Graphs with no induced $K_{2,t}$

Freddie Illingworth*
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge, United Kingdom
fci21@cam.ac.uk

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Abstract

Consider a graph G on n vertices with $\alpha\binom{n}{2}$ edges which does not contain an induced $K_{2,t}$ ($t \geqslant 2$). How large must α be to ensure that G contains, say, a large clique or some fixed subgraph H? We give results for two regimes: for α bounded away from zero and for $\alpha = o(1)$.

Our results for $\alpha = o(1)$ are strongly related to the Induced Turán numbers which were recently introduced by Loh, Tait, Timmons and Zhou. For α bounded away from zero, our results can be seen as a generalisation of a result of Gyárfás, Hubenko and Solymosi and more recently Holmsen (whose argument inspired ours).

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1 Introduction

Fix an integer $t \geqslant 2$ and consider a graph G on n vertices with $\alpha\binom{n}{2}$ edges which does not contain an induced $K_{2,t}$. How large does α have to be to ensure that G contains some substructure (like a large clique or a fixed subgraph H)? We consider two regimes: α is bounded away from zero and α goes to zero as n goes to infinity.

In the regime where α is bounded away from zero, G will contain substructures that grow with n (so for example the clique number of G, $\omega(G)$, will go to infinity). Gyárfás, Hubenko and Solymosi [7] dealt with the clique number in the case when $t = 2$ (that is, G contains no induced C_4), confirming a conjecture of Erdős.

Proposition 1 (Gyárfás-Hubenko-Solymosi, [7])**.** *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges. If G does not contain an induced $K_{2,2}$, then $\omega(G) \geqslant \alpha^2 n/10$.*

This was recently improved by Holmsen [8] (note that $1 - \sqrt{1-\alpha} \geqslant \alpha/2$ for $\alpha \in [0, 1]$).

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Proposition 2 (Holmsen, [8])**.** *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges. If G does not contain an induced $K_{2,2}$, then $\omega(G) \geqslant (1-\sqrt{1-\alpha})^2 n$.*

This result has the added advantage that $(1-\sqrt{1-\alpha})^2 \to 1$ as $\alpha \to 1$, so it is approximately tight as $\alpha \to 1$. The arguments in this paper are motivated by Holmsen's.

Our main result is Theorem 10, which is an extension to the situation where G does not contain an induced $K_{2,t}$ and also considers whether G contains some general subgraph (in place of a clique). For comparison with Proposition 2, we state the special case of the clique (we believe this result is also in a sense tight as $\alpha \to 1$ – see Remark 12). First, it will be convenient to define a constant β depending on α and t.

Definition 3. Given $\alpha \in [0,1]$ and an integer $t \geqslant 2$, define

$$\beta_t(\alpha) = \frac{t}{2\sqrt{t-1}}\Big[\sqrt{1-\big(1-\tfrac{2}{t}\big)^2\alpha} - \sqrt{1-\alpha}\Big].$$

Note that $\beta_2(\alpha) = 1-\sqrt{1-\alpha}$ so Proposition 2 can be stated as: if G is a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,2}$, then $\omega(G) \geqslant \beta_2(\alpha)^2 n$.

Theorem 4. *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ and let $\beta = \beta_t(\alpha)$. For any positive integer r with $R(t,r) \leqslant \beta^2 n$, we have $\omega(G) \geqslant r+1$.*

Here $R(t,r)$ denotes the usual Ramsey number. It is natural for Ramsey numbers to appear in the statement. The class of graphs with "no induced $K_{2,t}$" includes those with "no independent t-set" and if $\omega(G) \geqslant r+1$ for all such graphs, then $R(t,r+1) \leqslant n$.

Since $R(2,r) = r$, Theorem 4 is exactly Holmsen's result when $t = 2$. In Section 3, using known Ramsey number bounds we prove explicit lower bounds for the clique number for all t. As an illustration, we state the case $t = 3$, which is particularly clean.

