r)$-co-critical. This is one of the most intriguing open problems proposed by Galluccio, Simonovits and Simonyi in [10]. One interesting observation made in [10] is that if G is $(H_1, \ldots, H_r)$-co-critical, then $\chi(G) \geqslant R(H_1, \ldots, H_r) - 1$. They also made some observations on the minimum degree of (K_3, K_3)-co-critical graphs and maximum number of possible edges of $(H_1, \ldots, H_r)$-co-critical graphs.

We want to point out here that Hanson and Toft [12] in 1987 also studied the minimum and maximum number of edges over all $(H_1, \ldots, H_r)$-co-critical graphs on n vertices when $H_1, \ldots, H_r$ are complete graphs, under the name of *strongly* $(|H_1|, \ldots, |H_r|)$*-saturated* graphs. Recently, this topic has been studied under the name of $\mathcal{R}_{\min}(H_1, \ldots, H_r)$-*saturated* graphs [5, 9, 14].

Definition 2. A graph G is $(H_1, \ldots, H_r)$*-Ramsey-minimal* if $G \to (H_1, \ldots, H_r)$, but for any proper subgraph G' of G, $G' \nrightarrow (H_1, \ldots, H_r)$.

We define $\mathcal{R}_{\min}(H_1, \ldots, H_r)$ to be the family of all $(H_1, \ldots, H_r)$-Ramsey-minimal graphs. A graph G is $\mathcal{R}_{\min}(H_1, \ldots, H_r)$*-saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_r)$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_r)$ is a subgraph of $G + e$. It can be easily checked that a non-complete graph is $(H_1, \ldots, H_r)$-co-critical if and only if it is $\mathcal{R}_{\min}(H_1, \ldots, H_r)$-saturated. From now on, we shall use the notion of $(H_1, \ldots, H_r)$-co-critical other than $\mathcal{R}_{\min}(H_1, \ldots, H_r)$-saturated, as the former is much simpler and straightforward.

Let $R = R(K_{t_1}, \ldots, K_{t_r})$ be the classical Ramsey number for $K_{t_1}, \ldots, K_{t_r}$. Hanson and Toft [12] proved that every $(K_{t_1}, \ldots, K_{t_r})$-co-critical on n vertices has at most $e(T_{R-1,n})$ edges and this bound is best possible, where $T_{R-1,n}$ is the Turán graph on n vertices without K_R. They also observed that for all $n \geqslant R$, the graph $K_{R-2}+\overline{K}_{n-R+2}$ is $(K_{t_1}, \ldots, K_{t_r})$-co-critical. They further made the following conjecture that no $(K_{t_1}, \ldots, K_{t_r})$-co-critical graph on n vertices can have fewer than $e(K_{R-2} + \overline{K}_{n-R+2})$ edges.

Conjecture 3 (Hanson and Toft [12])**.** Let G be a $(K_{t_1}, \ldots, K_{t_r})$-co-critical graph on n vertices. Then

$$e(G) \geqslant (R-2)(n-R+2) + \binom{R-2}{2}.$$

This bound is best possible for every n.

Conjecture 3 remains wide open, except that the first nontrivial case, (K_3, K_3)-co-critical graphs, has been settled in [5] for $n \geqslant 56$. Structural properties of (K_3, K_4)-co-critical graphs are given in [2]. Motivated by Conjecture 3, we study the following problem. Let $\mathcal{T}_k$ denote the family of all trees on k vertices. For all $t, k \geqslant 3$, we write $G \to (K_t, \mathcal{T}_k)$ if for every 2-coloring $\tau : E(G) \to \{\text{red, blue}\}$, G has either a red K_t or a blue tree $T_k \in \mathcal{T}_k$. A non-complete graph G is $(K_t, \mathcal{T}_k)$*-co-critical* if $G \nrightarrow (K_t, \mathcal{T}_k)$, but $G + e \to (K_t, \mathcal{T}_k)$ for all e in $\overline{G}$. The main purpose of this paper is to study the structural properties of $(K_t, \mathcal{T}_k)$-co-critical graphs on n vertices in order to obtain the minimum size among all such graphs. By a classic result of Chvátal [4], $R(K_t, \mathcal{T}_k) = (t-1)(k-1)+1$. Hence, every $(K_t, \mathcal{T}_k)$-co-critical graph has at least $R(K_t, \mathcal{T}_k) = (t-1)(k-1)+1$ vertices. Following the observation made in both [10] and [12], every $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices has at most $e(T_{R(K_t, \mathcal{T}_k)-1,n})$ edges. We focus on studying the minimum number of possible edges over all $(K_t, \mathcal{T}_k)$-co-critical graphs on n vertices. Very recently, Rolek and the first author [14] proved the following.

Theorem 4 (Rolek and Song [14]). *Let $n, k \in \mathbb{N}$.*

(i) Every $(K_3, \mathcal{T}_4)$-co-critical graph on $n \geqslant 18$ vertices has at least $\lfloor 5n/2 \rfloor$ edges. This bound is sharp for every $n \geqslant 18$.

(ii) For all $k \geqslant 5$, if G is $(K_3, \mathcal{T}_k)$-co-critical on $n \geqslant 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$ vertices, then

$$e(G) \geqslant \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c(k),$$

where $c(k) = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right) k - 2$. This bound is asymptotically best possible.

To state our results, we need to introduce more notation. Given a family $\mathcal{F}$, a graph is $\mathcal{F}$*-free* if it does not contain any graph $F \in \mathcal{F}$ as a subgraph. We simply say a graph is F*-free* when $\mathcal{F} = \{F\}$. Erdős, Hajnal and Moon [7] in 1964 initiated the study of the minimum number of edges over all K_t-saturated graphs on n vertices (see the dynamic survey [8] on the extensive studies on K_t-saturated graphs). Theorem 5 below is a result of Day [6] on K_t-saturated graphs with prescribed minimum degree. It confirms a conjecture of Bollobás [1] when $t = 3$. It is worth noting that Day applied r-neighbor bootstrap percolation on a K_t-saturated graph to prove Theorem 5, where graph bootstrap percolation was introduced in [3]. Theorem 6 is a result of Hajnal [11] on K_t-saturated graphs.

Theorem 5 (Day [6]). *Let $q \in \mathbb{N}$. There exists a constant $c = c(q)$ such that, for all $3 \leqslant t \in \mathbb{N}$ and all $n \in \mathbb{N}$, if G is a K_t-saturated graph on n vertices with $\delta(G) \geqslant q$, then $e(G) \geqslant qn - c$.*

Theorem 6 (Hajnal [11]). *Let $t, n \in \mathbb{N}$. Let G be a K_t-saturated graph on n vertices. Then either $\Delta(G) = n - 1$ or $\delta(G) \geqslant 2(t-2)$.*

For a $(K_t, \mathcal{T}_k)$-co-critical graph G, let $\tau : E(G) \to \{\text{red, blue}\}$ be a 2-coloring of $E(G)$ and let E_r and E_b be the color classes of the coloring τ. We use G_r and G_b to denote the spanning subgraphs of G with edge sets E_r and E_b, respectively. We define τ to be a *critical coloring* of G if G has neither a red K_t nor a blue $T_k \in \mathcal{T}_k$ under τ, that is, if G_r is K_t-free and G_b is $\mathcal{T}_k$-free. For every $v \in V(G)$, we use $d_r(v)$ and $N_r(v)$ to denote the degree and neighborhood of v in G_r, respectively. Similarly, we define $d_b(v)$ and $N_b(v)$ to be the degree and neighborhood of v in G_b, respectively. One can see that if G is $(K_t, \mathcal{T}_k)$-co-critical, then G admits at least one critical coloring but $G + e$ admits no critical coloring for every edge e in $\overline{G}$.

In this paper, we first establish a number of important structural properties of $(K_t, \mathcal{T}_k)$-co-critical graphs in the hope that the method we develop here may shed some light on attacking Conjecture 3. Theorem 7(h) below is crucial in the proof of Theorem 8. Following Day [6], we apply q-neighbor bootstrap percolation on a not necessarily K_t-saturated graph, to prove Theorem 7(h), but with more involved rules.

