Circuit Covers of Signed Eulerian Graphs

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Abstract

A signed circuit cover of a signed graph is a natural analog of a circuit cover of a graph, and is equivalent to a covering of its corresponding signed-graphic matroid with circuits. It was conjectured that a signed graph whose signed-graphic matroid has no coloops has a 6-cover. In this paper, we prove that the conjecture holds for signed Eulerian graphs.

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1 Introduction

Let G be a graph. A *signed graph* is a pair (G,Σ) with $\Sigma\subseteq E(G)$, each edge in Σ is labelled by -1 and other edges are labelled by 1. The graph G can be viewed as the signed graph $(G,\varnothing)$. A *circuit* is a connected 2-regular graph. A circuit C of G is *balanced* if $|C\cap\Sigma|$ is even, otherwise it is *unbalanced*. We say that a subgraph of (G,Σ) is *unbalanced* if it contains an unbalanced circuit, otherwise it is *balanced*. Signed graphs is a special class of "biased graphs", which was defined by Zaslavsky in [7, 8]. Just as biased graphs, there are two interesting classes of matroids, the class of signed-graphic matroids and the class of even-cycle matroids, associated with signed graphs, which in fact are special classes of "frame matroids" and "lifted-graphic matroids" associated with biased graphs, respectively.

A *barbell* is a union of two unbalanced circuits sharing exactly one vertex or a union of two vertex-disjoint unbalanced circuits together with a minimal path joining them. A *signed circuit* of (G,Σ) is a balanced circuit or a barbell. We say the matroid with $E(G)$

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as its ground set and with the set of all signed circuits as its circuit set is the *signed-graphic* matroid defined on (G, Σ). We say that (G, Σ) is *flow-admissible* if each element of $E(G)$ is in a circuit of its signed-graphic matroid, that is, each edge of G is in a signed circuit of (G, Σ).

For a positive integer k, we say that a signed graph (G, Σ) has a *k-cover* if there is a family $\mathcal{C}$ of signed circuits of (G, Σ) such that each edge of G belongs to exactly k members of $\mathcal{C}$. For ordinary graphs G (signed graph (G, Σ) with $\Sigma = \varnothing$), a k-cover of G is just a family of circuits which together covers each edge of G exactly k times. In [1], Bermond, Jackson and Jaeger proved that every bridgeless graph G has a 4-cover. Fan [4] proved that every bridgeless graph G has a 6-cover. Together it follows that every bridgeless graph G has a k-cover, for every even integer k greater than 2. The only left case that $k = 2$ is the famous Circuit Double Cover Conjecture: every bridgeless graph G has a 2-cover, which is still open and believed to be very hard. It is somehow a surprise that it is even unknown whether there is an integer k such that every signed graph (G, Σ) has a k-cover.

Let A and B be two vertex-disjoint unbalanced circuits of length $2m + 1$. Let G be the signed graph obtained from A and B by joining A and B with two internally disjoint paths of length $2m + 1$ such that the two paths form an unbalanced circuit. Then each signed circuit of G is a barbell of $6m + 3$ edges. Any k-cover of G contains $k|E(G)| = k(8m+4) = 4k(2m+1)$ edges, which must be divisible by $6m+3 = 3(2m+1)$. That is, $4k$ must be divisible by 3, which means that k cannot be 2 or 4. Thus G has neither 2-covers nor 4-covers. Consider the singed graph H consisting of three unbalanced circuits of length $2m+1$ with exactly one vertex in common. Then each signed circuit of H is a barbell of $4m+2$ edges. Any k-cover of H contains $k|E(G)| = k(6m+3) = 3k(2m+1)$ edges, which must be divisible by $4m + 2 = 2(2m + 1)$. That is, $3k$ must be divisible by 2, which means that k cannot be odd. Thus H has no k-cover for any odd k. These counterexamples were first given by Fan [5], who also proposed the following conjecture.

Conjecture 1.1. *Every flow-admissible signed graph has a* 6*-cover.*

In this paper, we prove

Theorem 1.2. *Conjecture 1.1 holds for signed Eulerian graphs.*

In [3], Cheng, Lu, Luo, and Zhang proved that each signed Eulerian graph with an even number of negative edges has a 2-cover. We will prove Theorem 1.2 from a different aspect, and our proof does not rely on their result.

This paper is organised as follows. Definitions and results needed in the proof of Theorem 1.2 are given in Section 2. Theorem 1.2 will be proved in Section 4 by contradiction. All "small" signed Eulerian graphs occurring in Section 4 in the proof by contradiction are dealt with in Section 3.