Theorem 5. *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges. If G does not contain an induced $K_{2,3}$, then*

$$\omega(G) \geqslant \left\lfloor \tfrac{2}{3}\alpha\sqrt{n} \right\rfloor \text{ for all } n, \text{ and}$$
$$\omega(G) \geqslant \tfrac{1}{3}\alpha\sqrt{n\log n} + 2 \text{ for large enough } n \text{ in terms of } \alpha.$$

The regime where α goes to zero is closely related to the following natural question first proposed by Loh, Tait, Timmons and Zhou [9]. Consider a graph G on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ – how large must α be to ensure that some fixed graph H is a subgraph of G? If we do not ban G from containing an induced $K_{2,t}$ then the answer follows from the theorem of Erdős and Stone [3] (see Erdős and Simonovits [2]): $\alpha = 1 - \frac{1}{\chi(H)-1} + o(1)$ where $\chi(H)$ is the chromatic number of H. However forbidding G from containing an induced $K_{2,t}$ (ruling out Turán-style graphs) changes the answer drastically. In particular we will see that the required α grows like $n^{-1/2}$, that is, the required number of edges grows like $n^{3/2}$.

Loh, Tait, Timmons and Zhou introduced the notion of an *induced Turán number*: define

$$\mathrm{ex}(n, \{H, F\text{-ind}\})$$

to be the maximum number of edges in a graph on n vertices which does not contain H as a subgraph and does not contain F as an induced subgraph. In this paper we focus on $F = K_{2,t}$, which was also considered by Loh, Tait, Timmons and Zhou. We will give some improvements to their results. The important case where H is an odd cycle has been resolved by Ergemlidze, Győri and Methuku [5].

Proposition 6 (Loh-Tait-Timmons-Zhou, [9]). *Let $t \geqslant 3$ be an integer and G be a graph on n vertices within minimum degree d. If G does not contain an induced $K_{2,t}$, then*

$$\omega(G) \geqslant \left(\frac{d^2}{2n(t-1)}(1-o(1))\right)^{\frac{1}{t-1}} - t + 1.$$

A graph with $\alpha\binom{n}{2}$ edges has average degree $\alpha(n-1)$ and has a subgraph of minimum degree at least $\alpha(n-1)/2$. Thus one should view d as being between $\alpha(n-1)/2$ and $\alpha(n-1)$. We improve the dependence upon t for all α as well as adding a $(\log n)^{1-\frac{1}{t-1}}$ factor for constant $\alpha > 0$.

Theorem 7. *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges. If G does not contain an induced $K_{2,t}$, then*

$$\omega(G) \geqslant \left\lfloor \tfrac{t-1}{4}(\alpha^2 n)^{\frac{1}{t-1}} \right\rfloor - t + 3 \text{ for all } n, \text{ and}$$
$$\omega(G) \geqslant \tfrac{1}{20t}\left(\alpha^2 n (\log n)^{t-2}\right)^{\frac{1}{t-1}} \text{ for large enough } n \text{ in terms of } \alpha.$$

Finally, Loh, Tait, Timmons and Zhou gave a general upper bound for $\mathrm{ex}(n, \{H, F\text{-ind}\})$ when $F = K_{2,t+1}$.

Proposition 8 (Loh-Tait-Timmons-Zhou, [9]). *Fix a graph H with v_H vertices. For any integer $t \geqslant 2$,*

$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < (\sqrt{2} + o(1))t^{\frac{1}{2}}(v_H + t)^{\frac{t}{2}} n^{\frac{3}{2}}.$$

They also noted that a corollary of Füredi [6] is that, for H not bipartite,

$$\tfrac{1}{4}t^{\frac{1}{2}}n^{\frac{3}{2}} - \mathcal{O}_t\left(n^{\frac{4}{3}}\right) \leqslant \mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}).$$

In particular, for non-bipartite H, $\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) = \Theta_t(n^{3/2})$ but the correct growth rate in t lies between $\frac{1}{4}t^{1/2}n^{3/2}$ and $C_H t^{(t+1)/2} n^{3/2}$. We give a slightly more general result (expressing the upper bound for the induced Turán number in terms of a Ramsey number involving H – see Corollary 15 and Theorem 18) followed by an improvement to the general upper bound.

Theorem 9. *Fix a graph H with v_H vertices. For any integer $t \geqslant 1$,*

$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < (t+1)^{\frac{v_H - 1}{2}} n^{\frac{3}{2}},$$
$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < e^{\frac{v_H}{2} - 1} 2^{t-1} n^{\frac{3}{2}}.$$

The first bound shows that, for non-bipartite H, the correct growth rate in t is a polynomial in t times $n^{3/2}$. The second bound is better when t and v_H are of comparable size.