Theorem 7. *For all $t, k \in \mathbb{N}$ with $t \geqslant 3$ and $k \geqslant 3$, let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices. Among all critical colorings of G, let $\tau : E(G) \to \{\text{red, blue}\}$ be a critical coloring of G with $|E_r|$ maximum. Let $D_1, \ldots, D_p$ be all components of G_b. Let $H := G \setminus (\bigcup_{i\in[p]} E(G[V(D_i)]))$. Then the following hold.*

(a) $\Delta(G_r) \leqslant n - 2$ *and* $\delta(G_r) \geqslant 2(t-2)$.

(b) *For all $i, j \in [p]$ with $i \neq j$, if there exist $u \in V(D_i)$ and $v \in V(D_j)$ such that $uv \notin E(H)$, then $H[N_H(u) \cap N_H(v)]$ contains K_{t-2} as a subgraph.*

(c) *For every $uv \in E(H)$, if v is contained in all K_{t-2} subgraphs of $H[N_H(u)]$ and $\{v\} = V(D_j)$ for some $j \in [p]$, then $|D_i| = k - 1$ for all D_i with $u \notin D_i$ and $D_i \setminus N_H(u) \neq \varnothing$, where $i \in [p]$.*

(d) *If $\delta(H) \leqslant 2t - 5$ and $k \geqslant t$, then for any vertex $u \in V(H)$ with $d_H(u) = \delta(H)$, no edge of $H[N_H(u)]$ is contained in all K_{t-2} subgraphs of $H[N_H(u)]$.*

(e) $k \geqslant 2t - 1 - \delta(H)$ *and* $\delta(H) \geqslant t - 1$.

(f) $\displaystyle\sum_{i=1}^{p} e_G(V(D_i)) > \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n - (t-1)(\lceil k/2\rceil - 1)).$

(g) *H is connected.*

(h) *For every $q \in \mathbb{N}$ with $q \geqslant t - 1$, there exists a constant $c(q, k)$ such that, if $\delta(H) \geqslant q$, then $e(H) \geqslant qn - c(q, k)$.*

We prove Theorem 7 in Section 2. We then apply Theorem 7 to study the size of $(K_t, \mathcal{T}_k)$-co-critical graphs. We prove Theorem 8 in Section 3.

Theorem 8. *Let $t, k \in \mathbb{N}$ with $t \geqslant 4$ and $k \geqslant \max\{6, t\}$. There exists a constant $\ell(t,k)$ such that, for all $n \in \mathbb{N}$ with $n \geqslant (t-1)(k-1)+1$, if G is a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, then*

$$e(G) \geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - \ell(t,k).$$

Finally we prove that the linear bound given in Theorem 8 is asymptotically best possible when $t \in \{4, 5\}$ and $k \geqslant 6$. Proof of Theorem 9 is given in Section 4.

Theorem 9. *For each $t \in \{4, 5\}$, all $k \geqslant 3$ and $n \geqslant (2t-3)(k-1) + \lceil k/2\rceil\lceil k/2\rceil - 1$, there exists a $(K_t, \mathcal{T}_k)$-co-critical graph G on n vertices such that*

$$e(G) \leqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n + C(t,k)$$

where $C(t,k) = \frac{1}{2}(t^2+t-5)k^2 - (2t^2+2t-11)k - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left((2t-3)(k-1) - \left\lceil\frac{k}{2}\right\rceil\right)$ when $k \geqslant 4$ and $C(t,3) = -2t^2 + 5t - 2$.

With the support of Theorem 4 and Theorem 9, we believe that the linear bound given in Theorem 8 is asymptotically best possible for all $t \geqslant 3$ and $k \geqslant 3$.

2 Structural properties of $(K_t, \mathcal{T}_k)$-co-critical graphs

We first prove the following lemma.

Lemma 10. *For all $t, k \in \mathbb{N}$ with $t \geqslant 3$ and $k \geqslant 3$, let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices. Let $\tau : E(G) \to \{red, blue\}$ be a critical coloring of G. Then the following hold.*

(a) For every component D of G_b, $|D| \leqslant k-1$ and $G[V(D)] = K_{|D|}$.

(b) If $D_1, \cdots, D_q$ are the components of G_b with $|D_i| < k/2$ for all $i \in [q]$, then $V(D_1)$, $\cdots$, $V(D_q)$ are complete to each other in G_r, and so $q \leqslant t-1$.

Proof. To prove (a), let D be a component of G_b. Since G_b is $\mathcal{T}_k$-free, we see that $|D| \leqslant k-1$. Suppose next that $G[V(D)] \neq K_{|D|}$. Let $u, v \in V(D)$ be such that $uv \notin E(G)$. We obtain a critical coloring of $G + uv$ from τ by coloring the edge uv blue, a contradiction.

To prove (b), suppose there exist vertices $u \in V(D_i)$ and $v \in V(D_j)$ such that $uv \notin E_r$, where $i, j \in [q]$ with $i \neq j$. Then $uv \notin E(G)$ and so we obtain a critical coloring of $G+uv$ from τ by coloring the edge uv blue, a contradiction. Thus $V(D_1), \cdots, V(D_q)$ are complete to each other in G_r. Since G_r is K_t-free, we have $q \leqslant t-1$. □

We are now ready to prove Theorem 7.

Proof of Theorem 7: Let G, τ, $D_1, \dots, D_p$ and H be given as in the statement. Then $n \geqslant (t-1)(k-1)+1$. By Lemma 10(a), $|D_i| \leqslant k-1$ for all $i \in [p]$. Hence, G_b has at least t components because $|G_b| = n \geqslant (t-1)(k-1)+1$. We first prove Theorem 7(a). Since τ was chosen so that $|E_r|$ is maximum, G_r is K_t-free but $G_r + e$ contains a copy of K_t for every $e \in E(\overline{G_r})$. Hence G_r is K_t-saturated. Suppose there exists a vertex $x \in V(G)$ such that $d_r(x) = n-1$. Note that $G_r \setminus x$ is K_{t-1}-free because G_r is K_t-free. Since $G \neq K_n$, there must exist $u, w \in N_r(x)$ such that $uw \notin E(G)$. By Lemma 10(a), u, w belong to different components of G_b. But then we obtain a critical coloring of $G + uw$ from τ by first coloring the edge uw red, and then recoloring xu blue and all edges incident with u in G_b red, contrary to the fact that $G + uw$ has no critical coloring. This proves that $\Delta(G_r) \leqslant n-2$. Since G_r is K_t-saturated, by Theorem 6, $\delta(G_r) \geqslant 2(t-2)$.

To prove Theorem 7(b), let $u \in V(D_i)$ and $v \in V(D_j)$ be such that $uv \notin E(H)$, where $i \neq j$. Suppose $H[N_H(u) \cap N_H(v)]$ is K_{t-2}-free. Since $|D_\ell| \leqslant k-1$ for all $\ell \in [p]$, we obtain a critical coloring of $G+uv$ from τ by first coloring the edge uv red, and then recoloring all red edges in $G[V(D_\ell)]$ blue for all $\ell \in [p]$, a contradiction. Therefore, $H[N_H(u) \cap N_H(v)]$ contains K_{t-2} as a subgraph. This proves Theorem 7(b).

To prove Theorem 7(c), let $uv \in E(H)$ be such that v is contained in all K_{t-2} subgraphs of $H[N_H(u)]$ and $\{v\} = V(D_j)$ for some $j \in [p]$. We may assume that $u \in V(D_p)$ and $\{v\} = V(D_{p-1})$. Note that $H[N_H(u)] \setminus v$ is K_{t-2}-free. Suppose there exists an $\ell \in [p-2]$ such that $D_\ell \setminus N_H(u) \neq \varnothing$ but $|D_\ell| \leqslant k-2$. Let $w \in V(D_\ell) \setminus N_H(u)$. Then $wv \in E_r$, else we obtain a critical coloring of $G + wv$ from τ by coloring the edge wv blue. Since $H[N_H(u)] \setminus v$ is K_{t-2}-free, we then obtain a critical coloring of $G + uw$ from τ by coloring the edge uw red, and then recoloring wv blue and all red edges incident with u in $G[V(D_p)]$ blue, a contradiction. This proves Theorem 7(c).