2 Preliminaries

Let G be a finite graph. Let *loops*(G) denote the set of loops in G. Let $\Delta(G)$ and $\delta(G)$ be the maximal and minimal degree of G, respectively. For a positive integer k, let $V_k(G)$ be the subset of $V(G)$ consisting of degree-k vertices of G. A subgraph H of G is *spanning* if $V(H) = V(G)$. In this paper, we will also use H to denote its edge-set. For example, we will let $G\backslash H$ denote $G\backslash E(H)$. If exactly one component of G has edges, then we say that G is *connected up to isolated vertices*. Evidently, a connected graph is also connected up to isolated vertices, but the converse maybe not true.

We say that G is *even* if every vertex of G is of even degree. If an even graph is connected, we say that it is *Eulerian*. A circuit C of G is *non-separating* if $G\backslash C$ is connected, otherwise, it is *separating*. A *theta graph* is a graph that consists of a pair of vertices joined by three internally vertex-disjoint paths. Let $\mathcal{C}$ be a circuit-decomposition of an Eulerian graph G. Let H be a graph with $\mathcal{C}$ as its vertex set, where two vertices in H are adjacent if and only if their corresponding circuits in G have a common vertex. We say that H is *determined* by $\mathcal{C}$.

Lemma 2.1. *Let G be an Eulerian graph with $\Delta(G) \geqslant 4$. Let C be a circuit of G. Then there is a circuit C' of G with $C \cap C' = \varnothing$ such that $G\backslash C'$ is connected up to isolated vertices.*

Proof. Since G is Eulerian, G has a circuit-decomposition $\mathcal{C}$ containing C. Let H be the graph determined by $\mathcal{C}$. Since G is connected with $\Delta(G) \geqslant 4$, the graph H is connected with at least two vertices. Let T be a spanning tree of H. Since T has at least two degree-1 vertex, T has a degree-1 vertex, say C', which is not C. Then C' is the circuit as required by the lemma. □

Lemma 2.2. *Let G be a 2-connected graph with $|V(G)| \geqslant 3$. For any vertex v of G, there is an edge e of $G - v$ such that $G - V(e)$ is connected.*

Proof. Let C be a circuit of G passing through v with $|C|$ as large as possible. Evidently, $|C| \geqslant 3$ as $|V(G)| \geqslant 3$ and G is 2-connected. Let e be an edge of C that is not incident with v. Then $G - V(e)$ is connected, otherwise we can find a longer circuit going through v. □

A set $\Sigma' \subseteq E(G)$ is a *signature* of (G, Σ) if (G, Σ) and (G, Σ') have the same balanced circuits and the same unbalanced circuits. Evidently, for any edge-cut C^* of G, the symmetric difference $\Sigma \triangle C^*$ is a signature of (G, Σ). We say that (G, Σ') is obtained from (G, Σ) by *switching*. The following three lemmas are well-known results on signed graphs, which will be frequently used in Section 3 without reference. Please refer to ([2], Lemma 3.5.), if the reader needs more detail about Lemma 2.3.

Lemma 2.3. *All edges of a balanced signed subgraph of (G, Σ) can be labelled by 1 by switching.*

Lemma 2.4. *Each signed theta-graph has a balanced circuit and can not have exactly two balanced circuits.*

Lemma 2.5. *Every 2-edge-connected signed graph containing two edge-disjoint unbalanced circuits is flow-admissible.*

In ([6], Theorem 4.2.), Máčajová and Škoviera proved that a flow-admissible signed Eulerian graph with an odd number of negative edges contains three edge-disjoint unbalanced circuits. On the other hand, since each unbalanced Eulerian signed graph with an even number of negative edges contains two edge-disjoint unbalanced circuits, we have

Lemma 2.6. *A flow-admissible unbalanced signed Eulerian graph contains two edge-disjoint unbalanced circuits.*

For simplicity, we will also use G to denote a signed graph defined on G.

3 Signed Eulerian graphs with special circuit decompositions

Let k be a positive integer. Let kG be the graph obtained from G by replacing each edge in G with exactly k parallel edges. Consider a graph constructed as follows. For $k \geqslant 3$, let G be a circuit of length k and N be a subdivision of $2G$. Let C be a circuit of N, we say that C is *small* if $|V(C) \cap V_4(N)| = 2$, otherwise, C is *long*. When C is small, we also say that each vertex in $V(C) \cap V_4(N)$ is an *end* of C. Let e_1, e_2 be edges in a small circuit of N such that $\{e_1, e_2\}$ is not an edge-cut of N. That is, $\{e_1, e_2\}$ separates the two ends of the small circuit. We say that the signed graph obtained from N by labelling $\{e_1, e_2\}$ by -1 and all other edges by 1 is a *necklace* of *length* k. Evidently, all small circuits in a necklace are balanced and all long circuits are unbalanced. Hence, the small circuits form a 1-cover in a necklace.