2 Notation, main result and organisation

If v is a vertex of a graph $G = (V, E)$ then $\Gamma(v) = \{u \in V : uv \in E\}$ is the neighbourhood of v. We set $G_v = G[\Gamma(v)]$. For a fixed graph H, let $\{H - x\}$ be the set of graphs obtained by removing a single vertex from H and let $\{H - \bar{e}\}$ be the set of graphs obtained from H by either removing a single vertex or two non-adjacent vertices. In particular the Ramsey number, $R(K_t, \{H - x\})$, is the least n such that any red-blue colouring of the edges of K_n contains either a red K_t or a blue graph which can be obtained from H by removing a single vertex.

Our main result is the following which applies for all values of α.

Theorem 10. *Fix a graph H. Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ ($t \geqslant 2$) and let $\beta = \beta_t(\alpha)$.*

If $R(K_t, \{H-x\}) \leqslant \beta^2 n$, then H is a subgraph of G. In particular, if $R(K_t, \{H-x\}) \leqslant \frac{t-1}{t^2} \cdot \alpha^2 n$, then H is a subgraph of G.

The sufficiency of $R(K_t, \{H - x\}) \leqslant \frac{t-1}{t^2} \cdot \alpha^2 n$ follows from the following lemma which relates β to α in a manageable way.

Lemma 11. *For all $\alpha \in [0, 1]$ and integers $t \geqslant 2$, $\beta = \beta_t(\alpha)$ satisfies*

$$\begin{aligned}
&(t-1)\big(\alpha - \beta^2\big)^2 = t^2(1-\alpha)\beta^2, \\
&\tfrac{\sqrt{t-1}}{t}\alpha \leqslant \beta \leqslant \alpha, \\
&\beta \to 1, \ \text{as } \alpha \to 1.
\end{aligned}$$

Proof. The equation $(t-1)(\alpha - \beta^2)^2 = t^2(1-\alpha)\beta^2$ is a quadratic in β^2. One can check that $\beta_t(\alpha)$ does indeed square to a solution of this quadratic.

Fix t and define the function $f(x) = \sqrt{1 - (1 - 2/t)^2 x} - \sqrt{1-x}$ for $x \in [0, 1]$. Then f is convex increasing with $f(0) = 0$ and $f(1) = \frac{2\sqrt{t-1}}{t}$. Thus $f(x) \leqslant \frac{2\sqrt{t-1}}{t}x$. Also the derivative of f at zero is $\frac{2}{t} - \frac{2}{t^2} = \frac{2(t-1)}{t^2}$ so $f(x) \geqslant \frac{2(t-1)}{t^2}x$. In particular $\beta = \frac{t}{2\sqrt{t-1}}f(\alpha)$ satisfies $\frac{\sqrt{t-1}}{t}\alpha \leqslant \beta \leqslant \alpha$.

Finally, f is continuous so, as α tends to 1, β tends to $\frac{t}{2\sqrt{t-1}}f(1) = 1$. □

We prove Theorem 10 in Section 5. Before that we use Ramsey estimates to obtain various corollaries. We normally give two versions of the results: one which holds for all values of n and a stronger bound which holds for large enough n (in terms of α). The latter is only really applicable in the regime where α is bounded away from zero.

In Section 3 we look at the special case where H is a complete graph, proving Theorems 4, 5 and 7. In Section 4 we consider general H for the Induced Turán problem (so α going to zero) and prove Theorem 9. Finally in Section 6 we exhibit a variation on our methods which gives a slight asymptotic improvement for the induced Turán number of H-free graphs with no induced $K_{2,t}$. This includes the observation that such graphs contain $\mathcal{O}(n^{27/14}) = o(n^2)$ triangles.

3 Clique numbers of graphs with no induced $K_{2,t}$

If we take $H = K_{r+1}$ in Theorem 10 then $\{H - x\} = \{K_r\}$ so Theorem 4 is immediate.