To prove Theorem 7(d,e), let $u \in V(H)$ with $d_H(u) = \delta(H)$. We may assume that $u \in V(D_p)$. By Theorem 7(b), $d_H(u) \geqslant t-2$. Let $N_H(u) := \{u_1, \dots, u_{\delta(H)}\}$. By Theorem 7(b) applied to u and any vertex in $V(H) \setminus (V(D_p) \cup N_H(u))$, we see that $H[N_H(u)]$ must contain K_{t-2} as a subgraph. We may assume that $H[\{u_1, \dots, u_{t-2}\}] = K_{t-2}$. Then we may further assume that $u_i \in V(D_{p-i})$ for all $i \in [t-2]$. Let $v \in V(H) \setminus (V(D_p) \cup N_H(u))$.

To proceed to prove Theorem 7(d), assume $d_H(u) \leqslant 2t-5$ and $k \geqslant t$. Suppose $H[N_H(u)]$ has an edge, say u_1u_2, that is contained in all K_{t-2} subgraphs of $H[N_H(u)]$. Then both $H[N_H(u) \setminus u_1]$ and $H[N_H(u) \setminus u_2]$ are K_{t-2}-free. By Theorem 7(b) applied to u and any vertex in $V(H) \setminus (V(D_p) \cup N_H(u))$, $V(H) \setminus (V(D_p) \cup N_H(u))$ must be complete to $\{u_1, u_2\}$ in H. Then $V(D_{p-1}) \cup V(D_{p-2}) \subseteq N_H(u) \setminus \{u_3, \dots, u_{t-2}\}$. Thus $|V(D_{p-1}) \cup V(D_{p-2})| = \delta(H) - (t-4) \leqslant t-1 \leqslant k-1$, because $\delta(H) \leqslant 2t-5$ and $t \leqslant k$. Then we obtain a critical coloring of $G + uv$ from τ by first coloring the edge uv

red, and then recoloring u_1u_2 blue and all red edges incident with u in $G[V(D_p)]$ blue, a contradiction. This proves Theorem 7(d).

To proceed to prove Theorem 7(e), note that $|N_r(u) \cap V(D_p)| = |N_r(u)| - d_H(u)$. By Theorem 7(a), $|N_r(u)| \geqslant 2t-4$. Since D_p is a component of G_b, we see that $N_b(u) \cap V(D_p) \neq \varnothing$. It follows that $|V(D_p)| = |\{u\}| + |N_b(u) \cap V(D_p)| + |N_r(u) \cap V(D_p)| \geqslant 1+1+(2t-4)-d_H(u) = 2t-2-d_H(u)$. By Lemma 10(a), $2t-2-d_H(u) \leqslant |V(D_p)| \leqslant k-1$, which yields $k \geqslant 2t-1-d_H(u)$. Suppose next that $\delta(H) = t-2 < 2t-5$. Then $k \geqslant t+1$. But then $H[\{u_1, \dots, u_{t-2}\}]$ is the only K_{t-2} subgraph of $H[N_H(u)]$, contrary to Theorem 7(d). This proves Theorem 7(e).

We next prove Theorem 7(f). By Lemma 10(a,b), $|D_i| \leqslant k-1$, $G[V(D_i)] = K_{|D_i|}$ for all $i \in [p]$, and at most $t-1$ of the D_i's have less than $k/2$ vertices. Since $n \geqslant (t-1)(k-1)+1$, we see that $n-(t-2)(\lceil k/2 \rceil - 1) \geqslant (t-2)\lfloor k/2 \rfloor + k$. Let r be the remainder of $n-(t-2)(\lceil k/2 \rceil - 1)$ when divided by $\lceil k/2 \rceil$, and let $s \geqslant 0$ be an integer such that

$$n-(t-2)(\lceil k/2 \rceil - 1) = s\lceil k/2 \rceil + r.$$

Then $0 \leqslant r \leqslant \lceil k/2 \rceil - 1$. Let $\ell = t-2+\min\{1, r\}$. It is straightforward to see that if $r \neq 0$, then $\ell = t-1$ and $\sum_{i=1}^{p} e_G(V(D_i))$ is minimized when: $t-1$ of the components, say $D_1, \dots, D_\ell$ are such that $|D_1|, \dots, |D_{t-1}| < k/2$ with $|D_1|+\dots+|D_{t-1}| = (t-2)(\lceil k/2 \rceil - 1)+r$, and the remaining s components, $D_t, \cdots, D_{t-1+s}$ are such that $|D_t| = \cdots = |D_{t-1+s}| = \lceil k/2 \rceil$; if $r = 0$, then $\ell = t-2$ and $\sum_{i=1}^{p} e_G(V(D_i))$ is minimized when: $t-1$ of the components, say $D_1, D_2, \dots, D_{t-1}$ are such that $|D_1|, \dots, |D_{t-1}| < k/2$ with $|D_1|+\dots+|D_{t-1}| = (t-2)(\lceil k/2 \rceil - 1)$, and the remaining s components, $D_t, \cdots, D_{t-1+s}$ are such that $|D_t| = \cdots = |D_{t-1+s}| = \lceil k/2 \rceil$. Using the facts that $s\lceil k/2 \rceil + r = n-(t-2)(\lceil k/2 \rceil - 1)$ and $r \leqslant \lceil k/2 \rceil - 1$, it follows that

$$\begin{aligned}\sum_{i=1}^{p} e_G(V(D_i)) > s\binom{\lceil k/2 \rceil}{2} &= \left(s \cdot \left\lceil \frac{k}{2} \right\rceil\right)\left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}\right) \\ &\geqslant (n-(t-1)(\lceil k/2 \rceil - 1))\left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}\right) \\ &= \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}\right)(n-(t-1)(\lceil k/2 \rceil - 1)).\end{aligned}$$

This proves Theorem 7(f).

To prove Theorem 7(g), suppose that H is disconnected. Let $x, y \in V(H)$ be such that x and y are in different components of H. By Theorem 7(b), $\{x, y\} \subseteq D_i$ for some $i \in [p]$, and there must exist a vertex $w \in D_j$ such that $xw \notin E(H)$ and $yw \in E(H)$, where $j \in [p]$ with $j \neq i$. By Theorem 7(b), x and w have at least $t-2$ common neighbors in H. But then x and y must be in the same component of H, a contradiction. This proves Theorem 7(g).

It remains to prove Theorem 7(h). By Theorem 7(g), H is connected. Let $q \in \mathbb{N}$ with $q \geqslant t-1$. Assume $\delta(H) \geqslant q$. Following Day [6], we next apply q-neighbor bootstrap percolation on H. Note that H is not necessarily K_t-saturated. Given a set $S \subseteq V(H)$ and any vertex $v \in V(H)$, let $N_S(v) := N_H(v) \cap S$ and $d_S(v) := |N_S(v)|$. Let $R \subseteq V(H)$ be any nonempty set. Let $R^0 := R$ and for $i \geqslant 1$, let

$$R^i := R^{i-1} \cup \{v \in V(H) \mid d_{R^{i-1}}(v) \geqslant q\}.$$

Let $\overline{R} := \bigcup_{i \geqslant 0} R^i$, the closure of R under the q-neighbor bootstrap percolation on H. Then

$$e(H[\overline{R}]) \geqslant q(|\overline{R}| - |R|),$$

because every vertex in $R^i \setminus R^{i-1}$ is adjacent to at least q vertices in R^{i-1}. Let $Y(R) := V(H) \setminus \overline{R}$. Finally, for any $v \in V(H)$, let

$$\omega_{_R}(v) := d_{\overline{R}}(v) + d_{Y(R)}(v)/2.$$

We call $\omega_{_R}(v)$ the weight of v (with respect to R). Then

$$e_H(\overline{R}, Y(R)) + e_H(Y(R)) = \sum_{v \in Y(R)} \omega_{_R}(v).$$

Within $Y(R)$, we define $B(R)$ to be the set $\{v \in Y(R) \mid \omega_{_R}(v) < q\}$, which we call the set of bad vertices. We next show that there exists a constant $c_1(q,k)$ and a nonempty set $R \subseteq V(H)$ with $|R| \leqslant c_1(q,k)$ such that $B(R) = \varnothing$.

