Conditions for a bigraph to be super-cyclic

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Abstract

A hypergraph $\mathcal{H}$ is *super-pancyclic* if for each $A \subseteq V(\mathcal{H})$ with $|A| \geqslant 3$, $\mathcal{H}$ contains a Berge cycle with base vertex set A. We present two natural necessary conditions for a hypergraph to be super-pancyclic, and show that in several classes of hypergraphs these necessary conditions are also sufficient. In particular, they are sufficient for every hypergraph $\mathcal{H}$ with $\delta(\mathcal{H}) \geqslant \max\{|V(\mathcal{H})|, \frac{|E(\mathcal{H})|+10}{4}\}$.

We also consider *super-cyclic* bipartite graphs: (X, Y)-bigraphs G such that for each $A \subseteq X$ with $|A| \geqslant 3$, G has a cycle C_A such that $V(C_A) \cap X = A$. Super-cyclic graphs are incidence graphs of super-pancyclic hypergraphs, and our proofs use the language of such graphs.

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1 Introduction

1.1 Longest cycles in bipartite graphs and hypergraphs

For positive integers n, m, and δ with $\delta \leqslant m$, let $\mathcal{G}(n,m,\delta)$ denote the set of all bipartite graphs with a bipartition (X,Y) such that $|X| = n \geqslant 2, |Y| = m$ and for every $x \in X$, $d(x) \geqslant \delta$. In 1981, Jackson [4] proved that if $\delta \geqslant \max\{n, \frac{m+2}{2}\}$, then every graph $G \in \mathcal{G}(n,m,\delta)$ contains a cycle of length $2n$, i.e., a cycle that contains all vertices of X. This result is sharp. Jackson also conjectured that if $G \in \mathcal{G}(n,m,\delta)$ is 2-connected, then the lower bound on δ can be weakened.

Conjecture 1 (Jackson [4])**.** Let m, n, δ be integers. If $\delta \geqslant \max\{n, \frac{m+5}{3}\}$, then every 2-connected graph $G \in \mathcal{G}(n,m,\delta)$ contains a cycle of length $2n$.

Recently, the conjecture was proved in [8]. The restriction $\delta \geqslant \frac{m+5}{3}$ cannot be weakened any further because of the following example:

Example 2. Let $n_1 \geqslant n_2 \geqslant n_3 \geqslant 1$ be such that $n_1 + n_2 + n_3 = n$. Let $G_3(n_1, n_2, n_3) \in \mathcal{G}(n, 3\delta - 4, \delta)$ be the bipartite graph obtained from $K_{\delta-2,n_1} \cup K_{\delta-2,n_2} \cup K_{\delta-2,n_3}$ by adding two vertices a and b that are both adjacent to every vertex in the parts of size n_1, n_2, and n_3. Then a longest cycle in $G_3(n_1, n_2, n_3)$ has length $2(n_1 + n_2) \leqslant 2(n-1)$.

Very recently [9], the bound was refined for 3-connected graphs in $G \in \mathcal{G}(n,m,\delta)$.

Theorem 3 ([9])**.** *Let m, n, δ be integers. If $\delta \geqslant \max\{n, \frac{m+10}{4}\}$, then every 3-connected graph $G \in \mathcal{G}(n,m,\delta)$ contains a cycle of length $2n$.*

A construction very similar to Construction 2 shows that the bound $\frac{m+10}{4}$ is sharp.

The results can be translated into the language of hypergraphs and hamiltonian Berge cycles.

Recall that a *hypergraph* $\mathcal{H}$ is a set of vertices $V(\mathcal{H})$ and a set of edges $E(\mathcal{H})$ such that each edge is a subset of $V(\mathcal{H})$. We consider hypergraphs with edges of any size. The degree, $d(v)$, of a vertex v is the number of edges that contain v.

A *Berge cycle* of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \dots, v_\ell\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that for every $i \in [\ell]$, $v_i, v_{i+1} \in e_i$ (indices are taken modulo ℓ). The vertices $v_1, \dots, v_\ell$ are the *base vertices* of the cycle.

Naturally, a *hamiltonian Berge cycle* in a hypergraph $\mathcal{H}$ is a Berge cycle whose set of base vertices is $V(\mathcal{H})$.

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph. The *incidence graph* of $\mathcal{H}$ is the bipartite graph $I(\mathcal{H})$ with parts (X,Y) where $X = V(\mathcal{H})$, $Y = E(\mathcal{H})$ such that for $e \in Y, v \in X$, $ev \in E(I(\mathcal{H}))$ if and only if the vertex v is contained in the edge e in $\mathcal{H}$.

If $\mathcal{H}$ has n vertices, m edges and minimum degree at least δ, then $I(\mathcal{H}) \in \mathcal{G}(n,m,\delta)$. There is a simple relation between the cycle lengths in a hypergraph $\mathcal{H}$ and its incidence graph $I(\mathcal{H})$: If $\{v_1, \dots, v_\ell\}$ and $\{e_1, \dots, e_\ell\}$ form a Berge cycle of length ℓ in $\mathcal{H}$, then $v_1 e_1 \dots v_\ell e_\ell v_1$ is a cycle of length 2ℓ in $I(\mathcal{H})$, and vice versa.

1.2 Super-pancyclic hypergraphs and super-cyclic bigraphs

Recall that an n-vertex graph is *pancyclic* if it contains a cycle of length ℓ for every $3 \leqslant \ell \leqslant n$. There are a number of interesting results on pancyclic graphs, see e.g. survey [10]. A similar notion for hypergraphs and a strengthening of it were recently considered in [8].

A hypergraph $\mathcal{H}$ is *pancyclic* if it contains a Berge cycle of length ℓ for every $\ell \geqslant 3$. Furthermore, $\mathcal{H}$ is *super-pancyclic* if for every $A \subseteq V(\mathcal{H})$ with $|A| \geqslant 3$, $\mathcal{H}$ has a Berge cycle whose set of base vertices is A.

While the notion of super-pancyclic graphs is useless, since only complete graphs have this property, the notion for general hypergraphs is nontrivial. For example, Jackson's proof [4] that for $\delta \geqslant \max\{n, \frac{m+2}{2}\}$, each graph $G \in \mathcal{G}(n, m, \delta)$ has a cycle of length $2n$ yields a stronger statement. In the language of hypergraphs, it implies the following.

Theorem 4 ([4])**.** *If $\delta \geqslant \max\{n, \frac{m+2}{2}\}$, then every n-vertex hypergraph with m edges and minimum degree at least δ is super-pancyclic.*

It is interesting to find broader conditions guaranteeing that a hypergraph is super-pancyclic. The notion of super-pancyclicity translates into the language of bipartite graphs as follows.

By an (X, Y)-bigraph we mean a bipartite graph G with a specified ordered bipartition (X, Y). An (X, Y)-bigraph is *super-cyclic* if for every $X' \subseteq X$ with $|X'| \geqslant 3$, G has a cycle C with $V(C) \cap X = X'$; we say that C *is based on* X'.

To state necessary conditions for an (X, Y)-bigraph to be super-cyclic, we need a new notion. For $A \subseteq X$, the *super-neighborhood* $\widehat{N}(A)$ is the set $\{y \in Y : |N(y) \cap A| \geqslant 2\}$.

If G is a super-cyclic (X, Y)-bigraph, $A \subseteq X$, and C is a cycle based on A, let $B = V(C) \cap Y$. Then $B \subseteq \widehat{N}(A)$ and $G[A \cup B]$ is 2-connected. Since adding a vertex of degree at least 2 to a 2-connected graph keeps the graph 2-connected, we conclude that every super-cyclic bigraph satisfies the following[1].

$$\text{For each } A \subseteq X \text{ with } |A| \geqslant 3\text{:} \begin{cases} |\widehat{N}(A)| \geqslant |A|, \text{ and} \\ G[A \cup \widehat{N}(A)] \text{ is 2-connected.} \end{cases} \tag{1}$$

We conjecture that these necessary conditions for a bigraph to be super-cyclic are also sufficient.

Conjecture 5. If G is an (X, Y)-bigraph satisfying (1), then G is super-cyclic.

To give partial support for Conjecture 5, let us somewhat refine the notion of super-cyclic bigraphs.

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[1]Jaehoon Kim [6] observed that to check condition (1), it is sufficient to verify that $G[A \cup \widehat{N}(A)]$ is 2-connected only when $|A| = 3$, though $|\widehat{N}(A)| \geqslant |A|$ still needs to be checked for all A. When $|A| > 3$, if $G[A \cup \widehat{N}(A)]$ is not 2-connected, there is a subset $A' \subseteq A$ with $|A'| = 3$ for which $G[A' \cup \widehat{N}(A')]$ is also not 2-connected.