Theorem 4. *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ and let $\beta = \beta_t(\alpha)$. For any positive integer r with $R(t, r) \leqslant \beta^2 n$, we have $\omega(G) \geqslant r + 1$.*

Remark 12. The following example illustrates why we believe this result is in a sense tight as $\alpha \to 1$. Consider a graph G on n vertices which has no independent t-set and smallest possible clique number (a Ramsey-like graph): that is, $R(t, \omega(G) + 1) > n \geqslant R(t, \omega(G))$. Now G has no independent t-set so does not contain an induced $K_{2,t}$. If there are such graphs with $(1 - o(1))\binom{n}{2}$ edges then these form a sequence of graphs for which $\alpha \to 1$ (and so $\beta \to 1$), but for which the statement becomes false if β is actually replaced by 1.

We do believe that such graphs have $(1 - o(1))\binom{n}{2}$ edges. This would follow, for example, from $\frac{R(t-1,m)}{R(t,m)} \to 0$ as $m \to \infty$ (true for $t = 3$ and 4 by standard Ramsey bounds but not known in general): the non-neighbours of a vertex in such a graph, G, cannot contain an independent $(t-1)$-set, so there are at most $R(t-1, \omega(G)+1)$ non-neighbours, and so $\delta(G)$ would be $(1 - o(1))n$.

The following corollary for $t = 3$ contains Theorem 5.

Corollary 13. *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges which contains no induced $K_{2,3}$. Let $\beta = \beta_3(\alpha) = \frac{3}{2\sqrt{2}}\left[\sqrt{1 - \frac{\alpha}{9}} - \sqrt{1 - \alpha}\right]$. Then*

$$\omega(G) \geqslant \lfloor \beta\sqrt{2n} \rfloor \geqslant \left\lfloor \tfrac{2}{3}\alpha\sqrt{n} \right\rfloor \text{ for all } n, \text{ and}$$
$$\omega(G) \geqslant \beta\sqrt{\tfrac{1}{2}n\log n} + 2 \geqslant \tfrac{1}{3}\alpha\sqrt{n\log n} + 2 \text{ for large enough } n, \text{ say } n \geqslant \exp(2e^2\beta^{-2}).$$

Proof. Firstly, the theorem of Erdős and Szekeres [4] gives that $R(3, r) \leqslant \binom{r+1}{2}$ for all positive r. Thus $r = \lfloor \beta\sqrt{2n} \rfloor - 1$ satisfies $R(3, r) \leqslant \frac{1}{2}\lfloor \beta\sqrt{2n} \rfloor^2 \leqslant \beta^2 n$ and so Theorem 4 gives the first result.

Secondly, $R(3, r) \leqslant \frac{(r-2)^2}{\log(r-1)-1}$ for all $r \geqslant 4$ (a corollary of Shearer's result on independent sets in triangle-free graphs, [10]). Thus $r = \left\lfloor \beta\sqrt{\frac{1}{2}n\log n} \right\rfloor + 2$ satisfies $R(3, r) \leqslant \beta^2 n$ provided $n \geqslant \exp(2e^2\beta^{-2})$. □

The following corollary (which contains Theorem 7) for t larger than three is obtained in exactly the same way, using known bounds for $R(t, r)$. Improvements in the upper bounds on Ramsey numbers would improve the results.

Corollary 14. *Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ and let $\beta = \beta_t(\alpha)$. Then*

$$\omega(G) \geqslant \left\lfloor \tfrac{t-1}{e}(\beta^2 n)^{\frac{1}{t-1}} \right\rfloor - t + 3 \text{ and } \omega(G) \geqslant \left\lfloor \tfrac{t-1}{4}(\alpha^2 n)^{\frac{1}{t-1}} \right\rfloor - t + 3 \text{ for all } n, \text{ and}$$
$$\omega(G) \geqslant \tfrac{1}{20}\left(\beta^2 n\right)^{\frac{1}{t-1}}\left(\tfrac{\log n}{t-1}\right)^{1-\frac{1}{t-1}} \geqslant \tfrac{1}{20t}\left(\alpha^2 n(\log n)^{t-2}\right)^{\frac{1}{t-1}} \text{ for large enough } n \text{ in terms of } \beta.$$