Assume $B(R) \neq \varnothing$ for our initial R. Our goal is to move a small number of vertices into R so that the remaining vertices in $B(R)$ have strictly larger weight. To achieve this, let

$$\mathcal{U}_R := \{U \subseteq R \mid U = N_R(v) \text{ for some } v \in B(R)\}.$$

Note that for every vertex $v \in B(R)$, $d_R(v) \leqslant q-1$. Thus

$$|\mathcal{U}_R| \leqslant 1 + |R| + \binom{|R|}{2} + \cdots + \binom{|R|}{q-1}.$$

Let $\mathcal{U}_R := \{U_1, U_2, \ldots, U_{|\mathcal{U}_R|}\}$ and let $u_i \in B(R)$ with $N_R(u_i) = U_i$ for all $i \in \{1, \ldots, |\mathcal{U}_R|\}$. Then $d_{\overline{R}}(u_i) < q$, and so $d_{Y(R)}(u_i) \geqslant 1$ because $d_H(u_i) \geqslant q$. Let $x_i \in Y(R)$ such that $u_i x_i \in E(H)$ for all $i \in \{1, \ldots, |\mathcal{U}_R|\}$, and let $X(R) := \{x_1, x_2, \ldots, x_{_{|\mathcal{U}_R|}}\}$. By the choice of $\mathcal{U}_R$ and $u_1, u_2, \ldots, u_{_{|\mathcal{U}_R|}}$, for every vertex $v \in B(R)$, we see that $N_R(v) = N_R(u_i)$ for some $i \in \{1, 2, \ldots, |\mathcal{U}_R|\}$. Finally, let

$$S(R) := \{v \in B(R) \mid N_R(v) = N_R(u_i) \text{ and } \{v, x_i\} \subseteq D_j \text{ for some } i \in [|\mathcal{U}_R|] \text{ and } j \in [p]\}.$$

We next show that **Algorithm** 1 below yields a nonempty set $R \subseteq V(H)$ with $B(R) = \varnothing$.

Algorithm 1: Building a nonempty set $R \subseteq V(H)$ with $B(R) = \varnothing$

Data: $H := G \setminus (\bigcup_{i\in[p]} E(G[V(D_i)]))$ is a spanning subgraph of G_r with $\delta(H) \geqslant q$

Result: A nonempty set $R \subseteq V(H)$ with $B(R) = \varnothing$

1 Set R to be a set containing an arbitrary vertex in H;

2 **while** $B(R) \neq \varnothing$ **do**

3   Set R to be $R \cup X(R) \cup S(R) \cup \bigcup_{j=1}^{|\mathcal{U}_R|} N_{\overline{R}}(x_j)$;

4 **end**

Let R_i be the set R obtained in the i-th iteration of **Line 2** when running **Algorithm** 1. Then for all $i \geqslant 1$, $R_{i-1} \subseteq R_i$, $\overline{R}_{i-1} \subseteq \overline{R}_i$, $Y(R_i) \subseteq Y(R_{i-1})$ and $B(R_i) \subseteq B(R_{i-1})$. To see why $\omega_{R_i}(v) \geqslant \omega_{R_{i-1}}(v)$ for all $v \in B(R_i)$, we next introduce a control function on $V(H)$, because dealing with $\omega_R(v)$ directly is difficult. Let $\phi_R(v) := \sum_{x\in N_H(v)} f_R(x)$ for all $v \in V(H)$, where for all $x \in V(H)$,

$$f_R(x) = \begin{cases} 1, & \text{if } x \in R, \\ 1/2, & \text{if } x \in \overline{R} \setminus R, \\ d_R(x)/(2q), & \text{if } x \in Y(R). \end{cases}$$

It is worth noting that $\phi_R(v) \leqslant \omega_R(v)$ for every vertex $v \in V(H)$, because $d_R(x) \leqslant q-1$ for all $x \in Y(R)$. Similarly, for all $i \geqslant 1$, $f_{R_{i-1}}(x) \leqslant f_{R_i}(x)$ for every $x \in V(H)$, because $Y(R_i) \subseteq Y(R_{i-1})$. We next claim that

$(*)$ for all $i \geqslant 1$ and every $v \in B(R_i)$, $\phi_{R_i}(v) \geqslant \phi_{R_{i-1}}(v) + 1/(2q)$.

Proof. Let $i \geqslant 1$ and $v \in B(R_i)$. Then $v \in B(R_{i-1})$, since $B(R_i) \subseteq B(R_{i-1})$. Let $\mathcal{U}_{R_{i-1}}$, $\{u_1, \ldots, u_{|\mathcal{U}_{R_{i-1}}|}\} \subseteq B(R_{i-1})$, and $\{x_1, \ldots, x_{|\mathcal{U}_{R_{i-1}}|}\} \subseteq Y(R_{i-1})$ be defined accordingly for R_{i-1}. Then $N_{R_{i-1}}(v) = N_{R_{i-1}}(u_j)$ for some $j \in \{1, 2, \ldots, |\mathcal{U}_{R_{i-1}}|\}$. To prove $\phi_{R_i}(v) \geqslant \phi_{R_{i-1}}(v) + 1/(2q)$, it suffices to show that $f_{R_i}(x) \geqslant f_{R_{i-1}}(x) + 1/(2q)$ for some $x \in N_H(v)$. Since $\{x_1, \ldots, x_{|\mathcal{U}_{R_{i-1}}|}\} \subseteq Y(R_{i-1}) \cap R_i$, we see that $f_{R_{i-1}}(x) = d_{R_{i-1}}(x)/(2q) \leqslant (q-1)/(2q) = 1/2 - 1/(2q)$, and $f_{R_i}(x) = 1 > f_{R_{i-1}}(x) + 1/(2q)$ for all $x \in \{x_1, \ldots, x_{|\mathcal{U}_{R_{i-1}}|}\}$. We may assume that $vx_\ell \notin E(H)$ for all $\ell \in \{1, \ldots, |\mathcal{U}_{R_{i-1}}|\}$, otherwise we are done. Since $v \in B(R_i)$, by the choice of x_j and $S(R_{i-1})$, we see that $\{v, x_j\} \nsubseteq V(D_\ell)$ for all $\ell \in [p]$. By Theorem 7(b) applied to v and x_j, $H[N_H(v) \cap N_H(x_j)]$ contains K_{t-2} as a subgraph. Let W be the vertex set of such a K_{t-2} subgraph. It follows that $W \nsubseteq R_{i-1}$, else $G_r[W \cup \{u_j, x_j\}] = K_t$, since $N_{R_{i-1}}(v) = N_{R_{i-1}}(u_j)$ and $u_j x_j \in E(H)$. Let $x \in W \setminus R_{i-1}$.

If $x \in \overline{R}_{i-1} \setminus R_{i-1}$, then $f_{R_{i-1}}(x) = 1/2$ and $f_{R_i}(x) = 1$, and so $f_{R_i}(x) \geqslant f_{R_{i-1}}(x) + 1/(2q)$, as desired. If $x \in Y(R_{i-1})$, then either $x \in \overline{R}_i$ or $x \in Y(R_i)$. In both cases, we have $f_{R_{i-1}}(x) = d_{R_{i-1}}(x)/(2q) \leqslant 1/2 - 1/(2q)$. If $x \in \overline{R}_i$, then $f_{R_i}(x) \geqslant 1/2$ and so $f_{R_i}(x) \geqslant f_{R_{i-1}}(x) + 1/(2q)$. Finally, if $x \in Y(R_i)$, then $d_{R_i}(x) \geqslant d_{R_{i-1}}(x) + 1$ because

$x_j \in R_i \setminus R_{i-1}$ and $R_{i-1} \subseteq R_i$. Hence, $f_{R_i}(x) = d_{R_i}(x)/(2q) \geqslant (d_{R_{i-1}}(x)+1)/(2q) = f_{R_{i-1}}(x) + 1/(2q)$.