For an integer $k \geqslant 3$, a bipartite graph G with partition (X, Y) is *k-cyclic* if for every $X' \subseteq X$ with $|X'| = k$, G has a cycle C that is based on X'. If G is k-cyclic for all $3 \leqslant k \leqslant |X|$, then it is super-cyclic.

In a series of claims, we prove the following.

Theorem 6. *If G is an (X, Y)-bigraph satisfying* (1)*, then G is k-cyclic for $k = 3, 4, 5, 6$.*

Another result supporting Conjecture 5 was proved in [8] (in slightly different terms).

Theorem 7 ([8])**.** *Let $\delta \geqslant \max\{n, \frac{m+5}{3}\}$. If $G \in \mathcal{G}(n, m, \delta)$ satisfies* (1)*, then G is super-cyclic.*

We use Theorem 6 and the ideas of the proof of Theorem 3 to strengthen Theorem 7 as follows.

Theorem 8. *Let $\delta \geqslant \max\{n, \frac{m+10}{4}\}$. If $G \in \mathcal{G}(n, m, \delta)$ satisfies* (1)*, then G is super-cyclic.*

In terms of hypergraphs, our result is as follows.

Corollary 9 (Hypergraph version of Theorem 8)**.** *Let $\delta \geqslant \max\{n, \frac{m+10}{4}\}$. If the incidence graph of an n-vertex hypergraph $\mathcal{H}$ with m edges and minimum degree $\delta(\mathcal{H})$ satisfies* (1)*, then $\mathcal{H}$ is super-pancyclic.*

We present the main proofs in the language of bipartite graphs. We will say that an (X, Y)-bigraph G is *critical* if the following conditions hold:

(a) G satisfies (1) but is not super-cyclic,

(b) $\widehat{N}(X) = Y$, and

(c) for every $X' \subset X$ with $X' \neq X$, $G[X' \cup Y]$ is super-cyclic.

Note that every graph satisfying (1) is either super-cyclic or has a critical subgraph.

Furthermore, we say that a critical (X, Y)-bigraph G is *saturated* if, after adding any X, Y-edge to G, the resulting graph is super-cyclic.

In Section 2 we prove basic properties of critical bigraphs. Based on this, in Section 3 we prove Theorem 6 for $k = 3, 4$, and 5. In Section 4 we discuss saturated critical graphs, which will be useful in the last two sections. In Section 5 we prove Theorem 6 for $k = 6$. In Section 6 we prove Theorem 8.

2 Properties of critical bigraphs

For all (X, Y)-bigraphs G below we assume $|X| \geqslant 3$, since G is trivially super-cyclic when $|X| \leqslant 2$.

Lemma 10. *Suppose that an (X, Y)-bigraph G satisfies* (1)*. Then $|N(x) \cap N(x')| \geqslant 1$ for all distinct $x, x' \in X$.*

Proof. Let x'' be any vertex in $X - \{x, x'\}$ and $A = \{x, x', x''\}$. If $N(x) \cap N(x') = \varnothing$, then $G[A \cup \widehat{N}(A)] - x''$ has no x, x'-path, contradicting (1). $\square$

Claim 11. *Let G be a critical (X, Y)-bigraph. Then G is 2-connected.*

Proof. This is by the fact that $Y = \widehat{N}(X)$ and by (1). $\square$

Recall that for a vertex $v \in V(G)$ and a set $U \subseteq V(G)$, a *v, U-fan of size t* is a set of t paths from v to U such that the only common vertex of any two distinct paths is v. In view of Claim 11, the classical Dirac's Fan Lemma [3, 11] implies the following fact.

Lemma 12. *Let G be a critical (X, Y)-bigraph, $v \in V(G)$, and $U \subseteq V(G)$ with $|U| \geqslant 2$. Then G has 2 paths from v to U having only the vertex v in common.*

Let G be a critical (X, Y)-bigraph with $|X| = k + 1$ and $x_0 \in X$, where $k \geqslant 2$. Note that if $k < 2$, then G cannot be critical, since it is trivially super-cyclic. By definition, $G - \{x_0\}$ is super-cyclic. In particular, it has a cycle $C = x_1y_1x_2y_2 \dots x_ky_kx_1$ based on $X - \{x_0\}$. We index the vertices of C modulo k; for example, $x_{k+1} = x_1$. We derive some properties of such triples (G, x_0, C).

Claim 13. *For all $y_i, y_j \in N(x_0)$, x_i and x_j have no common neighbor outside C. Similarly, x_{i+1} and x_{j+1} have no common neighbor outside C.*

Proof. If x_i and x_j have a common neighbor $y \notin V(C)$, then the cycle

$$x_1y_1 \dots x_iyx_jy_{j-1} \dots y_ix_0y_jx_{j+1} \dots x_1$$

is based on X, contrary to assumption. If x_{i+1} and x_{j+1} have such a common neighbor, consider the cycle C in reverse and apply the same argument. $\square$

Claim 14. *For every $y_i \in N(x_0)$, x_i and x_0 have no common neighbor outside C; similarly, x_{i+1} and x_0 have no common neighbor outside C.*

Proof. If x_i and x_0 have a common neighbor $y \notin V(C)$, then we may extend C to a cycle based on X by replacing the edge x_iy_i with the path $x_iyx_0y_i$. The proof for x_{i+1} is similar. $\square$

Claim 15. *For every i, if x_i has a common neighbor y with x_0 outside C, then x_{i+1} has no common neighbor with x_0 outside C, except possibly for y.*

Proof. If x_{i+1} and x_0 have a common neighbor $y' \notin V(C)$, with $y' \neq y$, then we may extend C to a cycle based on X by replacing the path $x_iy_ix_{i+1}$ with the path $x_iyx_0y'x_{i+1}$. $\square$

Lemma 16. *The vertex x_0 has at least two neighbors in C.*

Proof. Let A be the subset of X consisting of x_0, together with all x_i that do not have a common neighbor with x_0 outside C.

If $|A| \geqslant 3$, then $G[A \cup \widehat{N}(A)]$ is 2-connected by (1), so x_0 has at least two neighbors in $\widehat{N}(A)$. Each of these neighbors must also be adjacent to at least one vertex in $A - \{x_0\}$. By our choice of A, these neighbors must be in C, and we are done.

If $|A| \leqslant 2$, then x_0 has a common neighbor outside C with all but at most one of $x_1, x_2, \ldots, x_k$. By Claim 15, two consecutive vertices x_i, x_{i+1} cannot have different common neighbors with x_0 outside C. Therefore there is a vertex y_0 outside C adjacent to x_0 and to all but at most one of $x_1, x_2, \ldots, x_k$.

By Claim 11, $d(x_0) \geqslant 2$. So there are two possibilities:

- If x_0 has a neighbor y_i in C, then at least one of x_i or x_{i+1} is adjacent to y_0; then it has a common neighbor with x_0 outside C, contradicting Claim 14.

- If x_0 has a neighbor y_0' outside C, then y_0' has a neighbor x_i in C because $\delta(G) \geqslant 2$. By Claim 15, x_{i-1} and x_{i+1} cannot have common neighbors with x_0 outside C except possibly for y_0'. If $x_{i-1} \neq x_{i+1}$ then at least one of them is adjacent to y_0, which is a contradiction. Otherwise, if $x_{i-1} = x_{i+1}$ then $k = 2$ and $|X| = 3$. If x_0 has a neighbor y_0'' that is adjacent to x_{i-1}, then we have a contradiction. If there is no such neighbor, then let $A = \{x_0, x_i, x_{i-1}\}$. We see that $G[A \cup \widehat{N}(A)]$ is not 2-connected, a contradiction.

Therefore the case $|A| \leqslant 2$ is impossible, completing the proof. □

3 3-, 4-, and 5-cyclic graphs

Theorem 6 makes four claims: for $k = 3, 4, 5, 6$. In this section, we prove three of them.

Claim 17. *All (X, Y)-bigraphs G satisfying (1) are 3-cyclic.*

Proof. Suppose the claim is false and take a vertex-minimal counter-example, so that $|X| = 3$ and $Y = \widehat{N}(X)$. Then G is critical. By Claim 11, G is 2-connected, so it contains a cycle.