Proof. The theorem of Erdős and Szekeres [4] gives that $R(t,r) \leqslant \binom{r+t-2}{t-1} \leqslant \frac{(r+t-2)^{t-1}}{(t-1)!}$ for all positive r. Thus $r = \lfloor \left(\beta^2 n(t-1)!\right)^{\frac{1}{t-1}} \rfloor - t + 2$ has $R(t,r) \leqslant \beta^2 n$ so, by Theorem 4,

$$\omega(G) \geqslant \lfloor \left(\beta^2 n(t-1)!\right)^{\frac{1}{t-1}} \rfloor - t + 3 \geqslant \lfloor \left(\tfrac{t-1}{t^2}\alpha^2 n(t-1)!\right)^{\frac{1}{t-1}} \rfloor - t + 3.$$

Furthermore $(t-1)! \geqslant \left(\frac{t-1}{e}\right)^{t-1}$ so $\left((t-1)!\right)^{\frac{1}{t-1}} \geqslant \frac{t-1}{e}$. That $\left(\frac{t-1}{t^2}(t-1)!\right)^{\frac{1}{t-1}} \geqslant \frac{t-1}{4}$ follows from $(t-1)! \geqslant \frac{(t-1)^{t-1/2}}{e^{t-1}}$ for $t \geqslant 4$ and can be checked directly for $t = 2, 3$.

Finally $R(t,r) \leqslant 2(20)^{t-3}\frac{r^{t-1}}{(\log r)^{t-2}}$ for r sufficiently large (see Bollobás [1, Thm 12.17]) so we obtain, for all large n, that

$$\omega(G) \geqslant \frac{1}{20}\left(\frac{\beta^2 n(\log n)^{t-2}}{(t-1)^{t-2}}\right)^{\frac{1}{t-1}} \geqslant \frac{1}{20}\left(\frac{\alpha^2 n(\log n)^{t-2}}{t^2(t-1)^{t-3}}\right)^{\frac{1}{t-1}}. \qquad \square$$

4 Turán number for no H and no induced $K_{2,t}$

We now focus on the regime where α goes to zero and consider the induced Turán numbers introduced by Loh, Tait, Timmons and Zhou.

Corollary 15. *Fix a graph H. For any integer $t \geqslant 2$,*

$$\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\}) < \tfrac{t}{2\sqrt{t-1}} R(K_t, \{H - x\})^{\frac{1}{2}} n^{\frac{3}{2}}.$$

Proof. Let G be a graph on n vertices containing no induced $K_{2,t}$ and no copy of H. By Theorem 10, $R(K_t, \{H-x\}) > \frac{t-1}{t^2} \cdot \alpha^2 n$ so $\alpha < \frac{t}{\sqrt{t-1}} n^{-\frac{1}{2}} R(K_t, \{H-x\})^{\frac{1}{2}}$. Therefore

$$e(G) = \alpha\binom{n}{2} < \tfrac{t}{2\sqrt{t-1}} R(K_t, \{H-x\})^{\frac{1}{2}} n^{\frac{1}{2}}(n-1). \qquad \square$$

We now use Theorem 7 and Corollary 15 to prove Theorem 9, restated here for convenience.

Theorem 9. *Fix a graph H with v_H vertices. For any integer $t \geqslant 1$,*

$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < (t+1)^{\frac{v_H - 1}{2}} n^{\frac{3}{2}},$$
$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < e^{\frac{v_H}{2} - 1} 2^{t-1} n^{\frac{3}{2}}.$$

Proof. Note that $R(K_t, \{H-x\}) \leqslant R(t+1, v_H - 1)$. For all positive integers a and b

$$\tbinom{a+b-2}{a-1} = \tfrac{a+b-2}{a-1} \cdot \tfrac{a+b-3}{a-2} \ \cdots \ \tfrac{b}{1} \leqslant b^{a-1},$$

and so Erdős and Szekeres's bound [4] gives $R(K_{t+1}, \{H-x\}) \leqslant (t+1)^{v_H - 2}$. By Corollary 15,

$$\mathrm{ex}(n, \{H, K_{2,t+1}\text{-ind}\}) < \tfrac{t+1}{2\sqrt{t}}(t+1)^{\frac{v_H}{2} - 1} n^{\frac{3}{2}} < (t+1)^{\frac{v_H - 1}{2}} n^{\frac{3}{2}}.$$

Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges and no induced $K_{2,t+1}$. If G does not contain H then $\omega(G) < v_H$ so, by Theorem 7, $v_H > \lfloor \frac{t}{4}(\alpha^2 n)^{\frac{1}{t}} \rfloor - t + 2$. $v_H + t - 2$ is an integer so

$$v_H + t - 2 > \tfrac{t}{4}(\alpha^2 n)^{\frac{1}{t}}.$$

Now rearranging and using $e(G) = \alpha\binom{n}{2} < \frac{\alpha}{2}n^2$ we get

$$e(G) < n^{\frac{3}{2}} 2^{t-1}\left(1 + \tfrac{v_H - 2}{t}\right)^{\frac{t}{2}} < e^{\frac{v_H}{2}-1} 2^{t-1} n^{\frac{3}{2}}. \qquad \square$$

5 Proof of main result

For convenience we restate the main result here. As mentioned earlier, the proof is motivated by that of Holmsen [8].

Theorem 10. *Fix a graph H. Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$ $(t \geqslant 2)$ and let $\beta = \beta_t(\alpha)$.*

If $R(K_t, \{H-x\}) \leqslant \beta^2 n$, then H is a subgraph of G. In particular, if $R(K_t, \{H-x\}) \leqslant \frac{t-1}{t^2} \cdot \alpha^2 n$, then H is a subgraph of G.

Proof. By Lemma 11, for $\alpha \in [0,1]$ we have $0 \leqslant \beta \leqslant \alpha \leqslant 1$ and also $\frac{t-1}{t^2}\left(\alpha - \beta^2\right)^2 = (1-\alpha)\beta^2$.

Suppose that G does not contain H. Let the set of missing edges in G be $M = \binom{V(G)}{2} - E(G)$, which has size $(1-\alpha)\binom{n}{2}$. For each $v \in V(G)$, let

m_v be the total number of missing edges in G_v,
$\bar{\Delta}_1, \ldots, \bar{\Delta}_{\gamma_v}$ be a maximal collection of pairwise vertex-disjoint independent t-sets in G_v.

By the maximality of γ_v, $G[\Gamma(v) \setminus \cup_j \bar{\Delta}_j]$ does not contain an independent t-set. Furthermore it does not contain any $H - x$ (else together with v we have a copy of H in G). Thus

$$\begin{aligned} &R(K_t, \{H-x\}) - 1 \geqslant |\Gamma(v)| - t\gamma_v = \deg(v) - t\gamma_v, \text{ and so} \\ &\gamma_v \geqslant \tfrac{1}{t}[\deg(v) - R(K_t, \{H-x\}) + 1] \geqslant \tfrac{1}{t}[\deg(v) - \beta^2(n-1)]. \end{aligned} \tag{1}$$

G contains no induced $K_{2,t}$ so at most one vertex in $\bar{\Delta}_i$ is adjacent to all of $\bar{\Delta}_j$ (for any $i \neq j$). In particular, between $\bar{\Delta}_i$ and $\bar{\Delta}_j$ there must be at least $t-1$ missing edges. These missing edges are in no $\bar{\Delta}_k$ (by vertex-disjointness) and each such edge corresponds to only one pair $(\bar{\Delta}_i, \bar{\Delta}_j)$. Considering these missing edges as well as the ones contained entirely in each $\bar{\Delta}_k$ gives

$$m_v \geqslant \tbinom{t}{2}\gamma_v + (t-1)\tbinom{\gamma_v}{2} = q(\gamma_v),$$

where

$$q(x) = \tfrac{t-1}{2} \cdot x(x + t - 1)$$

is convex and increasing for non-negative x. Averaging (1) over $v \in G$ we have

$$\frac{1}{n}\sum_{v\in G}\gamma_v \geqslant \tfrac{1}{tn}[2e(G)-\beta^2 n(n-1)] = \tfrac{1}{t}\big(\alpha-\beta^2\big)(n-1).$$