In all cases, we have shown that there exists some vertex $x \in N_H(v)$ such that $f_{R_i}(x) \geqslant f_{R_{i-1}}(x) + 1/(2q)$. Hence, $\phi_{R_i}(v) \geqslant \phi_{R_{i-1}}(v) + 1/(2q)$ for all $i \geqslant 1$ and $v \in B(R_i)$. $\square$

By (∗), **Algorithm** 1 stops after $m \leqslant 2q^2$ iterations of **Line 2** because $\phi_R(v) \leqslant \omega_R(v) < q$ for each $v \in B(R)$. Hence $R_m \subseteq V(H)$ with $R_m \neq \varnothing$ but $B(R_m) = \varnothing$. For all $i \geqslant 0$,

$$\begin{aligned} |R_{i+1}| &\leqslant |R_i| + |X(R_i)| + |S(R_i)| + \Big|\bigcup_{j=1}^{|\mathcal{U}_{R_i}|} N_{\overline{R}_i}(x_j)\Big| \\ &\leqslant |R_i| + |\mathcal{U}_{R_i}| + (k-2)|\mathcal{U}_{R_i}| + (q-1)|\mathcal{U}_{R_i}| \\ &= |R_i| + (k+q-2)|\mathcal{U}_{R_i}| \\ &\leqslant |R_i| + (k+q-2)\left(1 + |R_i| + \binom{|R_i|}{2} + \cdots + \binom{|R_i|}{q-1}\right), \end{aligned}$$

which only depends on q and k. It follows that by **Algorithm** 1, there exists a constant $c_1(q,k)$ and a non-empty set $R \subseteq V(H)$ with $|R| \leqslant c_1(q,k)$ such that $B(R) = \varnothing$. Then $\omega_R(v) \geqslant q$ for all $v \in Y(R)$ and so

$$e_H(\overline{R}, Y(R)) + e_H(Y(R)) = \sum_{v \in Y(R)} \omega_R(v) \geqslant q|Y(R)|.$$

Therefore,

$$\begin{aligned} e(H) &= e(H[\overline{R}]) + e_H(\overline{R}, Y(R)) + e_H(Y(R)) \\ &\geqslant q(|\overline{R}| - |R|) + q|Y(R)| \\ &\geqslant q(|\overline{R}| - c_1(q,k)) + q|Y(R)| \\ &= q(n - c_1(q,k)) \\ &= qn - c(q,k) \end{aligned}$$

where $c(q,k) = qc_1(q,k)$. This proves Theorem 7(h).

This completes the proof of Theorem 7. $\square$

3 Lower bound on the size of $(K_t, \mathcal{T}_k)$-co-critical graphs

We begin this section with a useful lemma, which may be of independent interest. We use $\alpha(G)$ and $\omega(G)$ to denote the independence number and clique number of G, respectively. It is worth noting that Lemma 12 is stronger than Theorem 11 when $\alpha(G) > |G|/2$: Theorem 11 yields that $\left|\bigcap_{S \in \mathcal{F}} S\right| > 0$, while Lemma 12 not only yields $\left|\bigcap_{S \in \mathcal{F}} S\right| \geqslant \delta(G)+1$ but also characterizes the case when $\left|\bigcap_{S \in \mathcal{F}} S\right| = 1$. For completeness, we include a proof here due to Hehui Wu. For a graph G, a set $A \subseteq V(G)$ is *stable* if $G[A]$ has no edges.

Theorem 11 (Hajnal [11])**.** *Let G be a graph and let $\mathcal{F}$ be the family of all maximum stable sets of G. Then*

$$\left|\bigcap_{S\in\mathcal{F}} S\right| + \left|\bigcup_{S\in\mathcal{F}} S\right| \geqslant 2\alpha(G).$$

Lemma 12. *Let G be a graph with $\alpha(G) > |G|/2$ and let $\mathcal{F}$ be the family of all maximum stable sets of G. Then*

$$\left|\bigcap_{S\in\mathcal{F}} S\right| \geqslant \delta(G) + 2\alpha(G) - |G| \geqslant \delta(G) + 1.$$

Moreover, if $\bigcap_{S\in\mathcal{F}} S = \{u\}$, then $\alpha(G) = (|G|+1)/2$ and u is an isolated vertex in G.

Proof. Let $X \in \mathcal{F}$ and $Y := V(G) \setminus X$. Then $|X| = \alpha(G) > |G|/2$, and so $|X| > |Y|$. Let $H := G[X, Y]$ be the bipartite subgraph of G with $V(H) = X \cup Y$ and $E(H) = \{xy \in E(G) : x \in X, y \in Y\}$. Let T be a stable set of H with $|T|$ maximum and let $X_1 := X \setminus T$, $Y_1 := Y \cap T$ and $Y_2 := Y \setminus T$. Then $|Y_1| + |X \setminus X_1| = |T| \geqslant |X| = |X_1| + |X \setminus X_1| > |Y| = |Y_1| + |Y_2|$, which implies that $|X_1| \leqslant |Y_1|$ and $|X \setminus X_1| > |Y_2|$. We next show that $H' := G[X \setminus X_1, Y_2]$ contains a matching that saturates Y_2. For any $S \subseteq Y_2$, we have $|N_{H'}(S)| \geqslant |S|$, else $T' := (T \setminus N_{H'}(S)) \cup S$ is a stable set of H with $|T'| > |T|$, a contradiction. By Hall's Theorem, there exists a matching, say M, of H' that saturates Y_2. Let $X_2 := V(M) \cap X$ and $X_3 := X \setminus (X_1 \cup X_2)$. Then

$$|X_3| = |X| - |X_1| - |X_2| \geqslant |X| - |Y| = 2\alpha(G) - |G| > 0,$$

because $|X_1| \leqslant |Y_1|$, $|X_2| = |Y_2|$ and $\alpha(G) > |G|/2$. Note that $X_1 \cup Y_1$ is anti-complete to $X \setminus X_1$ in H. By the choice of T, $\alpha(H[X_1 \cup Y_1]) \leqslant |X_1|$. Moreover, $\alpha(H[X_2 \cup Y_2]) \leqslant |X_2|$ because M is a perfect matching of $G[X_2, Y_2]$. Then for any $S \in \mathcal{F}$, $|S \cap (X_1 \cup Y_1)| \leqslant |X_1|$ and $|S \cap (X_2 \cup Y_2)| \leqslant |X_2|$. Therefore, $|X_3| \geqslant |S \cap X_3| = |S| - |S \cap (X_1 \cup Y_1)| - |S \cap (X_2 \cup Y_2)| \geqslant |X| - |X_1| - |X_2| = |X_3|$. It follows that $|S \cap X_3| = |X_3|$. Then $X_3 \subseteq S$. Hence, $X_3 \subseteq \bigcap_{S\in\mathcal{F}} S$ by the arbitrary choice of S.

Next, suppose there exists a vertex $u \in X_3$ with $d_G(u) = d > 0$. Let $N_G(u) := \{v_1, \ldots, v_d\}$. Then $\{v_1, \ldots, v_d\} \subseteq Y_2$. Let $u_1, \ldots, u_d \in X_2$ be such that $u_i v_i \in E(M)$ for all $i \in [d]$. For each $i \in [d]$, let $M^i := (M \setminus u_i v_i) \cup \{uv_i\}$, $X_2^i := V(M^i) \cap X$ and $X_3^i := X \setminus (X_1 \cup X_2^i)$. Then $u_i \in X_3^i$ and M^i is a perfect matching of $G[X_2^i, Y_2]$. By the arbitrary choice of M, $u_i \in \bigcap_{S\in\mathcal{F}} S$. Therefore, $|\bigcap_{S\in\mathcal{F}} S| \geqslant |\{u_1, \ldots, u_d\} \cup X_3| \geqslant d + (2\alpha(G) - |G|) \geqslant \delta(G) + 2\alpha(G) - |G| \geqslant \delta(G) + 1$, as desired.