In the rest of this section, we will always let G denote a 2-connected flow-admissible signed Eulerian graph with $\delta(G) \geqslant 4$, and $\mathcal{C}$ a circuit-decomposition of G, and let H be the graph determined by $\mathcal{C}$. We say that $\mathcal{C}$ is *optimal* if it satisfies the following properties:

(CD1) $\mathcal{C}$ is chosen with the number of unbalanced circuits as large as possible.

(CD2) subject to (CD1), $\mathcal{C}$ is chosen with $|\mathcal{C}|$ as large as possible.

In the rest of this section, we will always assume that $\mathcal{C}$ is optimal. For any $C \in \mathcal{C}$, we say that C is a *balanced* vertex of H if C is a balanced circuit of G, otherwise it is *unbalanced*.

The following lemma follows immediately from (CD1), (CD2), and Lemma 2.6.

Lemma 3.1. *For every pair of adjacent vertices C_i and C_j in H, if C_i is balanced, we have*

1. $1 \leqslant |V_G(C_i) \cap V_G(C_j)| \leqslant 2$,
2. *$C_i \cup C_j$ is balanced when C_j is balanced, and*
3. *$C_i \cup C_j$ is not flow-admissible when C_j is unbalanced.*

Lemma 3.2. *For every pair of adjacent unbalanced vertices C_i and C_j in H, if $|V_G(C_i) \cap V_G(C_j)| \geqslant 3$, then $C_i \cup C_j$ is a necklace.*

Proof. Since C_i and C_j are unbalanced, for any circuit decomposition $\mathcal{C}'$ of $C_i \cup C_j$, either all circuits in $\mathcal{C}'$ are balanced or at least two of them are unbalanced. If $C_i \cup C_j$ has an unbalanced circuit avoiding some vertex in $V_4(C_i \cup C_j)$, then $C_i \cup C_j$ can be decomposed into at least three circuits and two of which are unbalanced, which is not possible as $\mathcal{C}$ is optimal. So each circuit in $C_i \cup C_j$ avoiding a vertex in $V_4(C_i \cup C_j)$ is balanced. Hence, $C_i \cup C_j$ is a necklace. □

We say that G is *cover-decomposable* if G can be decomposed into two proper edge-disjoint flow-admissible signed Eulerian subgraphs.

Lemma 3.3. *If H is isomorphic to a graph pictured as Figure 1 and G has no balanced loops, then G is cover-decomposable or has a 6-cover.*

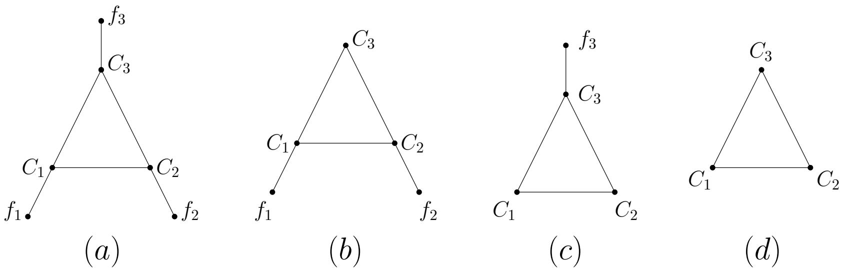

Figure 1: All degree-3 vertices are balanced, and others are unbalanced. All f_i's are loops of G.

Proof. Assume otherwise. Assume that H is isomorphic to the graph pictured as Figure 1 (d). For any $1 \leqslant i < j \leqslant 3$, when $|V(C_i) \cap V(C_j)| \leqslant 2$, it is obvious that $C_i \cup C_j$ has a 1-cover; when $|V(C_i) \cap V(C_j)| \geqslant 3$, it follows from Lemma 3.2 that $C_i \cup C_j$ is a necklace, so $C_i \cup C_j$ has a 1-cover too. Then G has a 2-cover. So H is isomorphic to a graph pictured as Figure 1 (a)-(c). Note that, $1 \leqslant |V_G(C_i) \cap V_G(C_j)| \leqslant 2$ when C_i is balanced by Lemma 3.1. When some C_i is a loop, implying that G is isomorphic to the graph pictured as Figure 1 (b) or (c), since each pair of adjacent circuits intersect in at most 2 vertices, there are a few cases to check that G has a 6-cover. So no C_i is a loop. When $|V_G(C_i) \cap V_G(C_j)| = 1$ for all $1 \leqslant i < j \leqslant 3$, since $C_1 \cup C_2 \cup C_3$ is isomorphic to a $2K_3$-subdivision, combined the fact that all f_i are unbalanced loops, it is easy to see that G has a 6-cover. Hence, $|V_G(C_i) \cap V_G(C_j)| \geqslant 2$ for some $1 \leqslant i < j \leqslant 3$.