Using Jensen, the monotonicity of q, and the fact that $\alpha \geqslant \beta \geqslant \beta^2$ gives

$$\begin{aligned}\frac{1}{n}\sum_{v\in G} m_v \geqslant \frac{1}{n}\sum_{v\in G} q(\gamma_v) \geqslant q\left(\frac{1}{n}\sum_{v\in G}\gamma_v\right) &\geqslant q\big(\tfrac{1}{t}\big(\alpha-\beta^2\big)(n-1)\big)\\ &= \tfrac{t-1}{2}\cdot\tfrac{1}{t}\big(\alpha-\beta^2\big)(n-1)\cdot\big(\tfrac{1}{t}\big(\alpha-\beta^2\big)(n-1)+t-1\big)\\ &\geqslant \tfrac{t-1}{2}\cdot\tfrac{1}{t}\big(\alpha-\beta^2\big)(n-1)\cdot\tfrac{1}{t}\big(\alpha-\beta^2\big)n\\ &= \tfrac{t-1}{t^2}\big(\alpha-\beta^2\big)^2\cdot\tbinom{n}{2}\\ &= \beta^2(1-\alpha)\tbinom{n}{2}.\end{aligned}$$

Now $\sum_{v\in G} m_v = \sum_{\bar{e}\in M}\#\{v \text{ with } \bar{e}\subset\Gamma(v)\}$ and $|M| = (1-\alpha)\binom{n}{2}$ so there is $\bar{e}\in M$ and $S\subset V(G)$ of size at least $\beta^2 n$ such that $\bar{e}\subset\Gamma(v)$ for each $v\in S$: that is, all vertices of S are in the common neighbourhood of the two end-vertices of the missing edge $\bar{e}$.

Now $G[S]$ contains no independent t-set (else together with $\bar{e}$ we have an induced $K_{2,t}$) and $|S| \geqslant \beta^2 n \geqslant R(K_t,\{H-x\})$ so $G[S]$ contains a copy of some $H-x$. Together with one end-vertex of $\bar{e}$ we have a copy of H in G. □

Remark 16. It is natural to ask whether the ideas of this argument could be extended to graphs which contain no induced $K_{s,t}$. The argument above is so clean partly because the number of independent 2-sets in G is determined by α (it is $|M| = (1-\alpha)\binom{n}{2}$). Extending to no induced $K_{s,t}$ would require some knowledge of the number of independent s-sets in G.

6 Improvement when there are few triangles

Corollary 15 says $\mathrm{ex}(n,\{H, K_{2,t}\text{-ind}\}) < \frac{t}{2\sqrt{t-1}}R(K_t,\{H-x\})^{\frac{1}{2}}n^{\frac{3}{2}}$. In this section we show that n-vertex H-free graphs with no induced $K_{2,t}$ contain $o(n^2)$ triangles. This asymptotically improves our lower bound on the number of missing edges in each neighbourhood and so improves Corollary 15 by a factor of $\sqrt{t}$ as well as reducing the Ramsey number used – see Theorem 18.

Theorem 17. *Fix a graph H and an integer $t \geqslant 2$. Every n-vertex graph which contains no copy of H and no induced $K_{2,t}$ has at most $\mathcal{O}(n^{27/14})$ triangles.*

Proof. By Corollary 15, there is a constant $C = C_{H,t}$ such that every m-vertex graph which contains no copy of H and no induced $K_{2,t}$ has at most $Cm^{3/2}$ edges.

Let G be a graph on n vertices containing no induced $K_{2,t}$ and no copy of H. For each vertex v of G, note that exactly $e(G_v)$ triangles in G contain v. As G has no copy of H

and no induced $K_{2,t}$,

$$e(G) \leqslant Cn^{3/2},$$
$$e(G_v) \leqslant C\deg(v)^{3/2}.$$

Let X be the set of vertices in G whose degree is at least $f(n)$ (a function of n whose value we give later). Firstly,

$$|X|f(n) \leqslant \sum_{v\in X}\deg(v) \leqslant 2e(G) \leqslant 2Cn^{3/2},$$

and so the number of triangles in G whose vertices are all in X is at most

$$\tbinom{|X|}{3} \leqslant \tfrac{1}{6}|X|^3 \leqslant \tfrac{4}{3}C^3n^{9/2}f(n)^{-3}.$$