Suppose the longest cycle $C = x_1y_1x_2y_2x_1$ of G has 4 vertices and does not include the vertex x_3. By Lemma 12, there are 2 paths from x_3 to $V(C)$ having only x_3 in common. Then G would contain a cycle of length 6 unless the paths are just x_3y_1 and x_3y_2. Suppose that $y_3 \in Y$ (note that $|Y| \geqslant 3$ by (1)). Again, by Lemma 12, there are 2 paths from y_3 to $V(C)$ having only y_3 in common. Then G would contain a cycle of length 6 unless the paths are just y_3x_1 and y_3x_2. Then we get a 6-cycle $x_1y_1x_3y_2x_2y_3x_1$. □

Claim 18. *All (X, Y)-bigraphs G satisfying (1) are 4-cyclic.*

Proof. Suppose the claim is false and take a vertex-minimal counter-example, so that $|X| = 4$ and $Y = \widehat{N}(X)$. Then G is critical. Let $x_0 \in X = \{x_0, x_1, x_2, x_3\}$ have maximum degree. By Claim 17, $G - \{x_0\}$ has a 6-cycle $C = x_1y_1x_2y_2x_3y_3x_1$.

Case 1: x_0 has a neighbor y_0 outside of C. By Lemma 16, x_0 is adjacent to at least two of $\{y_1, y_2, y_3\}$; since $\delta(G) \geqslant 2$, y_0 is adjacent to at least one of $\{x_1, x_2, x_3\}$. Then there is an edge of C both of whose endpoints are adjacent to x_0 or y_0; without loss of generality, it's x_1y_1. We can replace x_1y_1 by $x_1y_0x_0y_1$, extending C.

Case 2: All of x_0's neighbors are in C. Then the neighbors of x in C have degree at least 3, and all other vertices in Y at least 2. Since $|Y| \geqslant |X|$, there are also vertices of X with degree 3, and since x_0 was chosen to have maximum degree in X, its degree is at least 3. Therefore x_0 is adjacent to all of $\{y_1, y_2, y_3\}$.

Since $|Y| \geqslant 4$, there is $y_0 \in Y$ outside C. By Claim 11, y_0 has at least two neighbors in X, and neither of them is x_0. Without loss of generality, y_0 is adjacent to x_1 and x_2, and so G has a cycle $x_0y_1x_2y_0x_1y_3x_3y_2x_0$.

In both cases, we get an 8-cycle, a contradiction. □

Claim 19. *All* (X, Y)*-bigraphs* G *satisfying* (1) *are* 5*-cyclic.*

Proof. Suppose the claim is false and take a vertex-minimal counter-example, so that $|X| = 5$ and $Y = \widehat{N}(X)$. Then G is critical. Let $x_0 \in X = \{x_0, x_1, x_2, x_3, x_4\}$ have maximum degree. By Claim 18, $G - \{x_0\}$ has an 8-cycle $C = x_1y_1x_2y_2x_3y_3x_4y_4$.

Case 1: x_0 has a neighbor y_0 outside of C. By Lemma 16, x_0 is adjacent to at least two of $\{y_1, y_2, y_3, y_4\}$; since $\delta(G) \geqslant 2$, y_0 is adjacent to at least one of $\{x_1, x_2, x_3, x_4\}$. In almost all cases, there is an edge of C both of whose endpoints are adjacent to x_0 or y_0, in which case we are done as before. The remaining case is unique up to relabeling C; without loss of generality, x_0 is adjacent to y_1 and y_2 and y_0 is adjacent to x_4.

If x_3y_4 is an edge, then there is a 10-cycle $x_1y_1x_2y_2x_0y_0x_4y_3x_3y_4x_1$, and similarly there is a 10-cycle if x_1y_3 is an edge. If neither is an edge, then $\widehat{N}(\{x_0, x_1, x_3\})$ contains y_1 and y_2, but not y_3 or y_4, so it needs a third vertex (call it y_5) which is outside C, adjacent to x_1 and either to x_0 or to x_3. In either case, we get a 10-cycle: one of

$$x_1y_5x_3y_2x_2y_1x_0y_0x_4y_4x_1 \text{ or } x_1y_5x_0y_1x_2y_2x_3y_3x_4y_4x_1.$$

Note that in Case 1, we did not use that x_0 has maximum degree.

Case 2: All of x_0's neighbors are in C. In this case, as before, we argue that x_0 must have degree at least 3. Say x_0 is adjacent to $\{y_1, y_2, y_3\}$; we make no assumption about whether x_0 is adjacent to y_4.

We can replace x_2 or x_3 by x_0 to get new cycles using the same vertices y_1, y_2, y_3, y_4 of Y. If x_2 or x_3 has a neighbor other than y_1, y_2, y_3, y_4, then we can apply Case 1.

So all the other vertices of Y (and there must be at least one) must be adjacent only to x_1 and x_4. Since they can be swapped in for y_4 to get a new cycle, if y_4 is adjacent to any of x_0, x_2, x_3, we can also reduce to a cycle C where Case 1 applies. Therefore y_4 is also adjacent only to x_1 and x_4.

But now $\widehat{N}(\{x_0, x_1, x_2, x_3\}) = \{y_1, y_2, y_3\}$ which violates (1). In all cases, we get a contradiction. □

4 Saturated critical bigraphs

Recall that a critical (X,Y)-bigraph G is *saturated* if adding to G any X,Y-edge results in a super-cyclic bigraph.

Lemma 20. *If G is a saturated critical (X,Y)-bigraph, then for every $y \in Y$, $|N(y)| \neq |X|-1$.*

Proof. Suppose G is a saturated critical (X,Y)-bigraph, and for $y_0 \in Y$ and $x_0 \in X$ we have $N(y_0) = X - \{x_0\}$. Since G is critical, $G - \{x_0\}$ is super-cyclic, but G has no cycles based on X. Let $|X| = k$. Since G is saturated, $G + y_0x_0$ has a $2k$-cycle $y_0x_1y_1x_2y_2 \dots x_ky_0$ where $x_k = x_0$. Then G contains path $P = y_0x_1y_1x_2 \dots x_k$.

By the choice of y_0, $\{x_1, \dots, x_{k-1}\} \subseteq N(y_0)$. Thus if x_k is adjacent to any y_j for $1 \leqslant j \leqslant k-2$, then G has cycle $x_ky_jx_jy_{j-1} \dots y_0x_{j+1}y_{j+1} \dots x_k$, a contradiction. Hence x_k has only one neighbor on P. Let $N_G(x_k) = \{y_{k-1}, z_1, z_2, \dots, z_s\}$. Since G is 2-connected, $s \geqslant 1$. Again, if any z_i is adjacent to any x_j for $j \leqslant k-2$, then G has cycle $x_kz_ix_jy_{j-1} \dots y_0x_{j+1}y_{j+1} \dots x_k$, a contradiction. Hence $N(z_i) = \{x_{k-1}, x_k\}$ for all $1 \leqslant i \leqslant s$. Switching z_1 with y_{k-1} we conclude that $N(y_{k-1}) = \{x_{k-1}, x_k\}$. So, the only vertex of $X - \{x_k\}$ at distance 2 from x_k is x_{k-1}, a contradiction to Lemma 10. □

Lemma 21. *If G is a saturated critical (X,Y)-bigraph and some $x_0 \in X$ has degree 2, then*

(a) each of its neighbors is adjacent to all vertices in X, and

(b) $d(x) \geqslant 4$ for every $x \in X - \{x_0\}$.

In particular, at most one vertex in X has degree 2.

Proof. Suppose G is a saturated critical (X,Y)-bigraph, and $d(x_0) = 2$ for some $x_0 \in X$. Let $N(x_0) = \{y_1, y_2\}$. We first prove part (a):

$$N(y_1) = N(y_2) = X. \tag{2}$$

Indeed, suppose $N(y_j) \neq X$ for some $j \in \{1,2\}$. Then by Lemma 20, $|X - N(y_j)| \geqslant 2$, say, $\{x, x'\} \subseteq X - N(y_j)$. Consider $A = \{x_0, x, x'\}$ and $B = \widehat{N}_G(A)$. Then $y_j \notin B$ and so $d_{G[A\cup B]}(x_0) \leqslant 1$, a contradiction to (1). This proves (a).

Suppose (b) does not hold and consider an $x \in X - \{x_0\}$ such that $d(x) \leqslant 3$. Note that by (a), x is adjacent to y_1 and y_2. For any $x' \in X - \{x, x_0\}$, Claim 17 for $A = \{x, x', x_0\}$ yields that there is a common neighbor $y(x')$ of x and x' distinct from y_1 and y_2. Since $d(x) \leqslant 3$, all $y(x')$ coincide, and hence there is a vertex y adjacent to all vertices in X apart from x_0, a contradiction to Lemma 20. This proves (b). □

Lemma 22. *If G is a saturated critical (X,Y)-bigraph, then for every $y \in Y$, $|N(y)| \neq |X|-2$.*

Proof. Suppose G is a saturated critical (X, Y)-bigraph, and $N(y_0) = X - \{x', x''\}$ for some $y_0 \in Y$. Let $|X| = k$.