Assume that H is isomorphic to a graph pictured as Figure 1 (a). By Lemma 3.1 (1) and symmetry we may assume that $V_G(C_2) \cap V_G(C_3) = \{u, v\}$. Let C be the circuit of $C_2 \cup C_3$ that is incident to neither f_2 nor f_3. Since C is balanced by Lemma 3.1 (2), $G \backslash C$ is not connected otherwise G is cover-decomposable, so $V_G(C_1) \cap V_G(C_2 \cup C_3) \subseteq V_G(C) - \{u, v\}$. When $|V_G(C_1) \cap V_G(C_2)| = |V_G(C_1) \cap V_G(C_3)| = 1$, the graph G has a 2-cover. When $|V_G(C_1) \cap V_G(C_i)| = 2$ for some $2 \leqslant i \leqslant 3$, since $G \backslash loops(G)$ is balanced by Lemma 3.1 (2), there is a non-separating balanced circuit C' contained in $C \cup C_i$, implying that G is cover-decomposable. Hence, H is isomorphic to a graph pictured as Figure 1 (b) or (c).

Assume that $V_G(C_2) \cap V_G(C_3) = \{u, v\}$. Since exactly one of $\{C_2, C_3\}$ is unbalanced, there is a (u, v)-path P of $C_2 \cup C_3$ such that a circuit in $C_2 \cup C_3$ is unbalanced if and only if it contains P. Since all degree-3 vertices in Figure 1 are balanced, P is not incident to f_2 or f_3. Let C the unique balanced circuit of $C_2 \cup C_3$ that is not incident to f_2 or f_3. Since $(C_2 \cup C_3) - C$ and f_3 are unbalanced, $G\backslash C$ is not connected, so $V_G(C_1) \cap V_G(C_2 \cup C_3) \subseteq V_G(C) - \{u, v\}$. When $C_1 \cup C_2$ is a necklace, implying that H is isomorphic to a graph pictured as Figure 1 (c) by Lemma 3.1, there is a non-separating small circuit C' of the necklace $C_1 \cup C_2$ with $C' \subseteq C_1 \cup (C_2 - P)$. Since $(C_2 \cup C_3) - C$ and f_3 are unbalanced, $G\backslash C'$ is flow-admissible, so G is cover-decomposable as C' is balanced. Hence, by Lemma 3.1 (1) or Lemma 3.2, we have $|V_G(C_1) \cap V_G(C_i)| \leqslant 2$ for each $2 \leqslant i \leqslant 3$. Moreover, since $V_G(C_1) \cap V_G(C_2 \cup C_3) \subseteq V_G(C) - \{u, v\}$, repeatedly using a similar strategy, we can find a 6-cover of G or a non-separating balanced circuit C such that $G\backslash C$ is flow-admissible, a contradiction.

By symmetry we may therefore assume that $|V_G(C_i) \cap V_G(C_3)| = 1$ for each $1 \leqslant i \leqslant 2$. Set $m = |V_G(C_1) \cap V_G(C_2)| \geqslant 2$. When $m = 2$, by simple computation, the lemma holds. So $m \geqslant 3$. By Lemmas 3.1 and 3.2, H is isomorphic to the graph pictured as Figure 1 (c) and $C_1 \cup C_2$ is a necklace of length m. Assume that G is a counterexample to the lemma with $|V(G)|$ as small as possible. When C_3 does not share a vertex with a small circuit C of $C_1 \cup C_2$, delete C and identify its two ends as a new vertex. Let G' be the new graph. Then G' is cover-decomposable or has a 6-cover by the choice of G, so is G since C is balanced. Hence, C_3 intersects all small circuits of $C_1 \cup C_2$. Moreover, since $m \geqslant 3$ and $|V_G(C_i) \cap V_G(C_3)| = 1$ for each $1 \leqslant i \leqslant 2$, there are edge-disjoint long circuits C_1', C_2' of $C_1 \cup C_2$ with $|V_G(C_1') \cap V_G(C_3)| = 2$ and $|V_G(C_2') \cap V_G(C_3)| \geqslant 1$. Since C_1', C_2' are unbalanced and $C_1' \cup C_2' = C_1 \cup C_2$, the graph determined by $\{C_1', C_2', C_3, \{f_3\}\}$ isomorphic to the graph pictured as Figure 1 (c). Since $|V_G(C_1') \cap V_G(C_3)| = 2$, the lemma holds by similar analysis in the second paragraph of the proof. □

Let C be a separating circuit of a graph G with $u, v \in V(C)$. Let P be an (u, v)-path on C. For a component G' of $G\backslash C$, if $V(G') \cap V(P) \neq \varnothing$ we say that G' *intersects* P; if $V(G') \cap V(C) \subseteq V(P) - \{u, v\}$ we say that G' *properly intersects* with P.