Finally, if $\bigcap_{S\in\mathcal{F}} S = \{u\}$, then $1 = |\bigcap_{S\in\mathcal{F}} S| \geqslant d + 2\alpha(G) - |G|$. It follows that $d = 0$ and $\alpha(G) = (|G|+1)/2$, because $2\alpha(G) - |G| > 0$.

This completes the proof of Lemma 12. □

We are now ready to prove Theorem 8.

Proof of Theorem 8: Let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, where $t \geqslant 4$ and $k \geqslant \max\{6, t\}$. Then $n \geqslant (t-1)(k-1)+1$ and G admits a critical coloring. Among all critical colorings of G, let $\tau : E(G) \to \{\text{red, blue}\}$ be a critical coloring of G with $|E_r|$ maximum. By the choice of τ, G_r is K_t-saturated and G_b is $\mathcal{T}_k$-free. By Theorem 7(a), $\delta(G_r) \geqslant 2t-4$. Let $D_1, \cdots, D_p$ be all components of G_b. By Lemma 10(a), $|D_i| \leqslant k-1$ for all $i \in [p]$. Then $(t-1)(k-1)+1 \leqslant n \leqslant p(k-1)$. This implies that $p \geqslant t$. Let $H := G \setminus (\bigcup_{i\in[p]} E(G[V(D_i)]))$. Then H is a spanning subgraph of G_r. Clearly, H is K_t-free.

Assume first that $\delta(H) \geqslant 2t-4$. By Theorem 7(h) applied to H and $q = 2t-4$, there exists a constant $c(2t-4, k)$ such that $e(H) \geqslant (2t-4)n - c(2t-4, k)$. This, together with Theorem 7(f), yields that

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geqslant (2t-4)n - c(2t-4,k) + \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n - (t-1)(\lceil k/2\rceil - 1)) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(2t-4,k) - \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2 \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_1(t,k),
\end{aligned}$$

as desired, where $c_1(t,k) = c(2t-4,k) + \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2$.

Assume next that $\delta(H) \leqslant 2t-5$. Note that $k \geqslant \max\{6, t\} \geqslant t$ for all $t \geqslant 4$. Let $u \in V(H)$ with $d_H(u) = \delta(H)$. We may assume that $u \in V(D_p)$. Let $N_H(u) = \{u_1, \ldots, u_{\delta(H)}\}$. By Theorem 7(b) applied to u and any vertex in $V(H) \setminus (V(D_p) \cup N_H(u))$, we see that $H[N_H(u)]$ must contain K_{t-2} as a subgraph. We may assume that $H[\{u_1, \ldots, u_{t-2}\}] = K_{t-2}$. Then we may further assume that $u_i \in V(D_{p-i})$ for all $i \in [t-2]$. Note that $H[N_H(u)]$ is K_{t-1}-free and $\omega(H[N_H(u)]) = t-2 > |N_H(u)|/2$. Let $\mathcal{F}$ be the family of all K_{t-2} subgraphs of $H[N_H(u)]$. By Theorem 7(d), $|\bigcap_{A\in\mathcal{F}} A| \leqslant 1$. By Lemma 12 applied to the complement of $H[N_H(u)]$, we have $|\bigcap_{A\in\mathcal{F}} A| = 1$. We may assume that $\bigcap_{A\in\mathcal{F}} A = \{u_1\}$. By Lemma 12 again, $|N_H(u)| = 2t-5$, u_1 is complete to $N_H(u) \setminus u_1$ in H and u_1 is contained in all K_{t-2} subgraphs of $H[N_H(u)]$. Then $H[N_H(u) \setminus u_1]$ is K_{t-2}-free. By Theorem 7(b) applied to u and any vertex in $V(H) \setminus (V(D_p) \cup N_H(u))$, $V(H) \setminus (V(D_p) \cup N_H(u))$ must be complete to u_1 in H. Thus $\{u_1\} = V(D_{p-1})$. By Theorem 7(h) applied to H and $q = 2t-5$, there exists a constant $c(2t-5, k)$ such that $e(H) \geqslant (2t-5)n - c(2t-5, k)$. Recall that $p \geqslant t$. If $p = t$, then $n = (t-1)(k-1)+1$

and $|V(D_i)| = k-1$ for $i \in [p]$ with $i \neq p-1$. In this case,

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geqslant ((2t-5)n - c(2t-5,k)) + (p-1)(k-1)(k-2)/2 \\
&= ((2t-5)n - c(2t-5,k)) + (n-1)(k-2)/2 \\
&= (2t-6+k/2)n - c(2t-5,k) - (k-2)/2 \\
&\geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_2(t,k)
\end{aligned}$$

for all $k \geqslant 6$, as desired, where $c_2(t,k) = c(2t-5,k) + (k-2)/2$.

Next assume $p \geqslant t+1$. Since $k \geqslant t$, $|N_H(u)| \leqslant 2t-5$, and G_r is K_t-free, by Lemma 10(b), there are at most $t-1$ many D_i's satisfying $u \notin V(D_i)$ and $D_i \setminus N_H(u) = \varnothing$. We may assume that for all $i \in [p-t]$, $D_1, \ldots, D_{p-t}$ are such that $u \notin V(D_i)$ and $D_i \setminus N_H(u) \neq \varnothing$. By Theorem 7(c), $|D_i| = k-1$ for all $i \in [p-t]$. Thus

$$\sum_{i=1}^{p} e(G[V(D_i)]) \geqslant (p-t)(k-1)(k-2)/2.$$

Note that $n \leqslant (p-1)(k-1)+1$ because $\{u_1\} = V(D_{p-1})$ and $|D_i| \leqslant k-1$ for all $i \in [p]$ with $i \neq p-1$. Therefore,

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geqslant ((2t-5)n - c(2t-5,k)) + (p-t)(k-1)(k-2)/2 \\
&\geqslant ((2t-5)n - c(2t-5,k)) + \frac{1}{2}\left(\frac{n-1}{k-1} - t + 1\right)(k-1)(k-2) \\
&= (2t-6+k/2)n - c(2t-5,k) - (k-2)(tk-t-k+2)/2 \\
&\geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(2t-5,k) - [(t-1)k^2 - (3t-4)k + 2t-4]/2 \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_3(t,k)
\end{aligned}$$

for all $k \geqslant 6$, as desired, where $c_3(t,k) = c(2t-5,k) + [(t-1)k^2 - (3t-4)k + 2t-4]/2$.

Let $\ell(t,k) := \max\{c_1(t,k), c_2(t,k), c_3(t,k)\}$. Then

$$e(G) \geqslant \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - \ell(t,k),$$

as desired. This completes the proof of Theorem 8. □

4 Proof of Theorem 9

Let $t \in \{4, 5\}$, $k \geqslant 3$ and $n \geqslant (2t-3)(k-1) + \lceil k/2 \rceil \lceil k/2 \rceil - 1$. We will construct a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices which yields the desired upper bound in Theorem 9.

Let r, s be the remainder and quotient of $n - (2t-3)(k-1)$ when divided by $\lceil k/2 \rceil$, and let $A := K_{k-1}$. For each $i \in [t-2]$, let $B_i := K_{k-2}$ and $C_i := K_{k-2}$. Let H_1 be obtained from disjoint copies of $A, B_1, \ldots, B_{t-2}, C_1, \ldots, C_{t-2}$ by joining every vertex in B_i to all vertices in $A \cup C_i \cup B_j$ for each $i \in [t-2]$ and all $j \in [t-2]$ with $j \neq i$. Let $H_2 := (s-r)K_{\lceil k/2 \rceil} \cup rK_{\lceil k/2 \rceil + 1}$ when $k \geqslant 4$, and $H_2 := sK_2 \cup rK_1$ when $k = 3$. Finally, let G be the graph obtained from $H := H_1 \cup H_2$ by adding $2t-4$ new vertices $x_1, \ldots, x_{t-2}, y_1, \ldots, y_{t-2}$, and then, for each $i \in [t-2]$, joining: x_i to every vertex in $V(H)$ and all x_j; and y_i to every vertex in $V(H) \setminus V(A)$ and all x_j, where $j \in [t-2]$ with $j \neq i$. The construction of G when $t = 4$ and $k \geqslant 4$ is depicted in Figure 1, and the construction of G when $t = 5$ and $k \geqslant 4$ is depicted in Figure 2.