Assume $d(x') \geqslant d(x'')$. By Lemma 21, $d(x') \geqslant 3$. Since G is saturated, it has a path $P = y_0x_1y_1x_2 \dots y_{k-1}x_k$ where $x_k = x'$. We may assume $x'' = x_j$ for some j.

If x_k is adjacent to any y_i for $i \in [k-2] - \{j-1\}$, then G has cycle

$$y_0x_{i+1}y_{i+1}x_{i+2} \dots x_ky_ix_i \dots y_0,$$

a contradiction. So $N(x_k) \cap V(P) \subseteq \{y_{k-1}, y_{j-1}\}$. Let $N(x_k) - V(P) = \{z_1, z_2, \dots, z_s\}$. Since $d(x_k) \geqslant 3$, $s \geqslant 1$. Let $T = X - \{x_k, x_{k-1}, x_{j-1}\}$. Again, if any z_ℓ is adjacent to any $x_i \in T$, then G has cycle $y_0x_{i+1}y_{i+1}x_{i+2} \dots x_kz_\ell x_i \dots y_0$, a contradiction. Hence

$$N(z_\ell) \cap T = \varnothing \qquad \textit{for all } 1 \leqslant \ell \leqslant s. \tag{3}$$

Since Claim 19 implies $k \geqslant 6$, $|T| \geqslant 3$. By Claim 17, for each $x_i, x_{i'} \in T$, G contains a 6-cycle C_1 with $V(C_1) \cap X = \{x_k, x_i, x_{i'}\}$, say $C_1 = x_kyx_iy'x_{i'}y''x_k$. By (3) and the fact that $N(x_k) \cap V(P) \subseteq \{y_{j-1}, y_{k-1}\}$, $\{y, y''\} \subseteq \{y_{k-1}, y_{j-1}\}$. In particular, $x_ky_{j-1} \in E(G)$.

Similarly, if there are $x_i, x_{i'} \in T$ both not adjacent to y_{k-1} or both not adjacent to y_{j-1}, then G does not contain a 6-cycle C_1 with $V(C_1) \cap A = \{x_k, x_i, x_{i'}\}$; however, G is 3-cyclic, a contradiction. This means $|N(y_{k-1}) \cap T| \geqslant |T| - 1$ and $|N(y_{j-1}) \cap T| \geqslant |T| - 1$. Since $|T| \geqslant 3$, this implies that there is $x_i \in T \cap N(y_{k-1}) \cap N(y_{j-1})$. Note that $i \notin \{j-1, k-1, k\}$, since $x_{j-1} \notin T$. Since $x_{i+1} \notin \{x_j, x_k\}$, y_0 is adjacent to x_{i+1}.

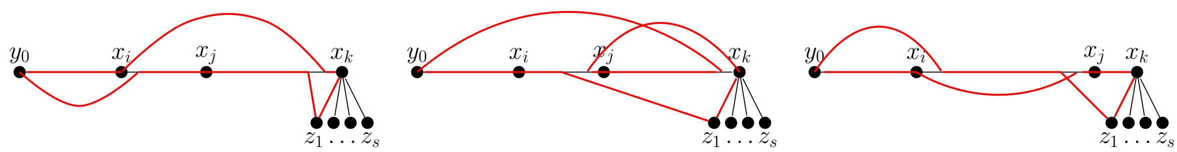

Figure 1: 3 configurations for Lemma 22

Since G is 2-connected, z_1 has a neighbor in $\{x_{k-1}, x_{j-1}\}$. If $z_1x_{k-1} \in E(G)$, then G has cycle $y_0x_1 \dots x_iy_{k-1}x_kz_1x_{k-1}y_{k-2}x_{k-2} \dots x_{i+1}y_0$. So $N(z_1) = \{x_k, x_{j-1}\}$ (see Fig. 1 (1)). If $j \neq k-1$, then G has the cycle $y_0x_1 \dots x_{j-1}z_1x_ky_{j-1}x_jy_jx_{j+1} \dots x_{k-1}y_0$ (see Fig. 1 (2)). Hence we may suppose $j = k-1$. Then by the definition of T, $i \leqslant k-3$. So G has the cycle $y_0x_1 \dots x_iy_{k-2}x_{k-1}y_{k-1}x_kz_1x_{k-2}y_{k-2} \dots x_{i+1}y_0$ (see Fig. 1 (3)), a contradiction. □

A critical (X, Y)-bigraph G is *Y-minimal* if every proper subgraph $G' = (X', Y'; E')$ of G satisfying (1) is super-cyclic.

Lemma 23. *If a saturated critical Y-minimal (X, Y)-bigraph G has vertices $y_1, y_2 \in Y$ of degree 2, then $N(y_1) \neq N(y_2)$.*

Proof. Suppose $N(y_1) = N(y_2) = \{x_1, x_2\}$, and consider the graph $G' := G - \{y_1\}$ with partite sets X and $Y' = Y - \{y_1\}$. Note that in G, each cycle of length at least 6 contains at most one vertex in $\{y_1, y_2\}$ since the neighbors of such a vertex on the cycle must be exactly x_1 and x_2. Hence for each cycle C of length at least 6 in G, there exists a cycle C' in G' with $C \cap X = C' \cap X$. We will show that (1) holds for G'.

Indeed, suppose there exists a set $A \subseteq X$ with $|\widehat{N}_{G'}(A)| < |A|$. Then $\{x_1, x_2\} \subseteq A$, $\widehat{N}_{G'}(A) = \widehat{N}_G(A) - \{y_1\}$, and hence $|\widehat{N}(A)| = |A|$. If $|A| \geqslant 4$, then $|\widehat{N}(A - \{x_1\})| \geqslant |A - \{x_1\}| = |A| - 1$. However, $\widehat{N}(A - \{x_1\}) \subseteq \widehat{N}(A) - \{y_1, y_2\}$, a contradiction. So $|A| = 3$, say $A = \{x_1, x_2, x_3\}$, $\widehat{N}_{G'}(A) = \{y_2, y_3\}$, and $\widehat{N}(A) = \{y_1, y_2, y_3\}$. But there is no 6-cycle in G based on A since $N(y_1) = N(y_2) = \{x_1, x_2\}$. This contradicts Claim 17.

Now suppose G' is not 2-connected. Then G' contains a cut vertex v, and $\{v, y_1\}$ is a cut set in G. This implies that x_1 and x_2 are in different components of $G - \{v, y_1\}$, and so $v = y_2$. Let $x_3 \in X - \{x_1, x_2\}$. Then there is no 6-cycle based on $\{x_1, x_2, x_3\}$ in G, a contradiction.

By the definition of critical Y-minimal bigraphs, G' is super-cyclic; but then G also is. □

Lemma 24. *If G is a saturated critical Y-minimal (X, Y)-bigraph, $x \in X$ and C is a cycle based on $X - \{x\}$, then x has at least two non-neighbors in $V(C) \cap Y$.*

Proof. Let $|X| = k$ and let $C = x_1 y_1 \dots x_{k-1} y_{k-1} x_1$. Suppose for the sake of contradiction that $|N(x) \cap V(C)| \geqslant k - 2$. If $N(x)$ contains a vertex y that is not in C, then because G is 2-connected, y has a neighbor in $V(C)$, say x_1. Then without loss of generality, $y_1 \in N(x)$, and we may replace the edge $x_1 y_1$ in C with the path $x_1 y x y_1$ to obtain a cycle based on X, a contradiction.

So we may assume $N(x) \subseteq V(C)$. Since $|\widehat{N}(X)| \geqslant |X|$, there exists a vertex $y \in \widehat{N}(X) \setminus V(C)$. Since G is 2-connected and $yx \notin E(G)$, y has some neighbors x_i and x_j in C. If $\{y_i, y_j\} \subseteq N(x)$ then we obtain the cycle $x_1 y_1 \dots x_i y x_j y_{j-1} \dots y_i x y_j x_{j+1} \dots x_1$, a contradiction. Similarly, we have that $\{y_{i-1}, y_{j-1}\} \not\subseteq N(x)$. The remaining case is $N(y) = \{x_i, x_{i+1}\}$ and $N(x) = V(C) - \{y_i\}$. By considering the cycle obtained by replacing y_i with y, we see that by symmetry, $N(y_i) = \{x_i, x_{i+1}\}$. But this contradicts Lemma 23. □

5 6-cyclic graphs

In this section, we complete the proof of Theorem 6 by proving that all (X, Y)-bigraphs satisfying (1) are 6-cyclic. We will use $N_C(x)$ to denote the neighborhoods of x that are in $V(C)$.

Lemma 25. *If G is a saturated critical (X, Y)-bigraph and $|X| = 6$, then X contains a vertex of degree at least 4.*

Proof. Suppose all vertices in X have degree at most 3.