Lemma 3.4. *Let C be a separating circuit of G such that all components of $G\backslash C$ are unbalanced. Let C' be a circuit-component of $G\backslash C$ with $\{u, v\} = V(C) \cap V(C')$. Let P_1 and P_2 be the (u, v)-paths of C. When C is balanced or $G\backslash C$ has at least three components, one of the following holds.*

(1) G is cover-decomposable, or

(2) $G\backslash C$ has exactly three components, none of which is flow-admissible and one of which properly intersects with P_i for each $1 \leqslant i \leqslant 2$.

Proof. Assume that (1) is not true. Without loss of generality we may assume that $C' = \{e, f\}$. Since C' is unbalanced, we may assume that $P_1 \cup \{e\}$ and $P_2 \cup \{x\}$ are balanced for some $x \in \{e, f\}$. Since $G\backslash C$ has at least two components, besides C', some component of $G\backslash C$ intersects with some P_i, say P_2. Since C is balanced or $G\backslash C$ has at least three components, $G\backslash(P_1 \cup \{e\})$ has two edge-disjoint unbalanced circuits. Since (1) does not hold, $G\backslash(P_1 \cup \{e\})$ is disconnected, so there exists some non-flow-admissible component of $G\backslash C$ properly intersecting with P_1. Repeating the analysis, there is also a non-flow-admissible component of $G\backslash C$ properly intersecting with P_2. So $G\backslash C$ has at least three components.

Let G_i be the union of the components of $G\backslash C$ that properly intersects with P_i for each $1 \leqslant i \leqslant 2$. Then G_1 and G_2 are not flow-admissible. Assume that G_1 is disconnected. Since G_1 contains two edge-disjoint unbalanced circuits, $G_1 \cup P_1 \cup \{x\}$ and $G\backslash(G_1 \cup P_1 \cup \{x\})$ are flow-admissible, implying that (1) holds. Hence, G_1 is connected, so is G_2 by symmetry. Besides C', G_1 and G_2, assume that $G\backslash C$ has another component G_3. Since G_3 is unbalanced and intersects $V(P_1)$ and $V(P_2)$ by the definition of G_1 and G_2 and the fact that G is 2-connected, both $G_1 \cup P_1 \cup \{f\}$ and $G\backslash(G_1 \cup P_1 \cup \{f\})$ are flow-admissible, a contradiction. So $G\backslash C$ has exactly three components C', G_1 and G_2, that is, (2) holds. □

For an (u, v)-path P of G, we say that P is *pendant* if $u \in V_1(G)$, $v \notin V_1(G) \cup V_2(G)$ and all internal vertices of P are in $V_2(G)$.

Lemma 3.5. *Let H be a tree with a unique vertex C of degree at least three, all leaf vertices are unbalanced, and all pendant paths have at most two edges. When C is balanced, $V_2(H) = \varnothing$. When C is unbalanced, all degree-2 vertices of H are balanced triangles and leaf vertices that are adjacent to degree-2 vertices are loops. Then G is cover-decomposable or has a 6-cover.*

Proof. Assume that the lemma is not true. Since G has a 6-cover when each component of $G\backslash C$ is a loop, there is a vertex C' in H adjacent to C with $|C'| \geqslant 2$. Set $m = |V_G(C) \cap V_G(C')|$. Since G is 2-connected and $\delta(G) \geqslant 4$, we have $m \geqslant 2$.

We claim that C' is balanced or $|C'| \neq 2$. Assume otherwise. Then C' is a component of $G\backslash C$ as all degree-2 vertices of H are balanced. Since $|C'| = 2$, we have $m = 2$. Let $\{u, v\} = V_G(C') \cap V_G(C)$, P_1 and P_2 be the (u, v)-paths of C. By Lemma 3.4, $G\backslash C$ has exactly three components C', G_1 and G_2, where G_1 and G_2 properly intersect P_1 and P_2, respectively. When $C \cup G_1$ is a necklace, there is a small circuit D of $C \cup G_1$ such that $G\backslash D$ is connected. Since C' and G_2 are unbalanced, $G\backslash D$ is flow-admissible, so G is cover-decomposable. Hence, G_1 is an unbalanced circuit of size at most 2 or G_1 consists of a balanced triangle and a loop, so is G_2 by symmetry. By simple computation, G is cover-decomposable or has a 6-cover.

Assume that C' is balanced. Then $C' \in V_2(H)$ is a triangle. So C is unbalanced and $|V_G(C') \cap V_G(C)| = 2$ by Lemma 3.1. Let u, v, P_1, P_2 be defined as above. Let e be the loop incident with C' and f the edge in C' whose ends are u, v. Since C is unbalanced, $P_1 \cup \{f\}$ is balanced and $P_2 \cup \{f\}$ is unbalanced. Evidently, (a) a component of $G\backslash C$ properly intersects with P_1, otherwise $P_1 \cup \{f\}$ and its complement are flow-admissible; and (b) no component of $G\backslash C$ intersects $P_2 - \{u, v\}$, otherwise the union G' of $P_2 \cup \{f\}$ and all components of $G\backslash C$ intersecting $P_2 - \{u, v\}$ and $G\backslash G'$ are flow-admissible. Then $P_2 \cup (C' - \{f\}) \cup \{e\}$ and its complement are flow-admissible, a contradiction.