The number of triangles of G containing at least one vertex in $V(G)\setminus X$ is at most

$$\sum_{v\notin X} e(G_v) \leqslant C\sum_{v\notin X}\deg(v)^{3/2}.$$

The function $x \mapsto x^{3/2}$ is convex and all $v \notin X$ satisfy $\deg(v) \leqslant f(n)$, so

$$\sum_{v\notin X}\deg(v)^{3/2} \leqslant \left(f(n)^{-1}\sum_{v\notin X}\deg(v)\right)f(n)^{3/2} = f(n)^{1/2}\sum_{v\notin X}\deg(v)$$
$$\leqslant 2f(n)^{1/2}e(G) \leqslant 2Cn^{3/2}f(n)^{1/2}.$$

Thus, the number of triangles in G is at most

$$\tfrac{4}{3}C^3n^{9/2}f(n)^{-3} + 2C^2n^{3/2}f(n)^{1/2}.$$

We minimise this last expression by taking $f(n) = 2^{4/7}C^{2/7}n^{6/7}$ which gives a value less than $3C^{15/7}n^{27/14}$. $\square$

Theorem 18. *Fix a graph H and an integer $t \geqslant 2$. Let $\Delta(n,H,t)$ denote the greatest number of triangles in a graph on n vertices containing no copy of H and no induced $K_{2,t}$. Let G be a graph on n vertices with $\alpha\binom{n}{2}$ edges containing no induced $K_{2,t}$. If*

$$\alpha^2(n-1) > R(K_t, \{H-\bar{e}\}) - 1 + 3\Delta(n,H,t)\tbinom{n}{2}^{-1},$$

then H is a subgraph of G. In particular,

$$\mathrm{ex}(n, \{H, K_{2,t}\text{-ind}\}) \leqslant \tfrac{1}{2}\big(R(K_t,\{H-\bar{e}\}) - 1 + o(1)\big)^{\frac{1}{2}}n^{\frac{3}{2}}.$$

Proof. $R(K_t,\{H-\bar{e}\}) \geqslant 2$ so we in fact have

$$\alpha[\alpha(n-1)-1] > (1-\alpha)(R(K_t,\{H-\bar{e}\})-1) + 3\Delta(n,H,t)\tbinom{n}{2}^{-1}.$$

We will use this to show H is a subgraph of G. Suppose for contradiction it is not. Let the set of missing edges in G be $M = \binom{V(G)}{2} - E(G)$ which has size $(1-\alpha)\binom{n}{2}$. For each $v \in V(G)$ let

$$\begin{aligned} e_v &= e(G_v), \\ m_v &= \text{ total number of missing edges in } G_v. \end{aligned}$$

First note that $e_v + m_v = \binom{|\Gamma(v)|}{2} = \binom{\deg(v)}{2}$, so, by Jensen's inequality,

$$\sum_{v\in G}(m_v + e_v) \geqslant n\binom{2e(G)/n}{2} = n\binom{\alpha(n-1)}{2} = \alpha[\alpha(n-1)-1]\binom{n}{2}.$$

Now e_v is also the number of triangles in G containing v so $\sum_{v\in G} e_v$ is three times the number of triangles in G which is at most $3\Delta(n,H,t)$. Thus

$$\sum_{v\in G} m_v \geqslant \alpha[\alpha(n-1)-1]\binom{n}{2} - 3\Delta(n,H,t) > (1-\alpha)\binom{n}{2}(R(K_t,\{H-\bar{e}\})-1).$$

Now $\sum_{v\in G} m_v = \sum_{\bar{e}\in M} \#\{v \text{ with } \bar{e} \subset \Gamma(v)\}$ and $|M| = (1-\alpha)\binom{n}{2}$ so there is some missing edge $\bar{e}$ and some $S \subset V(G)$ of size $R(K_t,\{H-\bar{e}\})$ with $\bar{e} \subset \Gamma(v)$ for each $v \in S$. $G[S]$ does not contain an independent t-set (else together with $\bar{e}$ we have an induced $K_{2,t}$ in G) so $G[S]$ contains a copy of some $H-x$ or some $H-\bar{e}$. Together with $\bar{e}$ we have that G contains a copy of H proving the first result.

By Theorem 17, $\Delta(n,H,t) = o(n^2)$. Suppose that G is a graph on n vertices with no H and no induced $K_{2,t}$. We must have

$$\alpha^2(n-1) \leqslant R(K_t,\{H-\bar{e}\}) - 1 + o(1).$$

Using $e(G) = \alpha\binom{n}{2}$ we get the required result. □

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