Case 1: There is a vertex $y \in Y$ with $d(y) \geqslant 4$. By Lemma 22, $d(y) \neq 4$, so $d(y) \geqslant 5$. Let x_1, x_2, x_3, x_4, x_5 be five neighbors of y; let $C = x_1y_1x_2y_2x_3y_3x_4y_4x_5y_5x_1$ be a cycle containing them.

If $y \notin V(C)$, then x_1, x_2, x_3, x_4, x_5 each have two neighbors on C and are also adjacent to y. Since the degree of each x_i is at most 3, they cannot have any other neighbors. In that case, the set $A = \{x_1, x_2, x_4\}$ contradicts (1), since $\widehat{N}(A) = \{y_1, y\}$.

Therefore $y \in V(C)$; say, $y = y_1$. Then x_3, x_4, x_5 have two neighbors on C and an edge to y_1, so they have degree 3. By (1) applied to $A = \{x_1, x_3, x_5\}$, x_1 must have an edge to one of y_2, y_3, y_4; symmetrically, x_2 must have an edge to one of y_3, y_4, y_5. This yields 3 edges incident to each of x_1, x_2, x_3, x_4, x_5; none of these can have any other neighbors.

By (1), $|\widehat{N}(X)| \geqslant 6$; however, since there is only one vertex in $X - V(C)$, $\widehat{N}(X) \subseteq N(X \cap V(C)) = Y \cap V(C)$. This only has size 5, a contradiction.

Case 2: All vertices in Y have degree at most 3. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Let $C_1 = x_1y_1x_2y_2x_3y_3x_1$ be a 6-cycle based on $\{x_1, x_2, x_3\}$ and let $C_2 = x_4y_4x_5y_5x_6y_6x_4$ be a 6-cycle based on $\{x_4, x_5, x_6\}$. We have $V(C_1) \cap V(C_2) = \varnothing$, since a vertex in $V(C_1) \cap V(C_2)$ would have degree at least 4.

In the cycle based on $\{x_1, x_2, x_4\}$, the vertex x_4 must have two common neighbors with $\{x_1, x_2\}$. Since $\Delta(G) \leqslant 3$, at least one of them is a neighbor of x_4 on C_2. Without loss of generality, let y_4 be a common neighbor with x_1, so that $x_1y_4 \in E(G)$.

Now consider the cycle based on $\{x_1, x_4, x_5\}$. By the same argument, either x_4 or x_5 must be adjacent to one of x_1's neighbors on C_1. Without loss of generality, let x_4y_1 be that edge; then the cycle $x_1y_4x_5y_5x_6y_6x_4y_1x_2y_2x_3y_3x_1$ is based on X, a contradiction. □

Claim 26. *All (X, Y)-bigraphs G satisfying (1) are 6-cyclic.*

Proof. Take a vertex-minimal counterexample G with the most edges, meaning in particular that $|X| = 6$ and $Y = \widehat{N}(X)$. By Claims 17–19, G is k-cyclic for $3 \leqslant k \leqslant 5$; therefore G is critical, saturated and Y-minimal.

Let $X = \{x_1, \ldots, x_6\}$ and x_6 be a vertex of maximum degree in X. By Lemma 25, $d(x_6) \geqslant 4$. Let $C = x_1y_1x_2y_2x_3y_3x_4y_4x_5y_5x_1$ be a cycle based on $X - \{x_6\}$. By Lemma 16 and Lemma 24, x_6 has either 2 or 3 neighbors on C, so it has at least one neighbor y_6 not on C.

By symmetry, the following two cases are exhaustive.

Case 1: $\{y_1, y_3\} \subseteq N_C(x_6)$. In this case, by Claim 14, no vertex $y \in N(x_6) - V(C)$ can be adjacent to x_1, x_2, x_3, or x_4, so it must be adjacent to x_5 and x_6 only. By Lemma 23, y_6 is the only such vertex. By Claim 14 again, x_6 cannot be adjacent to y_4 or y_5. To have $d(x_6) \geqslant 4$, x_6 must also be adjacent to y_2, and therefore $d(x_6) = 4$.

If $x_2y_4 \in E(G)$, then the cycle $x_2y_4x_4 \ldots y_2x_6y_6x_5y_5x_1y_1x_2$ is based on X, and if $x_2y_5 \in E(G)$, the cycle $x_2y_2 \ldots x_5y_6x_6y_1x_1y_5x_2$ is based on X. Thus, $x_2y_4 \notin E(G)$ and $x_2y_5 \notin E(G)$. A similar argument shows that $x_3y_5, x_3y_4 \notin E(G)$. However, applying Claim 17 to $A = \{x_2, x_3, x_5\}$, we find distinct vertices $y' \in N(x_2) \cap N(x_5)$ and $y'' \in N(x_3) \cap N(x_5)$ such that $y', y'' \notin \{y_4, y_5, y_6\}$. Therefore x_5 is adjacent to y_4, y_5, y_6, y', y'', and $d(x_5) \geqslant 5 > d(x_6)$, contradicting the choice of x_6.

Case 2: $N_C(x_6) = \{y_1, y_2\}$. In this case, in order to have $d(x_6) \geqslant 4$, x_6 must have neighbors y, y' outside C. By Claim 14, y and y' can only have x_4 and x_5 as neighbors. If y is adjacent to x_4 and y' is adjacent to x_5, or vice versa, we contradict Claim 15; if both are adjacent only to x_4 or both only to x_5, we contradict Lemma 23. □

6 Bigraphs with high minimum degree

6.1 Properties of smallest counterexamples

Throughout this subsection, we assume that G is a vertex-minimal counterexample to Theorem 8 with the most edges; let $G \in \mathcal{G}(n, m, \delta)$ where $\delta \geqslant \max\{n, \frac{m+10}{4}\}$. Then for each $X' \subset X$ with $X' \neq X$, $G[X' \cup Y]$ also satisfies the conditions of Theorem 8 and hence is super-cyclic.

Let $G' = G[X \cup \widehat{N}(X)]$, i.e., G' is obtained by removing only the degree-1 vertices of G. Then G' is critical and saturated. In particular, for every $x \in X$, there exists a cycle C in G' (and therefore in G) such that $V(C) \cap X = X - \{x\}$. By Lemma 12, G' has an $x, V(C)$-fan F of size 2.

Among the triples (C, x, F) where $x \in X$, C is a cycle with $V(C) \cap X = X - \{x\}$ and F is an $x, V(C)$-fan, choose a triple such that the size of F is maximized, and subject to this, $|V(F)|$ is minimized. Let $|V(C)| = 2\ell$ (so $|X| = \ell + 1$). Let t be the size of F, and let $T = V(C) \cap V(F) = \{u_1, \ldots, u_t\}$.

Fix a clockwise direction of C. For every vertex u of C, $x_C^+(u)$ (respectively, $x_C^-(u)$) denotes the closest to u clockwise (respectively, counterclockwise) vertex of X distinct from u. For a set $U \subset V(C)$, $X_C^+(U) = \{x_C^+(u) \ : \ u \in U\}$. When C is clear from the content, the subscripts could be omitted. The vertices $y^+(u), y^-(u)$ and the sets $X^-(U), Y^+(U), Y^-(U)$ are defined similarly.

Viewing F as a tree (spider) with root x, any two vertices $u, v \in V(F)$ define the unique u, v-path $F[u, v]$ in F. For $u, v \in V(C)$, let $C[u, v]$ be the clockwise u, v-path in C and let $C^-[u, v]$ be the counterclockwise u, v-path in C.

Lemma 27. $t \leqslant \ell - 2$.

Proof. We first show that

$$t \leqslant \ell - |T \cap X|. \tag{4}$$

If $w \in T \cap X$ and $y^+(w) \in T$, then the cycle $wF[w, y^+(w)]y^+(w)C[y^+(w), w]w$ is based on X, a contradiction. Similarly, $y^-(w), x^+(w), x^-(w) \notin T$. Thus, $|T \cap X| \leqslant \ell/2$ and $|T \cap Y| \leqslant \ell - 2|T \cap X|$. This proves (4).

For the remainder of the proof, note that if Claims 13–15 are applied to G', then the conclusions hold for G as well, since they are unaffected by the addition of vertices of degree 1 in Y.

Let $C = x_1y_1 \ldots x_\ell y_\ell x_1$, and suppose $t \geqslant \ell - 1$. By (4), $|T \cap X| \leqslant 1$. If $T \cap X = \varnothing$, we may assume that $xy_i \in E(G)$ for all $1 \leqslant i \leqslant \ell - 1$. By (1), $|\widehat{N}(X)| \geqslant \ell + 1$, so there is $y \in Y - V(C)$ with at least two neighbors in X. This will contradict one of Claims 13–15 (possibly, in reversed orientation of C), unless all such y are adjacent to only $x_\ell = x^-(y_\ell)$ and $x_1 = x^+(y_\ell)$. Fix such a vertex y. Let $A = X - \{x_\ell\}$. There exists a vertex $y' \in (Y - V(C)) \cup \{y_\ell\}$ such that $y' \in \widehat{N}(A)$, i.e., y' has two neighbors other than x_ℓ (so $y' \neq y$). Let C' be the cycle obtained by replacing y_ℓ with y. Then the vertex y' violates one of Claims 13–15 with respect to C'.