We may therefore assume that C' is unbalanced with $|C'| \geqslant 3$, implying that C is unbalanced by Lemma 3.1. By the choice of C', for each component G' of $G \backslash C$, either G' is a loop or $|G'| \geqslant 3$. When $|G'| \geqslant 3$, $C \cup G'$ is a necklace by Lemma 3.2. Let D be a small circuit of $C \cup C'$. Since $G \backslash D$ has two edge-disjoint unbalanced circuits, $G \backslash D$ is disconnected, so a component G_D of $G \backslash C$ properly intersects in $C \cap D$. Since $C \cup C'$ has three small circuits, G_D is the unique component of $G \backslash C$ properly intersecting in $C \cap D$ and $C \cup C'$ has exactly three small circuits, implying $|C'| = 3$, otherwise G is cover-decomposable. When G_D is not a loop, there is a small circuit D' of $C \cup G_D$ such that $G \backslash D'$ is connected, so G is cover-decomposable. Hence, G_D is a loop. By the choice of C', each component G' of $G \backslash C$ that is not a loop is an unbalanced triangle. When C' is the unique component of $G \backslash C$ that is not a loop, G has a 3-cover. When there is another component G_1 of $G \backslash C$ that is not a loop, let D be a small circuit of $C \cup C'$ intersecting G_1. Let G' be the union of $D \cup G_1$ and the loop incident with D. Then G' and $G \backslash G'$ are flow-admissible, so G is cover-decomposable. □

4 Proof of Theorem 1.2.

In this section, we prove Theorem 1.2, which is restated here in a slightly different way.

Theorem 4.1. *Every flow-admissible signed Eulerian graph has a* 6*-cover.*

Proof. Assume that the result is not true. Let G be a counterexample with $|V(G)|$ as small as possible. Evidently, the following statements hold.

4.1.1.

- *G is unbalanced with $\delta(G) \geqslant 4$;*
- *G has no balanced loops; and*
- *G is not cover-decomposable, in particular, if C is a non-separating balanced circuit of G, then $G \backslash C$ is not flow-admissible.*

4.1.2. *G is* 2*-connected.*

Subproof. Assume otherwise. There are edge-disjoint Eulerian subgraphs G_1, G_2 of G with $|E(G_1)|, |E(G_2)| \geqslant 2$, with $\{v\} = V(G_1) \cap V(G_2)$, and with $E(G) = E(G_1) \cup E(G_2)$. Since G is a minimal counterexample and not cover-decomposable, G_1 and G_2 are unbalanced. Let G_i^+ be a signed graph obtained from G_i by adding an unbalanced loop e_i incident with v for each integer $1 \leqslant i \leqslant 2$. Since G_1^+ and G_2^+ are flow-admissible, both of them have 6-covers by the choice of G. Since $|V(G_1) \cap V(G_2)| = 1$, we can obtain a 6-cover of G by combining 6-covers of G_1^+ and G_2^+, a contradiction. □

Let $\mathcal{C}$ be an optimal circuit decomposition of G and H the graph determined by $\mathcal{C}$. Since G is connected, so is H. By Lemma 2.6, at least two members of $\mathcal{C}$ are unbalanced. Hence, by Lemma 3.2, $|V(H)| \geqslant 3$ and the following holds. If a block of H contains exactly one cut-vertex of H, we say the block is a *leaf block*.

4.1.3. *Each balanced vertex of H is a cut-vertex, in particular, each vertex in a leaf block of H that is not a cut-vertex is unbalanced.*

By 4.1.3 or the third part of 4.1.1, for any vertex C of H, all components of $G\backslash C$ are unbalanced. For a subgraph H' of H, each vertex $v \in V(H')$ is labeled by a circuit C_v in $\mathcal{C}$. We say that the subgraph $G' = \cup_{v\in V(H')} E(C_v)$ corresponds to H'.