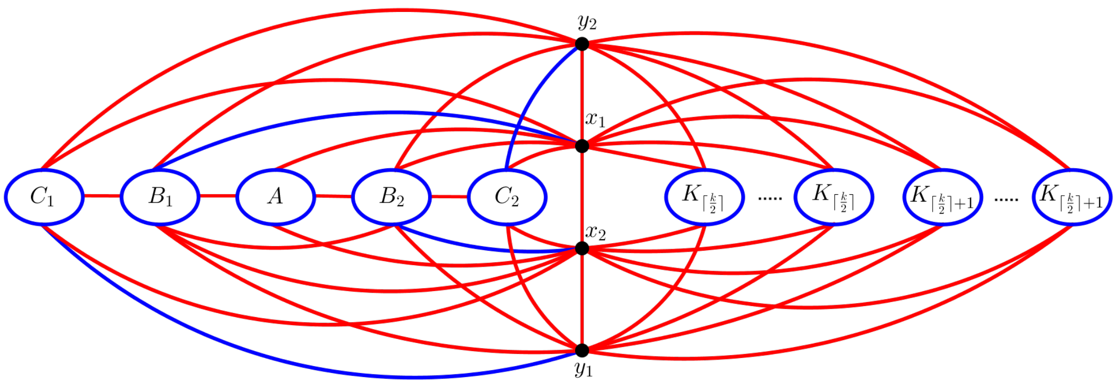

Figure 1: A $(K_4, \mathcal{T}_k)$-co-critical graph for all $k \geqslant 4$.

Let $\sigma : E(G) \to \{\text{red, blue}\}$ be defined as follows: all edges in $A, B_1, \ldots, B_{t-2}$, $C_1, \ldots, C_{t-2}$ and H_2 are colored blue; for every $i \in [t-2]$, all edges between x_i and B_i are colored blue and all edges between y_i and C_i are colored blue; the remaining edges of G are all colored red. Note that the $\{\text{red, blue}\}$-coloring of G depicted in Figure 1 (resp. Figure 2) is σ when $t = 4$ (resp. $t = 5$) and $k \geqslant 4$. It is simple to check that σ is a critical coloring of G. We next show that σ is the unique critical coloring of G up to symmetry.

Let $X := \{x_1, \ldots, x_{t-2}\}$ and $Y := \{y_1, \ldots, y_{t-2}\}$. Let $\tau : E(G) \to \{\text{red, blue}\}$ be an arbitrary critical coloring of G. It suffices to show that $\tau = \sigma$ up to symmetry. Let G_r^τ and G_b^τ be G_r and G_b under the coloring τ, respectively. Note that $G[V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X] = K_{(t-1)(k-1)}$. By Lemma 10(a) and the fact that G_r^τ is K_t-free,

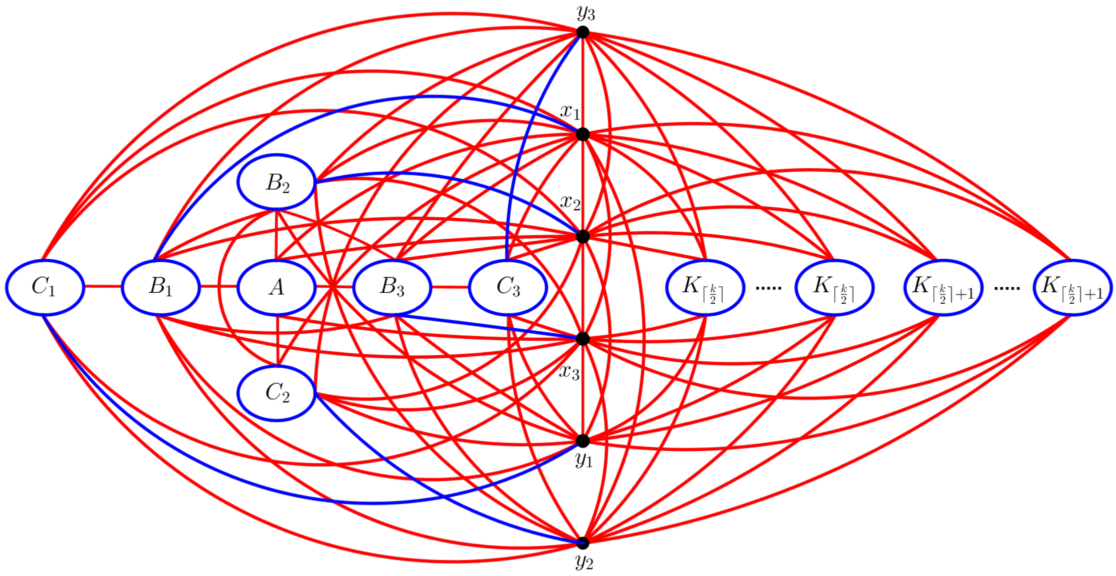

Figure 2: A $(K_5, \mathcal{T}_k)$-co-critical graph for all $k \geqslant 4$.

$G_b^\tau[V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X]$ has exactly $t-1$ components, say $D_1, \ldots, D_{t-1}$, such that $V(D_i)$ is complete to $V(D_j)$ in G_r^τ for all $i, j \in [t-1]$ with $i \neq j$. Then each D_i is isomorphic to K_{k-1} in G_b^τ for all $i \in [t-1]$. Since every vertex in $V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X$ belongs to a blue K_{k-1} in G_b^τ, it follows that: for each $i \in [t-2]$, y_i is complete to $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup (X \setminus x_i)$ in G_r^τ; and $V(C_i)$ is complete to $V(B_i) \cup X$ in G_r^τ. We next prove three claims.

Claim 1. $A = D_i$ for some $i \in [t-1]$.

Proof. Suppose $A \neq D_i$ for all $i \in [t-1]$. Then for each $i \in [t-1]$, we see that $(V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X) \cap V(D_i) \neq \varnothing$. Let $d_i \in (V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X) \cap V(D_i)$ for each $i \in [t-1]$. Then $d_1, \ldots, d_{t-1}$ are pairwise distinct and $G_r^\tau[\{d_1, \ldots, d_{t-1}\}] = K_{t-1}$. Note that either $X \subseteq \{d_1, \ldots, d_{t-1}\}$ or $x_i \notin \{d_1, \ldots, d_{t-1}\}$ for some $i \in [t-2]$. It follows that in the former case, $G_r^\tau[\{d_1, \ldots, d_{t-1}, u\}] = K_t$ for any $u \in C_1$, and in the latter case, $G_r^\tau[\{d_1, \ldots, d_{t-1}, y_i\}] = K_t$, because y_i is complete to $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup (X \setminus x_i)$ in G_r^τ. In both cases we obtain a contradiction because G_t^τ is K_t-free. □

By Claim 1, we may assume that $A = D_{t-1}$. Then $V(A)$ is complete to $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X$ in G_r^τ. For each $i \in [t-2]$, since G_b^τ is $\mathcal{T}_k$-free, there must exist a vertex $c_i \in V(C_i)$ such that c_i is adjacent to at most one vertex of Y in G_b^τ. Then c_i is adjacent to at least $t-3$ vertices of Y in G_r^τ. We next show that

Claim 2. For each $i \in [t-2]$, $|X \cap V(D_i)| = 1$.