If $|T \cap X| = 1$, then by (4), we may assume that $xy_i \in E(G)$ for all $1 \leqslant i \leqslant \ell - 2$ and that x has a common neighbor $y \in Y - V(C)$ with x_ℓ. By (1), $|\widehat{N}(X - x_\ell)| \geqslant \ell$, so there is $y_0 \in (Y - V(C)) \cup \{y_{\ell-1}, y_\ell\}$ with at least two neighbors in $X - x_\ell$. If $y_0 \in (Y - V(C))$, this again will contradict one of Claims 13–15, unless $N(y_0) = \{x_{\ell-1}, x_1\}$. In this case, we obtain the longer cycle $y_1C[y_1, y_{\ell-2}]y_{\ell-2}xyx_\ell y_{\ell-1}x_{\ell-1}y_0x_1$. So suppose without loss of generality $y_0 = y_\ell$ has a neighbor $z \in X - \{x_\ell, x_1\}$. By the case, $z \neq x$, so suppose $z = x_j$ for some $2 \leqslant j \leqslant \ell - 1$. Then G has cycle $y_\ell C[y_\ell, y_{j-1}]y_{j-1}xyx_\ell C^-[x_\ell, x_j]x_jy_\ell$ based on X, a contradiction. □

Given a cycle C and distinct $x_1, x_2, x_3 \in X \cap V(C)$, we say that x_1 *and* x_2 **cross** *at* x_3 if the cyclic order is x_1, x_3, x_2 and $x_1y^+(x_3), x_2y^-(x_3) \in E(G)$ or if the cyclic order is x_1, x_2, x_3 and $x_1y^-(x_3), x_2y^+(x_3) \in E(G)$. In this case, we also say that x_3 *is* **crossed** *by* x_1 *and* x_2.

The following is Lemma 2.8 in [9]. It holds for each bipartite graph G (no restrictions).

Lemma 28 ([9])**.** *Let C be a cycle of an (X,Y)-bigraph G, and let $u, v \in V(C) \cap X$. If u and v have at most r crossings, then $d_C(u) + d_C(v) \leqslant |V(C)|/2 + 2 + r$.*

Proof. We induct on r. Suppose $r = 0$. Consider the two paths $P_1 = C[u, v]$ and $P_2 = C^-[u, v]$. In $P_1 = v_1 \dots v_k$ ($v_1 = u, v_k = v$), each $v_i \in X$ satisfies at most one of the following: $v_{i+1}u \in E(G)$ or $v_{i-1}v \in E(G)$. So $d_{P_1}(u) + d_{P_1}(v) \leqslant |V(P_1) \cap X|$. Similarly, $d_{P_2}(u) + d_{P_2}(v) \leqslant |V(P_2) \cap X|$. Since $(X \cap V(P_1)) \cap (X \cap V(P_2)) = \{u, v\}$ and $V(P_1) \cup V(P_2) = V(C)$, we get $d_C(u) + d_C(v) \leqslant |V(C)|/2 + 2$.

For $r \geqslant 1$, delete an edge incident to u that is used in a crossing, and induct. □

Lemma 29. *If $u_i \in X \cap T$, then $y^+(u_i)$ has no neighbors in $(F - V(C)) \cup X^+(T) \setminus \{x^+(u_i)\}$.*

Proof. Suppose $y^+(u_i)$ has a neighbor z in $F - V(C)$. Then the cycle

$$u_iF[u_i, z]zy^+(u_i)C[y^+(u_i), u_i]u_i$$

is based on X, a contradiction.

Suppose now that $y^+(u_i)$ has a neighbor x_1 in $X^+(T) \setminus \{x^+(u_i)\}$, where $u \in T$ satisfies $x^+(u) = x_1$. Then the cycle $x_1y^+(u_i)C[y^+(u_i), u]uF[u, u_i]u_iC^-[u_i, x_1]x_1$ is based on X, a contradiction. □

Lemma 30. *If $x_1 \in X^+(T)$, then x_1 cannot have a neighbor in $F - V(C)$.*

Proof. Suppose x_1 has a neighbor y' in $F - V(C)$. Let $u_1 \in T$ be such that $x_1 = x^+(u_1)$ and z be a neighbor of u_1 in F. Let P be a z, y'-path in F and the cycle C' be defined by $C' = x_1C[x_1, u_1]u_1zPy'x_1$. If $y' \neq z$, then C' is based on X and we are done. Thus $z = y'$ and hence $u_1 \in X$. Let $F' = F - u_1$. Note that F' is an $x, V(C')$-fan such that $|V(F \cap C)| = |V(F' \cap C')|$, but $|V(F')| < |V(F)|$, contradicting the choice of C and F. □

Lemma 31. *Suppose that $x_1, x_2 \in X^+(T)$. Then*

(i) x_1 and x_2 share no neighbors in $Y - V(C)$;

(ii) neither x_1 nor x_2 share a neighbor in $Y - V(C)$ with x.

Proof. Part (i) follows from Claim 13. From Lemma 30, if x_1 and x have a common neighbor outside of C, it is not in F. Suppose they share some neighbor $y \in Y - V(C)$. Let $x_1 = x^+(u_1)$. Then we have a longer cycle $x_1C[x_1, u_1]u_1F[u_1, x]xyx_1$. The same holds for x_2 and x. This proves (ii). □

Lemma 32. *Suppose $u_1, u_2 \in T$. If $x_1 = x^+(u_1)$ and $x_2 = x^+(u_2)$ cross at $x_3 \in X \cap V(C)$, then*

(i) $x_3 \notin T$;

(ii) G has a cycle C' containing $(X \cap V(C) - \{x_3\}) \cup \{x\}$ such that $|C'| \geqslant |C|$;

(iii) x_3 shares no neighbors in $Y - V(C)$ with any vertex in the set $\{x\} \cup X^+(T)$;

(iv) x_3 has at most t neighbors on C.

Proof. Suppose that the cyclic order is x_1, x_3, x_2 and $x_1y^+(x_3), x_2y^-(x_3) \in E(G)$ (the other case is symmetric).

For part (i), let y be a neighbor of x_3 in F. Let z be a neighbor of u_1 in F. Let P be a z,y-path in F and the cycle C' be defined by

$$C' := x_1y^+(x_3)C[y^+(x_3), u_1]u_1zPyx_3C^-[x_3, x_1]x_1.$$

Then C' is based on X. This contradiction proves (i).

The cycle

$$C_1 := x_1y^+(x_3)C[y^+(x_3), u_2]u_2F[u_2, u_1]u_1C^-[u_1, x_2]x_2y^-(x_3)C^-[y^-(x_3), x_1]x_1$$

proves (ii).

To prove (iii), assume that $y \in Y - V(C)$ is a common neighbor of x_3 and a vertex in $\{x\} \cup X^+(T)$, and consider all cases. If $yx \in E(G)$, let

$$C' = x_1y^+(x_3)C[y^+(x_3), u_1]u_1F[u_1, x]xyx_3C^-[x_3, x_1]x_1.$$

If y is not in $F[x, u_1]$, then C' is a cycle based on X, a contradiction. Otherwise, let F'' be $F - F[u_1, y]$. Note F'' is an $x, V(C'')$-fan where

$$C'' = x_1y^+(x_3)C[y^+(x_3), u_1]u_1F[u_1, y]yx_3C^-[x_3, x_1]x_1,$$

and $|V(F \cap C)| = |V(F'' \cap C'')|$, but $|V(F'')| < |V(F)|$, contradicting the choice of C and F.

If $u_j \in T$, $x_j = x^+(u_j)$, $yx_j \in E(G)$, and $x_j \in C[y^+(x_3), u_1]$, then the cycle

$$C' := x_1 C[x_1, x_3] x_3 y x_j C[x_j, u_1] u_1 F[u_1, u_j] u_j C^-[u_j, y^+(x_3)] y^+(x_3) x_1$$

is based on X. Similarly, if $x_j \in C[u_1, y^-(x_3)]$, then the cycle

$$C' := x_2 C[x_2, u_j] u_j F[u_j, u_2] u_2 C^-[u_2, x_3] x_3 y x_j C[x_j, y^-(x_3)] y^-(x_3) x_2$$

is based on X, a contradiction. This proves (iii).