4.1.4. *Let e be a cut-edge of H whose ends are C_i and C_j. If e is not a leaf edge and $H - \{C_i, C_j\}$ has exactly two components, then C_i or C_j is unbalanced.*

Subproof. Assume to the contrary that C_i and C_j are balanced. Let G_1 and G_2 be the subgraphs of G corresponding to the two components of $H - \{C_i, C_j\}$ with $V(G_1) \cap V_G(C_i) \neq \varnothing$. It follows from 4.1.3 that G_1, G_2 are unbalanced. Moreover, since G is 2-connected, by Lemma 3.1, we have $|V_G(C_i) \cap V_G(C_j)| = 2$. Let $u \in V_G(G_1) \cap V_G(C_i)$ and $v \in V(G_2) \cap V_G(C_j)$. Since $|V_G(C_i) \cap V_G(C_j)| = 2$, the graph $C_i \cup C_j$ has a circuit C avoiding u and v such that $(C_i \cup C_j)\backslash C$ is connected up to isolated vertices. Since $H - \{C_i, C_j\}$ has exactly two components, $G\backslash C$ is connected, so $G\backslash C$ is flow-admissible. Moreover, since $C_i \cup C_j$ is balanced by Lemma 3.1, C is balanced, so G is cover-decomposable, a contradiction. □

4.1.5. *For any separating circuit $C \in \mathcal{C}$, if G' is a component of $G\backslash C$ that is not flow-admissible, then one of the following holds.*

(1) G' is an unbalanced circuit such that $|G'| \leqslant 2$ or $C \cup G'$ is a necklace. In particular, when C is balanced, $|G'| \leqslant 2$.

(2) G' consists of a loop and a balanced triangle.

Subproof. When G' is a circuit, since $\delta(G) \geqslant 4$, by Lemmas 3.1 and 3.2, (1) holds. Assume that G' is not a circuit. When G' consists of exactly two edge-disjoint circuits that share exactly one vertex, since C only shares vertices with the balanced circuit of G' and $\delta(G) \geqslant 4$, the unbalanced circuit C' in G' has at most two edges. When $|C'| = 2$, there is a non-separating balanced circuit of G contained in $C \cup C'$, a contradiction. So C' is a loop. By Lemma 3.1 and 4.1.2, the balanced circuit in G' is a triangle, so (2) holds. Hence, we may assume that $\Delta(G') \geqslant 6$ or $|V_4(G')| \geqslant 2$.

Since G' is not flow-admissible, by switching we may assume that there is a unique edge e of G' labelled by -1 and all other edges in G' are labelled by 1. When e is a loop, let v be the end of e, and B a block of $G'\backslash\{e\}$ containing v, and let C' be a circuit of B containing v; otherwise, let $\{v\} = \varnothing$, and B the block containing e, and let C' be a circuit of B with $e \in C'$. If possible, we may further assume that C' is chosen with $V_G(C') \cap V_2(G') \neq \varnothing$. By Lemma 2.1, there is a circuit C_1 of $G'\backslash loops(G')$ with $C' \cap C_1 = \varnothing$ such that $G'\backslash C_1$ is connected up to isolated vertices. Since C_1 is balanced and $G\backslash C_1$ has two edge-disjoint unbalanced circuits, $G\backslash C_1$ is not connected. Hence, $V_G(C) \cap V(G') \subseteq V_G(C_1)$ and $\varnothing \neq V_2(G') \subseteq V_G(C_1)$ as e is the only edge in G' which has a chance to be a loop. By the choice of C', the set $V_2(G')$ is contained in another block B' of G' with $B \neq B'$ as C' contains no vertex in $V_2(G')$. Since $V_G(C) \cap V(G') \subseteq V(B')$ and G is 2-connected, $|B| = 1$, a contradiction to the choice of B. □

4.1.6. *For any $C \in \mathcal{C}$, the graph $G\backslash C$ has at most two components.*

Subproof. Assume that $G\backslash C$ has three components. Since each component G' of $G\backslash C$ is unbalanced and $G\backslash G'$ is flow-admissible, G' is not flow-admissible. By 4.1.5, H is a tree with C as a unique vertex of degree at least three, and all its pendant paths have at most two edges. When C is balanced, 4.1.4 implies that $V_2(H) = \varnothing$. Hence, by 4.1.5 and Lemma 3.5, G is cover-decomposable or has a 6-cover, a contradiction. □

4.1.7. *For any balanced vertex C of H, each degree-1 vertex of H adjacent with C is a loop of G.*

Subproof. Let C' be a degree-1 vertex of H adjacent with C. Assume that C' is not a loop of G. Then $|C'| = |V_G(C) \cap V_G(C')| = 2$ by 4.1.5. It follows from Lemma 3.4 and 4.1.6 that G is cover-decomposable, a contradiction. □