Proof. Suppose $|X \cap V(D_i)| \neq 1$ for some $i \in [t-2]$. Since $|X| = t-2$, we may assume that $|X \cap V(D_1)| \geqslant 2$ and $X \cap V(D_{t-2}) = \varnothing$. We may further assume that $x_1, x_2 \in V(D_1)$. Then $x_1x_2 \in E_b$. Since $X \cap V(D_{t-2}) = \varnothing$ and for all $i \in [t-2]$, $|V(B_i)| = k-2 < k-1 = |V(D_{t-2})|$, we may assume that $V(B_i) \cap V(D_{t-2}) \neq \varnothing$ for $i \in [2]$. Let $b_1 \in V(B_1) \cap V(D_{t-2})$. We see that $c_1y_i \in E_r$ for some $i \in [2]$, because c_1 is adjacent to at least $t-3$ vertices of Y in G_r^τ. If $t = 4$, then $G_r^\tau[\{b_1, c_1, y_i, x_{3-i}\}] = K_4$, a contradiction. Thus $t = 5$. Suppose $V(B_1) \cap V(D_2) \neq \varnothing$. Let $b_2 \in V(B_1) \cap V(D_2)$. Then $G_r^\tau[\{b_1, b_2, c_1, y_i, x_{3-i}\}] = K_5$, a contradiction. Thus $V(B_1) \cap V(D_2) = \varnothing$. By symmetry, $V(B_2) \cap V(D_2) = \varnothing$. Then $V(D_2) = V(B_3) \cup \{x_3\}$. But then $G_r^\tau[\{b_1, c_1, y_i, x_{3-i}, x_3\}] = K_5$, a contradiction. □

Claim 3. For each $i \in [t-2]$, $V(B_i) \subseteq V(D_j)$ for some $j \in [t-2]$.

Proof. Suppose there exists an $i \in [t-2]$ such that $V(B_i) \nsubseteq V(D_j)$ for every $j \in [t-2]$. We may assume $i = 1$. Since $V(B_1) \subseteq V(D_1) \cup \cdots \cup V(D_{t-2})$, we see that $k-2 = |B_1| \geqslant 2$. Thus $k \geqslant 4$. We claim that $V(B_1) \cap V(D_j) = \varnothing$ for some $j \in [t-2]$. Suppose $V(B_1) \cap V(D_j) \neq \varnothing$ for all $j \in [t-2]$. Let $d_j \in V(B_1) \cap V(D_j)$ for all $j \in [t-2]$. But then $G_r^\tau[\{d_1, \ldots, d_{t-2}, c_1, y_\ell\}] = K_t$, where $c_1y_\ell \in E_r$ for some $\ell \in [t-2]$, a contradiction. Thus $V(B_1) \cap V(D_j) = \varnothing$ for some $j \in [t-2]$, as claimed. We may assume that $V(B_1) \cap V(D_{t-2}) = \varnothing$. Since $V(B_1) \nsubseteq V(D_j)$ for every $j \in [t-2]$, it follows that $t = 5$, $V(B_1) \subseteq V(D_1) \cup V(D_2)$, and $V(B_1) \cap V(D_1) \neq \varnothing$ and $V(B_1) \cap V(D_2) \neq \varnothing$. Let $d_1 \in V(B_1) \cap V(D_1)$ and $d_2 \in V(B_1) \cap V(D_2)$. By Claim 2, let $x_i \in X \cap V(D_3)$. Then $G_r^\tau[\{d_1, d_2, x_i, c_1, y_j\}] = K_5$, where $c_1y_j \in E_r$ for some $j \in [3]$ with $j \neq i$, a contradiction. □

By Claim 2 and Claim 3, $V(B_i) \cup V(B_j) \nsubseteq D_\ell$ for any $i \neq j \in [t-2]$ and all $\ell \in [t-2]$. By symmetry, we may assume that $V(B_i) \subseteq V(D_i)$ for all $i \in [t-2]$. Then $V(B_i) \cup \{x_j\} = V(D_i)$ for some $j \in [t-2]$ since $|V(D_i)| = |V(B_i)| + 1$ and $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X = V(D_1) \cup \cdots \cup V(D_{t-2})$. By symmetry, we may assume that $V(B_i) \cup \{x_i\} = V(D_i)$ for all $i \in [t-2]$. It follows that for all $i, j \in [t-2]$ with $i \neq j$, B_i is complete to B_j in G_r^τ, x_i is complete to $X \setminus x_i$ and B_j in G_r^τ, y_i is complete to C_i in G_b^τ, y_i is complete to $C_j \cup (X \setminus x_i)$ in G_r^τ, x_i is complete to B_i in G_b^τ, $\{x_i, y_i\}$ is complete to H_2 in G_r^τ, all edges in $A, B_1, \ldots, B_{t-2}, C_1, \ldots, C_{t-2}$ and H_2 are colored blue under τ. This proves that $\tau = \sigma$ and thus σ is the unique critical coloring of G up to symmetry. It can be easily checked that adding any edge $e \in E(\overline{G})$ to G creates a red K_t if e is colored red, and a blue T_k if e is colored blue. Hence, G is $(K_t, \mathcal{T}_k)$-co-critical. Note that

$$\begin{aligned} e_G(X \cup Y, V(G) \setminus (X \cup Y)) &= (t-2)(n-(2t-4)) + (t-2)(n-(2t-4+k-1)) \\ &= (t-2)(2n-4t-k+9); \end{aligned}$$

$e_G(X \cup Y) = \binom{t-2}{2} + (t-2)(t-3)$; $e_G(V(B_1) \cup \cdots \cup V(B_{t-2}), V(C_1) \cup \cdots \cup V(C_{t-2})) = (t-2)(k-2)^2$; $e_G(V(C_1) \cup \cdots \cup V(C_{t-2})) = (t-2)\binom{k-2}{2}$; $e_G(V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2})) = \binom{(t-2)(k-2)+k-1}{2}$. Using the facts that $s\lceil k/2 \rceil + r = n - (2t-3)(k-1)$ and $r \leqslant \lceil k/2 \rceil - 1$, we see that when $k \geqslant 4$,

$$\begin{aligned}
e(G) &= (t-2)(2n-4t-k+9) + \binom{t-2}{2} + (t-2)(t-3) + (t-2)(k-2)^2 + \\
&\quad (t-2)\binom{k-2}{2} + \binom{(t-2)(k-2)+k-1}{2} + (s-r)\binom{\lceil k/2 \rceil}{2} + r\binom{\lceil k/2 \rceil + 1}{2} \\
&= (2t-4)n - (t-2)k - \frac{1}{2}(t-2)(5t-9) \\
&\quad + (k-2)\big((t-2)(k-2) + (t-2)(k-3)/2 + (t-1)(tk-k-2t+3)/2\big) \\
&\quad + \frac{s-r}{2}\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1\right) + \frac{r}{2}\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \\
&= (2t-4)n - (t-2)\big(k - \frac{1}{2}(5t-9)\big) + \frac{1}{2}(k-2)\big((t^2+t-5)k - 2t^2 - 2t + 11\big) \\
&\quad + \frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil - 1\right)\left(s\left\lceil \frac{k}{2} \right\rceil + r\right) + \frac{r}{2}\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \\
&\leqslant (2t-4)n + \frac{1}{2}\big((t^2+t-5)k^2 - (4t^2+6t-25)k - t^2 + 23t - 40\big) \\
&\quad + \frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil - 1\right)(n - (2t-3)(k-1)) + \frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil - 1\right)\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right)n + \frac{1}{2}(t^2+t-5)k^2 - (2t^2+2t-11)k \\
&\quad - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left((2t-3)(k-1) - \left\lceil \frac{k}{2} \right\rceil\right) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right)n + C(t,k),
\end{aligned}$$

where $C(t,k) = \frac{1}{2}(t^2+t-5)k^2 - (2t^2+2t-11)k - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left((2t-3)(k-1) - \left\lceil \frac{k}{2} \right\rceil\right)$.

When $k = 3$, we have $2s \leqslant n - 2(2t-3)$ and

$$\begin{aligned}
e(G) &= (t-2)(2n-4t-3+9) + \binom{t-2}{2} + (t-2)(t-3) + (t-2) + \binom{(t-2)+2}{2} + s \\
&\leqslant (2t-4)n - (t-2)(3t-4) + \binom{t-2}{2} + \binom{t}{2} + \frac{n-2(2t-3)}{2} \\
&= (2t-7/2)n + \binom{t-2}{2} + \binom{t}{2} - (3t^2-8t+5) \\
&= (2t-7/2)n + C(t,3),
\end{aligned}$$

where $C(t,3) = \binom{t-2}{2} + \binom{t}{2} - (3t^2-8t+5) = -2t^2+5t-2$.

This completes the proof of Theorem 9. □

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