By the choice of (C, x, F) and (ii), x_3 has at most t neighbors on C_1. The only vertices in $Y \cap V(C) - V(C_1)$ are $y^-(x_1)$ and $y^-(x_2)$. If $x_3 y^-(x_1) \in E(G)$, then the cycle

$$y^-(x_1) C[y^-(x_1), y^-(x_3)] y^-(x_3) x_2 C[x_2, u_1] u_1 F[u_1, u_2] u_2 C^-[u_2, x_3] x_3 y^-(x_1)$$

is based on X. If $x_3 y^-(x_2) \in E(G)$, then the cycle

$$x_1 C[x_1, x_3] x_3 y^-(x_2) C[y^-(x_2), u_1] u_1 F[u_1, u_2] u_2 C^-[u_2, y^+(x_3)] y^+(x_3) x_1$$

is based on X. This proves (iv). ☐

Lemma 33. *Suppose $u_1, u_2 \in T$, $x_1 = x^+(u_1)$, and $x_2 = x^+(u_2)$. Then no two vertices $x_3, x_4 \in V(C)$ crossed by x_1 and x_2 have a shared neighbor in $Y - V(C)$.*

Proof. Suppose vertices $x_3, x_4 \in V(C) \cap X$ are crossed by x_1 and x_2 and there is some $y \in (N(x_3) \cap N(x_4)) - V(C)$. By Lemma 32, $y \notin V(F)$.

We consider two cases. If x_3 and x_4 both are on $C[x_1, x_2]$ or both are on $C[x_2, x_1]$, then we may assume that their cyclic order is x_1, x_3, x_4, x_2. In this case, the cycle

$$x_1 C[x_1, x_3] x_3 y x_4 C[x_4, u_2] u_2 F[u_2, u_1] u_1 C^-[u_1, x_2] x_2 y^-(x_4) C^-[y^-(x_4), y^+(x_3)] y^+(x_3) x_1$$

is based on X.

If one of x_3 and x_4 is on $C[x_1, x_2]$ and the other is on $C[x_2, x_1]$, then we may assume that their cyclic order is x_1, x_3, x_2, x_4. In this case, the cycle

$$x_1 C[x_1, x_3] x_3 y x_4 C^-[x_4, x_2] x_2 y^+(x_4) C[y^+(x_4), u_1] u_1 F[u_1, u_2] u_2 C^-[u_2, y^+(x_3)] y^+(x_3) x_1$$

is based on X. This proves the lemma. ☐

Lemma 34. *Let $A \subseteq X^+(T)$. Then $\sum_{w \in A} d_C(w) \leqslant |A|(\ell - 2) + 2$.*

Proof. Let $x_1, x_2 \in A$ such that $x_1 = x^+(u_1)$ and $x_2 = x^+(u_2)$ for some $u_1, u_2 \in T$. We first prove that

$$\text{if } u_2 \in Y \text{ and } y^+(x_2) x_1 \in E(G), \text{ then } d_C(x_2) \leqslant \ell - 2. \tag{5}$$

The cycle $C' = x_1 y^+(x_2) C[y^+(x_2), u_1] u_1 F[u_1, u_2] u_2 C^-[u_2, x_1] x_1$ contains all vertices in C except x_2 and possibly $y^+(u_1)$ (in the case that $u_1 \in X$). By Lemma 29 and Lemma 30, $N_C(x_2) = N_{C'}(x_2)$. By Lemma 27 applied to C' and x_2, $d_C(x_2) = d_{C'}(x_2) \leqslant \ell - 2$.

In particular, if $d_C(x_1) = \ell$, i.e., x_1 is adjacent to every y vertex in C, then by Lemma 29, each $x_2 \in X^+(T) - \{x_1\}$ satisfies $u_2 \in Y$. Therefore by (5), $d_C(x_2) \leqslant \ell - 2$. It follows that $\sum_{w \in A} d_C(w) \leqslant |A|(\ell - 2) + 2$.

So suppose every $w \in A$ has $d_C(w) \leqslant \ell - 1$, and there exist two vertices $x_1, x_2 \in A$ with equality. Define u_1, u_2 as before. Then for every $x_3 \in X^+(T) - \{x_1, x_2\}$, either $u_1 \in Y$ and $x_3 y^+(x_1) \notin E(G)$ by (5), or $u_1 \in X$ and $x_3 y^+(x_1) \notin E(G)$ by Lemma 29. The same holds for x_3 and x_2. Therefore $d_C(x_3) \leqslant \ell - 2$, and again $\sum_{w \in A} d_C(w) \leqslant |A|(\ell - 2) + 2$. □

Lemma 35. *Suppose $t \geqslant 4$, $u_1, u_2 \in T$, $x_1 = x^+(u_1)$, and $x_2 = x^+(u_2)$. Then at most one vertex in C is crossed by x_1 and x_2.*

Proof. Suppose vertices $x_3, x_4 \in V(C) \cap X$ are crossed by x_1 and x_2.

Let $A = X^+(T) \cup \{x, x_3, x_4\}$ (possibly, $X^+(T) \cap \{x_3, x_4\} \neq \varnothing$), and $A' = A - \{x, x_3, x_4\}$. Note that $|A'| \geqslant t - 2$, and by Lemma 34 applied to A', $\sum_{w \in A'} d_C(w) \leqslant |A'|(\ell - 2) + 2$.

Since x can have at most t neighbors on C, $|N(x) - V(C)| \geqslant \delta - t$. By Lemma 32(iv), $|N(x_3) - V(C)| \geqslant \delta - t$ and $|N(x_4) - V(C)| \geqslant \delta - t$. By Claims 13–15 (applied to G') and Lemmas 32(iii) and 33, no two distinct vertices in A have a common neighbor in $Y - V(C)$. Thus, remembering the ℓ vertices in $Y \cap V(C)$, we get

$$
\begin{aligned}
|Y| &\geqslant \ell + \sum_{u \in A} |N(u) - V(C)| \\
&= \ell + |N(x) - V(C)| + |N(x_3) - V(C)| + |N(x_4) - V(C)| + \sum_{u \in A'} |N(u) - V(C)| \\
&\geqslant \ell + 3(\delta - t) + \delta|A'| - (\ell - 2)|A'| - 2 \\
&\geqslant \ell + 3\delta - 3t + (\delta - \ell + 2)|A'| - 2 \\
&\geqslant \ell + 3\delta - 3t + (\delta - \ell + 2)(t - 2) - 2 \\
&\geqslant \ell + 3\delta - 3t + (\delta - \ell + 2) + (\delta - \ell + 2)(t - 3) - 2 \\
&\geqslant \ell + 3\delta - 3t + (\delta - \ell + 2) + 3(t - 3) - 2 \\
&= 4\delta - 3t + 2 + 3(t - 3) - 2 = 4\delta - 9,
\end{aligned}
$$

as claimed. □

Lemma 36. *For any $x_1, x_2 \in X$, x_1 and x_2 cannot be separated by a set of two vertices.*

Proof. Recall that G is a vertex-minimum counterexample to Theorem 8, and $G' = G[X \cup \widehat{N}(X)]$ is critical and saturated.

Suppose that for some $x_1, x_2 \in X, u_1, u_2 \in V(G)$, x_1 and x_2 are in different components of $G - \{u_1, u_2\}$. Note that $u_1, u_2 \in V(G')$, since $V(G) - V(G')$ contains only vertices of degree 1 in Y.

If there also exists $x_3 \in X - \{x_1, x_2\}$ such that x_3 is in a different component of $G - \{u_1, u_2\}$ than both x_1 and x_2, then G cannot contain a 6-cycle based on $\{x_1, x_2, x_3\}$, since these vertices are separated by a set of size two. Hence we may assume $G - \{u_1, u_2\}$ contains exactly two components containing vertices in X. Call these components D_1 and D_2 where $x_1 \in V(D_1)$ and $x_2 \in V(D_2)$.

Choose any two vertices $x, x' \in X - \{u_1, u_2\}$; then choose a third vertex $x'' \in X - \{u_1, u_2\}$ such that not all three of x, x', x'' are in the same component of $G - \{u_1, u_2\}$. Let C be a cycle based on $A = \{x, x', x''\}$.

Since $\{u_1, u_2\}$ separates one of the vertices of A from the others, $u_1, u_2 \in V(C)$; since $V(C) \cap X = A$ and neither u_1 nor u_2 is in A, we must have $u_1, u_2 \in Y$.

Moreover, u_1 must have an edge to either x or x' in C, and therefore in G. Since $x, x' \in X$ were arbitrary, $|N(u_1)| \geqslant |X| - 1$. By Lemma 20 applied to G', $N_{G'}(u_1) = X$, and therefore $N(u_1) = X$. By symmetry, we also obtain $N(u_2) = X$.