4.1.8. *H is not a tree.*

Subproof. Assume otherwise. By 4.1.6, H is a path. Evidently, at most one vertex in $V_2(H)$ is unbalanced, otherwise, G is cover-decomposable. By 4.1.4, no balanced vertices of H are adjacent, so $|V(H)| \leqslant 5$. Moreover, if $|V(H)| \geqslant 4$, then exactly one vertex in $V_2(H)$ is unbalanced. Assume that H has two adjacent vertices C_1, C_2 with $|V_G(C_1) \cap V_G(C_2)| \geqslant 3$. Then $C_1 \in V_1(H)$, $|V(H)| \leqslant 4$ and $C_1 \cup C_2$ is a necklace by Lemma 3.2. Let C_3 be the other vertex adjacent to C_2 in H. When $V_G(C_2) \cap V_G(C_3)$ is in a small circuit of $C_1 \cup C_2$, the graph G has a 6-cover. When $V_G(C_2) \cap V_G(C_3)$ is not in a small circuit of $C_1 \cup C_2$, implying $|V_G(C_2) \cap V_G(C_3)| = 2$, since $V_G(C_1) \cap V_G(C_3) = \varnothing$, the graph $C_1 \cup C_2$ can be decomposed to two long circuits C_1', C_2', where both share exactly one vertex with C_3. Note that the circuit decomposition $(\mathcal{C} - \{C_1, C_2\}) \cup \{C_1', C_2'\}$ is still optimal. Hence, the graph determined by $(\mathcal{C} - \{C_1, C_2\}) \cup \{C_1', C_2'\}$ is isomorphic to a graph pictured as Figure 1 (c) or (d). Lemma 3.3 implies that G is cover-decomposable or has a 6-cover. Therefore, combined with Lemma 3.1 we can assume that every pair of adjacent vertices in H share at most two vertices in G. Note that each degree-1 vertex of H adjacent to a balanced vertex is a loop by 4.1.7. By simple computation, G has a 6-cover, a contradiction. □

4.1.9. *H is not 2-connected and whose leaf blocks are isomorphic to K_2.*

Subproof. Assume otherwise. When H is not 2-connected, let B be a leaf block of H that is not isomorphic to K_2, and v be the unique cut-vertex of H in $V(B)$. When H is 2-connected, let $B = H$ and v any vertex of B. By Lemma 2.2, there is an edge e in $B - v$ such that $B - V_H(e)$ is connected, so $H - V_H(e)$ is also connected. Without loss of generality assume that C_1 and C_2 are the ends of e. Then $C_1 \cup C_2$ and $G\backslash(C_1 \cup C_2)$ are connected. Since $C_1 \cup C_2$ is flow-admissible by 4.1.3, the graph $G\backslash(C_1 \cup C_2)$ is not flow-admissible. Since H is not isomorphic to the graph pictured as Figure 1 (d) by Lemma 3.3, H has exactly three unbalanced vertices and exactly two leaf blocks, one of which is B that is isomorphic to K_3 and the other is isomorphic to K_2. Let $C_1C_2C_3 \ldots C_n$ be a longest path in H. It follows from 4.1.4 that $n = 4$. By 4.1.7, the circuit C_4 is a loop of G. That is, H is isomorphic to the graph pictured as Figure 1 (c). Hence, G is cover-decomposable or has a 6-cover by Lemma 3.3, a contradiction. □

Let B be a block of H with $|V(B)| \geqslant 3$. By 4.1.8 and 4.1.9, such B exists and B is not a leaf block. When H has two blocks that are not isomorphic to K_2, it follows from 4.1.3 and 4.1.9 that G is cover-decomposable. Hence, B is the unique block of H that is not isomorphic to K_2. By 4.1.3, each vertex in B that is not a cut-vertex of H is unbalanced.

Let $u \in V(B)$ be a cut vertex of H. When u is unbalanced or H has two pendant paths using u, let H_1 be the union of all pendant paths containing u, and G_1 the subgraph of G corresponding to H_1. Since $|V(B)| \geqslant 3$, by 4.1.3 and 4.1.9, both G_1 and $G \backslash G_1$ are flow-admissible, a contradiction. Hence, u is balanced and H has exactly one pendant path using u. By the arbitrary choice of u, all cut-vertices of H in B are balanced. Using a similar strategy, all vertices in $V_2(H) - V(B)$ are balanced. Combined with 4.1.4, we have $V_2(H) - V(B) = \varnothing$. That is, each pendant path of H has exactly one edge. By 4.1.7, each vertex in $V_1(H)$ is a loop of G.

When there is a vertex in $V(B)$ that is not a cut-vertex of H, let v denote such a vertex. Otherwise, let v be any vertex of B. By Lemma 2.2, there is an edge $e \in B - v$ such that $B - V(e)$ is connected. Let H_1 be the union of e and all pendant paths of H using an end of e, and G_1 be the subgraph of G corresponding to H_1. Since each vertex in B that is not a cut-vertex of H is unbalanced, H_1 contains two unbalanced vertices, so G_1 is flow-admissible. Since $H - V(H_1)$ is connected and has an unbalanced vertex, H is isomorphic to a graph pictured as Figure 1 (a) or (b). Lemma 3.3 implies that G is cover-decomposable or has a 6-cover, a contradiction. □

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