Now suppose each component of $G - \{u_1, u_2\}$ has at least 2 vertices in X. For $i \in \{1, 2\}$, set $X_i = X \cap D_i$. By the minimality of G, there exists a cycle C_1 of G based on $X_1 \cup \{x_2\}$ and a cycle C_2 based on $X_2 \cup \{x_1\}$. Since D_1 and D_2 are separated by $\{u_1, u_2\}$, $N_{C_1}(x_2) = N_{C_2}(x_1) = \{u_1, u_2\}$. Therefore $(C_1 - \{x_2\}) \cup (C_2 - \{x_1\})$ is a cycle in G which is based on X, a contradiction.

Thus we may assume without loss of generality that $V(D_1) \cap X = \{x_1\}$. Note that this implies all neighbors of x_1 other than u_1 and u_2 have degree 1. Let G_1 be obtained from G by deleting x_1 and all of its neighbors except for u_1.

We will show that G_1 is a counterexample that has fewer vertices than G. Set $X' = X - \{x_1\} = X \cap V(G_1)$. If there exists $A \subseteq X'$ with $|A| \geqslant 3$ such that $|\widehat{N}_{G_1}(A)| < |A|$, then in G, $\widehat{N}_G(A \cup \{x_1\}) = \widehat{N}_{G_1}(A) \cup \{u_2\} < |A \cup \{x_1\}|$, a contradiction.

Next, we will show that for all A with $|A| \geqslant 3$, $G_1[A \cup \widehat{N}_{G_1}(A)]$ is 2-connected. Recall that $G_1 - \{u_1\} = D_2$. The subgraph of D_2 obtained by removing all vertices in Y of degree 1 is still connected. Call this subgraph H. If $A = X'$, then $G_1[A \cup \widehat{N}_{G_1}(A)] = G_1[H \cup \{u_1\}]$. Since H is connected and u_1 is adjacent to all vertices in X', $G_1[H \cup \{u_1\}]$ is 2-connected. Now suppose $A \neq X'$. Then by the choice of G as a minimum counterexample, there exists a cycle C in G with $V(C) \cap X = A \cup \{x_1\}$, where $N_C(x_1) = \{u_1, u_2\}$. In particular, $P := C - \{x_1, u_1, u_2\}$ is a path containing all vertices of A. In G_1, $G_1[A \cup \widehat{N}_{G_1}(A)]$ can be obtained from P by adding u_1, which is adjacent to all of $V(P) \cap X$, and possibly adding some additional vertices in Y with degree at least 2. Hence it is 2-connected.

Next, suppose that it is super-cyclic. By the minimality of G, G contains no cycle C based on X; however, because G_1 is super-cyclic, we may find a cycle $C' = v_1 v_2 \dots v_{2|X'|} v_1$ in G_1 (and therefore in G) based on $X - \{x_1\}$ such that $u_2 \notin V(C')$. If $u_1 \notin V(C')$, then we may replace in C' any segment $v_i v_{i+1} v_{i+2}$ (for $v_i \in X$) with the path $v_i u_1 x_1 u_2 v_{i+2}$ to obtain a contradiction. Otherwise, if $u_1 = v_i$ for some i, we replace the path $v_{i-1} u_1 v_{i+2}$ with $v_{i-1} u_1 x u_2 v_{i+2}$.

Finally, we have $|Y \cap G_1| \leqslant |Y| - (\delta - 1) \leqslant (4\delta - 10) - (\delta - 1) \leqslant 4(\delta - 1) - 10$. The last inequality holds because we may assume that $|X| \geqslant 7$ and therefore $\delta \geqslant 7$, since Theorem 6 implies Theorem 8 for $|X| \leqslant 6$. This shows that G_1 is a counterexample for Theorem 8 (with $\delta' = \delta - 1$) which has fewer vertices than G, contradicting the choice of G. □

6.2 Proof of Theorem 8

Proof of Theorem 8. As in the previous subsection, suppose for the sake of contradiction that G is a vertex-minimum, edge-maximal counterexample to Theorem 8. By the choice of G, for each $x \in X$, there exists some cycle C with $V(C) \cap X = X - \{x\}$. We may also assume that $|X| \geqslant 7$ and therefore $\delta \geqslant 7$, since Theorem 6 implies Theorem 8 for $|X| \leqslant 6$.

Letting $G' = G[X \cup \widehat{N}(X)]$, it follows from our choice of G that G' is critical and saturated.

If there exists a pair (x, C) with an $x, V(C)$-fan F of size at least 4, then choose such a triple which maximizes $t = |V(F) \cap V(C)|$, and subject to this, minimizes $|V(F)|$. Let $T = V(F) \cap V(C)$. By Lemmas 31 and 35, no two vertices in $X^+(T) \cup \{x\}$ share a neighbor outside of $V(C)$, and no two vertices in $X^+(T)$ cross more than one time. By Lemma 28, for each pair $x_1, x_2 \in X^+(T)$, $d_C(x_1) + d_C(x_2) \leqslant |V(C) \cap Y| + 3 = |X| + 2$. Therefore

$$\begin{aligned} |Y| &\geqslant |V(C) \cap Y| + \sum_{w \in X^+(T) \cup \{x\}} d_{Y-V(C)}(w) \\ &\geqslant |X| - 1 + \delta(t+1) - \sum_{w \in X^+(T) \cup \{x\}} d_C(w) \\ &\geqslant |X| - 1 + \delta(t+1) - t - t(|X|+2)/2. \end{aligned}$$

Since the coefficient at t is at least $\delta - 1 - (\delta+2)/2 > 0$ (assuming, as we do, that $\delta > 4$), this quantity is minimized whenever t is minimized, i.e., $t = 4$. We obtain $|Y| \geqslant |X| - 1 + 5\delta - 4 - 2(|X|+2)$, which is minimized when $|X| = \delta$. So $|Y| \geqslant 4\delta - 9$, a contradiction.

Now suppose that for all $x \in X$ and cycles C with $V(C) \cap X = X - \{x\}$, the largest $x, V(C)$-fan has size at most 3. Choose $x \in X$ with the maximum number of neighbors of degree at least 2. If every $x \in X$ has at most 3 neighbors of degree at least 2 (and at least $\delta - 3$ neighbors of degree 1), then we have $|Y| \geqslant |X|(\delta - 3) + 3$; since $|X| \geqslant 4$, $|Y| \geqslant 4\delta - 9$, a contradiction.

Therefore x has at least 4 neighbors of degree at least 2. Let F be a maximum $x, V(C)$-fan of G and set $T = F \cap V(C)$. By Lemma 36, $|T| \geqslant 3$, since x cannot be separated from $X - x'$ by a set of size 2. So $|T| = 3$.

By Lemma 16, $|T \cap Y| \geqslant 2$ (we apply this lemma to G', but the conclusion carries over to G). If $|T \cap Y| = 3$, then since x has at least 4 neighbors of degree at least 2, there exists $y \in N(x) - V(C)$. Since all vertices in $X - \{x\}$ are contained in C, y has a neighbor $x' \in C$. But then $F \cup xyx'$ is an $x, V(C)$-fan of size 4, a contradiction.

Finally, we may assume $T \cap V(C) = \{x_1, y_1, y_2\}$, where x and x_1 have at least 2 common neighbors outside C. In particular, $\{y_1, y_2\} \subset N(x)$, and for any $x' \neq x, x_1$, we have $N(x) \cap N(x') \subseteq \{y_1, y_2\}$, otherwise we could find a larger $x, V(C)$-fan. We will show that $N(y_1) = N(y_2) = X$. If there exists $x', x'' \in X - \{x_1\}$ such that $x'y_i, x''y_i \notin E(G)$ for some $i \in \{1, 2\}$, then there cannot exist a 6-cycle based on $\{x, x', x''\}$, a contradiction.

Hence $|N(y)| \geqslant |X| - 2$ which implies $N(y) = X$ by Lemma 22 (again, we apply this lemma to G', but the conclusion carries over to G).

Consider $y \in \widehat{N}(x) - \{y_1, y_2\}$. Since there is no $x, V(C_x)$-fan of size 4, $N(y) \subseteq T \cup \{x\}$. That is, $N(y) = \{x, x_1\}$ and so $\widehat{N}(x) \subseteq N(x_1)$. Recall that we chose x to have a maximum number of neighbors of degree at least 2. Additionally, note that $V(C) \cap Y \subseteq \widehat{N}(X)$. Thus $N_C(x_1) = \{y_1, y_2\}$, since otherwise $|\widehat{N}(x_1)| > |\widehat{N}(x)|$. But then $\{y_1, y_2\}$ separates $\{x, x_1\}$ from the rest of the vertices in X, contradicting Lemma 36. □

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